Shear-rate-dependent transport coefficients for inelastic Maxwell models

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Abstract

The Boltzmann equation for d-dimensional inelastic Maxwell models is considered to analyse transport properties in spatially inhomogeneous states close to the simple shear flow. A normal solution is obtained via a Chapman–Enskog-like expansion around a local shear flow distribution $f^{(0)}$ that retains all the hydrodynamic orders in the shear rate. The constitutive equations for the heat and momentum fluxes are obtained to first order in the deviations of the hydrodynamic field gradients from their values in the reference state and the corresponding generalized transport coefficients are *exactly* determined in terms of the coefficient of restitution α and the shear rate a. Since $f^{(0)}$ applies for arbitrary values of the shear rate and is not restricted to weak dissipation, the transport coefficients turn out to be nonlinear functions of both parameters a and α. A comparison with previous results obtained for inelastic hard spheres from a kinetic model of the Boltzmann equation is also carried out.

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1. Introduction

Granular gases are usually modelled as a gas of hard spheres dissipating part of their kinetic energy during collisions. In the simplest model, the grains are taken to be smooth so that the inelasticity of collisions is characterized only through a constant coefficient of normal restitution $\alpha \leqslant 1$. For a low-density gas, the Boltzmann kinetic equation has been conveniently modified to account for inelastic binary collisions [1, 2] and the Navier–Stokes transport coefficients [3] for states with small hydrodynamic gradients have been computed by means of the Chapman–Enskog method [4] around the local version of the homogeneous cooling state. As in the case of elastic collisions, the transport coefficients are given in terms of the solutions of exact linear integral equations which are approximately solved by considering the leading terms in a Sonine polynomial expansion of the velocity distribution function. On

the other hand, in spite of this approach, the Sonine predictions compare in general quite well with computer simulations, even for quite strong dissipation conditions [5].

Needless to say, the mathematical difficulties of solving the Boltzmann equation for inelastic hard spheres (IHS) increase considerably when one considers situations for which large gradients occur and more complex constitutive equations than the Navier–Stokes ones (which are linear in the spatial gradients) are required. For this kind of situations one has to resort to alternative approaches, such as the use of simplified kinetic models, where the true Boltzmann collision operator is replaced by a simpler collision model (e.g., the BGK model [6]) that preserves its most relevant physical features. In the case of elastic collisions, this route has been shown to be very fruitful since most of the exact solutions derived from the BGK model agree rather well [7] with results obtained by numerically solving the Boltzmann equation by means of the direct simulation Monte Carlo (DSMC) method [8]. Nevertheless, much less is known for inelastic collisions although some exact results obtained in the simple or uniform shear flow (USF) problem [9] and in the nonlinear Couette flow state [10] support the reliability of the inelastic version of the BGK model [9] for granular gases as well. The USF state is perhaps the simplest flow problem since the only nonzero hydrodynamic gradient is $\partial u_x/\partial y \equiv a = $ const, where $\mathbf{u}$ is the flow velocity and a is the constant shear rate. Due to its simplicity, this state has been widely studied in the past for elastic [7] and inelastic gases [11] as an ideal testing ground to shed light on the intricacies associated with the response of the system to strong shear rates. Very recently [12], the inelastic BGK model has been applied to determine the transport coefficients for a granular gas in spatially inhomogeneous states close to the USF. The heat and momentum fluxes were evaluated to first order in the deviations of the hydrodynamic field gradients from their values in the reference USF state. Given that the system is strongly sheared, the corresponding transport coefficients turn out to be *nonlinear* functions of both the shear rate and the coefficient of restitution. This is the main new ingredient of these constitutive equations. In addition, due to the mathematical difficulties involved in the general problem, results were restricted to a particular kind of perturbations for which the steady state conditions of the USF hold [12]. This allowed us to perform a linear stability analysis of the hydrodynamic equations with respect to the USF state.

As happens for elastic collisions, a possibility of analyzing transport around USF retaining the explicit form of the Boltzmann collision operator is to consider inelastic Maxwell models (IMM), i.e., models for which the collision rate is independent of the relative velocity of the two colliding particles. Although these IMM do not correspond to any microscopic interaction, it has been shown by several authors [13] that the cost of sacrificing physical realism can be in part compensated by the amount of exact analytical results derived from this interaction model. This is the main reason why IMM have attracted the attention of physicists and mathematicians since the beginning of the century. On the other hand, beyond its academic interest, it must be remarked that recent experiments [14] for magnetic grains with dipolar interactions in air in a two-dimensional geometry have been accurately described by IMM. The excellent quantitative agreement found between the magnetic particles experiments [14] and IMM suggests that magnetic particles can be considered as an ideal experimental probe for the predictions of this analytically tractable kinetic theory.

The main advantage of using IMM instead of IHS is that a velocity moment of order k of the Boltzmann collision operator only involves moments of order less than or equal to k. This allows one to evaluate the Boltzmann collision moments without the explicit knowledge of the velocity distribution function [15]. Thanks to this simplification, the velocity moments of IMM under USF can be computed as functions of the shear rate. Very recently [16], the first nontrivial shear-rate-dependent moments of USF for IMM have been explicitly computed: the second-degree moments which are directly related to rheological properties

and the fourth-degree moments whose knowledge is necessary to get the heat flux around USF. The knowledge of these moments allows one to re-examine the problem studied in [12] in the context of the Boltzmann equation and without any restriction on the kind of perturbations to USF considered. In addition, the comparison between the transport coefficients derived here with those obtained from IHS [12] can be used again as a test to assess the degree of usefulness of IMM as a prototype model for granular flows. Previous comparisons between IMM and IHS for inhomogeneous situations [17–19] have shown in general good agreement (especially for low order moments), confirming the reliability of IMM as a good approximation to model granular systems.

Since I am interested in studying heat and momentum transport in a strongly sheared granular gas, the physical situation is such that the gas is in a state that deviates from the USF by *small* spatial gradients. Under these conditions and taking the USF state $f^{(0)}$ as the reference one, the Boltzmann equation is solved by means of a Chapman–Enskog-like expansion [20] around the distribution $f^{(0)}$. Since the latter applies for arbitrary values of the shear rate a, the successive approximations in the Chapman–Enskog expansion will retain all the hydrodynamic orders in a. Therefore, the non-equilibrium problem studied here deals with two kinds of spatial gradients: *small* gradients due to perturbations of the USF and arbitrary *large* gradients due to the background shear flow. In this paper, the calculations will be restricted to first order in the spatial gradients of density, temperature and flow velocity. At this level of approximation, the momentum transport is characterized by a viscosity tensor $\eta_{ijk\ell}$, while the heat flux is expressed in terms of a thermal conductivity tensor κ_{ij} and a new tensor μ_{ij} (not present in the elastic case). The set of generalized transport coefficients $\eta_{ijk\ell}$, κ_{ij} and μ_{ij} are nonlinear functions of a and α. The determination of this dependence for IMM is the primary target of this paper.

The plan of the paper is as follows. In section 2, the Boltzmann equation for IMM is introduced and a brief summary of relevant results derived for the USF problem is given. The Chapman–Enskog expansion around USF is described in section 3 and the linear integral equations defining the generalized transport coefficients are displayed. Section 4 deals with the explicit evaluation of the transport coefficients associated with the momentum and heat fluxes. The details of the calculations are displayed along several appendices. The dependence of some of the above coefficients on the shear rate and on the coefficient of restitution is illustrated and compared with known results obtained for IHS [12]. The comparison shows in general qualitative good agreement, especially in the case of the viscosity tensor $\eta_{ijk\ell}$. The paper is closed in section 6 with a brief discussion on the results reported in this paper.

2. Inelastic Maxwell models and uniform shear flow

Let us consider a granular fluid modelled as an inelastic Maxwell gas. The inelasticity of collisions among all pairs is accounted for by a *constant* coefficient of restitution $0 \leqslant \alpha \leqslant 1$ that only affects the translational degrees of freedom of the grains. At a kinetic level, all the relevant information on the state of the system is provided by the one-particle velocity distribution function $f(\mathbf{r}, \mathbf{v}, t)$. In the low-density regime the *inelastic* Boltzmann equation [1, 2] gives the time evolution of $f(\mathbf{r}, \mathbf{v}, t)$. The corresponding Boltzmann equation for inelastic Maxwell models (IMM) can be obtained from the inelastic Boltzmann equation for inelastic hard spheres (IHS) by replacing the rate for collisions between two particles (which is proportional to the relative velocity in the case of IHS) by an average velocity-independent collision rate. With this simplification and in the absence of an external force, the Boltzmann equation for IMM reads

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) f(\mathbf{r}, \mathbf{v}, t) = J[\mathbf{v}|f(t), f(t)], \tag{2.1}$$

where the Boltzmann collision operator is

$$J[\mathbf{v}_1|f, f] = \frac{\omega}{n\Omega_d} \int d\mathbf{v}_2 \int d\widehat{\boldsymbol{\sigma}}[\alpha^{-1} f(\mathbf{v}_1') f(\mathbf{v}_2') - f(\mathbf{v}_1) f(\mathbf{v}_2)]. \tag{2.2}$$

Here, n is the number density, $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the total solid angle in d dimensions, $\widehat{\boldsymbol{\sigma}}$ is a unit vector along the line of the two colliding spheres, and $\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity of the colliding pair. In addition, the primes on the velocities denote the initial values $\{\mathbf{v}_1', \mathbf{v}_2'\}$ that lead to $\{\mathbf{v}_1, \mathbf{v}_2\}$ following a binary collision:

$$\mathbf{v}_1' = \mathbf{v}_1 - \tfrac{1}{2}(1 + \alpha^{-1})(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})\widehat{\boldsymbol{\sigma}}, \qquad \mathbf{v}_2' = \mathbf{v}_2 + \tfrac{1}{2}(1 + \alpha^{-1})(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})\widehat{\boldsymbol{\sigma}}. \tag{2.3}$$

The effective collision frequency ω is independent of velocity but depends on space and time through its dependence on density and temperature. Here, I will assume that $\omega \propto nT^q$, with $q \geqslant 0$. The case $q = 0$ will be referred here as model A, while the case $q \neq 0$ will be called model B. Model A is closer to the original Maxwell model of elastic particles, while model B is closer to hard spheres when $q = \tfrac{1}{2}$. Furthermore, the collision frequency ω can be also seen as a free parameter of the model, determined to optimize the agreement with some property of interest of the original Boltzmann equation for IHS. As in previous works [17–19], ω is chosen here to guarantee that the cooling rate of IMM be the same as that of IHS (evaluated at the local equilibrium approximation) [17]. With this choice, one obtains

$$\omega = \frac{d + 2}{2}\nu_0, \qquad \nu_0 = A(q)nT^q, \tag{2.4}$$

where the value of the quantity $A(q)$ is irrelevant for our purposes. Henceforth, I will take this choice for ω.

There is another more refined version of IMM [21] where the collision rate has the same dependence on the scalar product $(\widehat{\boldsymbol{\sigma}} \cdot \widehat{\mathbf{g}})$ as in the case of IHS. However, both versions of IMM lead to similar results in problems as delicate as the high energy tails and so I will consider here the simplest version given by equations (2.2) and (2.3).

The first $d + 2$ velocity moments of f define the number density

$$n(\mathbf{r}, t) = \int d\mathbf{v} \, f(\mathbf{r}, \mathbf{v}, t), \tag{2.5}$$

the flow velocity

$$\mathbf{u}(\mathbf{r}, t) = \frac{1}{n(\mathbf{r}, t)} \int d\mathbf{v} \, \mathbf{v} f(\mathbf{r}, \mathbf{v}, t), \tag{2.6}$$

and the *granular* temperature

$$T(\mathbf{r}, t) = \frac{m}{dn(\mathbf{r}, t)} \int d\mathbf{v} \, V^2(\mathbf{r}, t) f(\mathbf{r}, \mathbf{v}, t), \tag{2.7}$$

where $\mathbf{V}(\mathbf{r}, t) \equiv \mathbf{v} - \mathbf{u}(\mathbf{r}, t)$ is the peculiar velocity. The macroscopic balance equations for density, momentum, and energy follow directly from equation (2.1) by multiplying with 1, $m\mathbf{v}$ and $\tfrac{1}{2}mv^2$ and integrating over $\mathbf{v}$:

$$D_t n + n\nabla \cdot \mathbf{u} = 0, \tag{2.8}$$

$$D_t u_i + (mn)^{-1}\nabla_j P_{ij} = 0, \tag{2.9}$$

$$D_t T + \frac{2}{dn}(\nabla \cdot \mathbf{q} + P_{ij}\nabla_j u_i) = -\zeta T, \tag{2.10}$$

where $D_t = \partial_t + \mathbf{u} \cdot \nabla$. The microscopic expressions for the pressure tensor P, the heat flux $\mathbf{q}$ and the cooling rate ζ are given, respectively, by

$$P(\mathbf{r}, t) = \int d\mathbf{v}\, m \mathbf{V} \mathbf{V} f(\mathbf{r}, \mathbf{v}, t), \tag{2.11}$$

$$\mathbf{q}(\mathbf{r}, t) = \int d\mathbf{v}\, \frac{1}{2} m V^2 \mathbf{V} f(\mathbf{r}, \mathbf{v}, t), \tag{2.12}$$

$$\zeta(\mathbf{r}, t) = -\frac{1}{d n(\mathbf{r}, t) T(\mathbf{r}, t)} \int d\mathbf{v}\, m V^2 J[\mathbf{r}, \mathbf{v}|f(t)]. \tag{2.13}$$

The balance equations (2.8)–(2.10) apply regardless of the details of the interaction model considered. The influence of the collision model appears through the α-dependence of the cooling rate and of the momentum and heat fluxes. In particular, the cooling rate ζ is given by [17]

$$\zeta = \frac{1 - \alpha^2}{2d} \omega = \frac{d+2}{4d} (1 - \alpha^2) \nu_0. \tag{2.14}$$

Let us assume that the gas is under USF. This idealized macroscopic state is characterized by a constant density, a uniform temperature and a simple shear with the local velocity field given by

$$u_i = a_{ij} r_j, \qquad a_{ij} = a \delta_{ix} \delta_{jy}, \tag{2.15}$$

where a is the *constant* shear rate. This linear velocity profile assumes no boundary layer near the walls and is generated by the Lees–Edwards boundary conditions [24], which are simply periodic boundary conditions in the local Lagrangian frame moving with the flow velocity [25]. Since the heat flux is zero in the USF problem, the balance equation for the energy (2.10) reads

$$\nu_0^{-1} \partial_t \ln T = -\zeta^* - \frac{2a^*}{d} P_{xy}^*, \tag{2.16}$$

where $\zeta^* = \zeta/\nu_0, a^* = a/\nu_0, P_{xy}^* = P_{xy}/p, p = nT$ being the hydrostatic pressure. Equation (2.16) shows that the temperature changes in time due to the competition between two (opposite) mechanisms: on the one hand, viscous (shear) heating and, on the other hand, energy dissipation in collisions. The *reduced* shear rate a^* is the nonequilibrium relevant parameter of the USF problem since it measures the departure of the system from equilibrium. Note that, except for model A ($q = 0$), $a^*(t) \propto T(t)^{-q}$ is a function of time through its dependence on temperature. Since in the hydrodynamic regime $P_{xy}^*(t)$ depends on time only through its dependence on $a^*(t)$ [7], then for $q \neq 0$ a steady state is eventually reached in the long time limit when both viscous heating and collisional cooling cancel each other and the fluid autonomously seeks the temperature at which the above balance occurs. In this situation, a^* and α are not independent quantities but they are related through the steady state condition:

$$a^* P_{xy}^* = -\frac{d}{2} \zeta^*. \tag{2.17}$$

However, when $q = 0$, the collision frequency ν_0 is independent of temperature and a^* remains constant in time, so that there is no steady state (except if a^* takes the specific value given by (2.17)). Consequently, only in the case of model A the reduced shear rate a^* and the coefficient of restitution α are in general *independent* parameters in the USF state. Note that, although the temperature changes in time, the distribution of velocities relative to the thermal speed $\sqrt{2T/m}$ is expected to reach, after a sufficient number of collisions per

particle, a stationary form that only depends on the control parameters a^* and α [22]. As a consequence, when the velocity moments are conveniently scaled with the thermal speed, they reach stationary values after a kinetic transient regime [16, 23]. These steady values are nonlinear functions of both the (reduced) shear rate and the coefficient of restitution. More details on the USF state for dissipative systems can be found in [16].

The USF problem is perhaps the nonequilibrium state most widely studied in the past few years both for granular and conventional gases [7, 11]. At a microscopic level, it becomes spatially homogeneous when the velocities of the particles are referred to the Lagrangian frame of reference co-moving with the flow velocity $\mathbf{u}$ [25]. In this frame, the one-particle distribution function adopts the *uniform* form, $f(\mathbf{r}, \mathbf{v}) \rightarrow f(\mathbf{V})$, and the Boltzmann equation (2.1) reads

$$\left(\partial_t - aV_y \frac{\partial}{\partial V_x}\right) f(\mathbf{V}) = J[\mathbf{V}|f, f]. \tag{2.18}$$

Upon writing equation (2.18), the following identity has been used:

$$\mathbf{v} \cdot \nabla f = v_y \frac{\partial f}{\partial u_x} \frac{\partial u_x}{\partial y} = -aV_y \frac{\partial f}{\partial V_x}, \tag{2.19}$$

where in the last step I have taken into account that f depends on $\mathbf{u}$ through the peculiar velocity $\mathbf{V}$. The elements of the pressure tensor provide information on the rheological properties of the system. These elements can be obtained by multiplying the Boltzmann equation (2.18) by $mV_i V_j$ and integrating over $\mathbf{V}$. The result is [15–17]

$$\partial_t P_{ij} + a_{i\ell} P_{j\ell} + a_{j\ell} P_{i\ell} = -\nu_{0|2}(P_{ij} - p\delta_{ij}) - \zeta p\delta_{ij}, \tag{2.20}$$

where

$$\nu_{0|2} = \zeta + \frac{(1+\alpha)^2}{4} \nu_0. \tag{2.21}$$

In the case of model A ($q = 0$), the set of first-order differential equations (2.20) can be exactly solved [16]. In terms of the reduced elements $P_{ij}^* = P_{ij}(t)/p(t)$, the solution can be written as

$$P_{xx}^* = \frac{1 + 2d\gamma(\widetilde{a})}{1 + 2\gamma(\widetilde{a})}, \qquad P_{yy}^* = P_{zz}^* = \frac{1}{1 + 2\gamma(\widetilde{a})}, \tag{2.22}$$

$$P_{xy}^* = -d\frac{\gamma(\widetilde{a})}{\widetilde{a}} = -\frac{\widetilde{a}}{[1 + 2\gamma(\widetilde{a})]^2}, \tag{2.23}$$

where

$$\widetilde{a} = \frac{a}{\omega_{0|2}}, \qquad \omega_{0|2} = \nu_{0|2} - \zeta = \frac{(1+\alpha)^2}{4} \nu_0, \tag{2.24}$$

and $\gamma(\widetilde{a})$ is the real root of the cubic equation

$$\gamma(1 + 2\gamma)^2 = \frac{\widetilde{a}^2}{d}, \tag{2.25}$$

namely,

$$\gamma(\widetilde{a}) = \frac{2}{3} \sinh^2\left[\frac{1}{6}\cosh^{-1}\left(1 + \frac{27}{d}\widetilde{a}^2\right)\right]. \tag{2.26}$$

Insertion of equations (2.22) and (2.23) into equation (2.20) yields

$$\partial_t \ln T = 2\lambda, \tag{2.27}$$

where I have called

$$\lambda = 2\gamma(\widetilde{a})\omega_{0|2} - \zeta. \tag{2.28}$$

In the elastic limit ($\alpha = 1$), $\widetilde{a} = a^*$, and equations (2.22)–(2.28) reduce to the well-known exact solution obtained long time ago by Ikenberry and Truesdell for Maxwell molecules [26]. Equation (2.27) shows that $T(t)$ either grows or decays exponentially. The first situation happens if $2\gamma(\widetilde{a})\omega_{0|2} > \zeta$. In that case, the imposed shear rate is sufficiently large (or the dissipation in collisions is sufficiently low) as to make the viscous heating effect dominate over the inelastic cooling effect. The opposite happens if $2\gamma(\widetilde{a})\omega_{0|2} < \zeta$.

Beyond rheological properties, the next nontrivial moments in the USF problem are the fourth-degree velocity moments. These moments are needed to determine the transport properties of the gas in states close to USF, which is the main goal of this paper. Very recently, the dependence of the fourth-degree moments on the shear rate and the coefficient of restitution has been explicitly obtained for IMM [16]. Their explicit expressions are displayed in appendix A for a three-dimensional gas ($d = 3$). In a similar way to the case of elastic Maxwell molecules [7], it has been shown that, for a given value of α, the fourth-degree moments are divergent for shear rates larger than a certain critical value $a_c(\alpha)$. This singular behaviour of the moments reflects the existence of an algebraic high-velocity tail in the distribution function. However, for practical reasons, since in general the numerical value $a_c(\alpha)$ is rather large, nonlinear shearing effects are still significant for $a < a_c$.

As said before, in the case of model B ($q \neq 0$), after a transient regime the system achieves a steady state, so that $\lambda = 0$ and equation (2.28) leads to

$$\gamma(\widetilde{a}) = \frac{d+2}{2d}\frac{1-\alpha}{1+\alpha}. \tag{2.29}$$

The steady solution for the pressure tensor is still given by equations (2.22) and (2.23), except that $\widetilde{a}$ (or, equivalently a^*) and α are not independent quantities. The explicit dependence of the steady-state value $a_s^*(\alpha)$ of the reduced shear rate can be easily obtained by inserting condition (2.29) into equation (2.25):

$$a_s^*(\alpha) = \sqrt{\frac{d+2}{2}(1-\alpha^2)}\frac{d+1-\alpha}{2d}. \tag{2.30}$$

It must be remarked that the steady-state results are 'universal' in the sense that they hold both for models A and B, regardless of the precise dependence $\nu_0(t)$ [16]. For elastic collisions, equation (2.30) yields $a^* = 0$ and so the equilibrium results are recovered, i.e., $P_{ij}^* = \delta_{ij}$. The analytical results obtained for the steady USF state in the case of model B [18] agree quite well with Monte Carlo simulations of the Boltzmann equation for IHS [27, 28], even for strong dissipation.

3. Small perturbations around the uniform shear flow state

Let us assume now that we disturb the USF by small spatial perturbations. The response of the system to these perturbations gives rise to additional contributions to the momentum and heat fluxes, which can be characterized by generalized transport coefficients. Since the system is strongly sheared, the corresponding transport coefficients are highly nonlinear functions of the shear rate. The goal here is to determine the shear-rate dependence of these coefficients for IMM.

To analyse this problem, one has to start from the Boltzmann equation (2.1) with a general time and space dependence. First, it is convenient to keep using the relative velocity $\mathbf{V} = \mathbf{v} - \mathbf{u}_0$, where $\mathbf{u}_0 = \mathbf{a} \cdot \mathbf{r}$ is the flow velocity of the *undisturbed* USF state. Here, the

only nonzero element of the tensor a is $a_{ij} = a\delta_{ix}\delta_{jy}$. On the other hand, in the *disturbed* state the true velocity $\mathbf{u}$ is in general different from $\mathbf{u}_0$, i.e., $\mathbf{u} = \mathbf{u}_0 + \delta\mathbf{u}$, $\delta\mathbf{u}$ being a small perturbation to $\mathbf{u}_0$. As a consequence, the true peculiar velocity is now $\mathbf{c} \equiv \mathbf{v} - \mathbf{u} = \mathbf{V} - \delta\mathbf{u}$. In the Lagrangian frame moving with velocity $\mathbf{u}_0$, the convective operator $\mathbf{v} \cdot \nabla$ can be written as

$$\mathbf{v} \cdot \nabla f = (\mathbf{V} + \mathbf{u}_0) \cdot \nabla f = -aV_y \frac{\partial}{\partial V_x} f + (\mathbf{V} + \mathbf{u}_0) \cdot \nabla f, \tag{3.1}$$

where the derivative ∇f in the last term must be taken now at constant $\mathbf{V}$. According to the identity (3.1), the Boltzmann equation reads

$$\frac{\partial}{\partial t} f - aV_y \frac{\partial}{\partial V_x} f + (\mathbf{V} + \mathbf{u}_0) \cdot \nabla f = J[\mathbf{V}|f, f]. \tag{3.2}$$

I am interested in computing the transport coefficients in a state that slightly deviates from the USF. For this reason, I assume that the spatial gradients of the hydrodynamic fields

$$A(\mathbf{r}, t) \equiv \{n(\mathbf{r}, t), T(\mathbf{r}, t), \delta\mathbf{u}(\mathbf{r}, t)\} \tag{3.3}$$

are small. Under these conditions, a solution to the Boltzmann equation (3.2) can be obtained by means of a generalization of the conventional Chapman–Enskog method [4], where the velocity distribution function is expanded about a *local* shear flow reference state in terms of the small spatial gradients of the hydrodynamic fields relative to those of USF. This type of Chapman–Enskog-like expansion has been considered in the case of elastic gases to obtain the set of shear-rate-dependent transport coefficients [7, 29] in a thermostatted shear flow problem and it has also been recently considered [12, 20, 30] in the context of inelastic gases.

The Chapman–Enskog method assumes the existence of a *normal* solution in which all space and time dependence of the distribution function occurs through a functional dependence on the fields $A(\mathbf{r}, t)$, i.e.,

$$f(\mathbf{r}, \mathbf{V}, t) \equiv f[\mathbf{V}|A(\mathbf{r}, t)]. \tag{3.4}$$

This solution expresses the fact that the space dependence of the shear flow is completely absorbed in the relative velocity $\mathbf{V}$ and all other space and time dependence occurs entirely through a functional dependence on the fields $\mathbf{A}(\mathbf{r}, t)$. This functional dependence can be made local by an expansion of the distribution function in powers of the hydrodynamic gradients:

$$f(\mathbf{r}, \mathbf{V}, t) = f^{(0)}(\mathbf{V}|A(\mathbf{r}, t)) + f^{(1)}(\mathbf{V}|A(\mathbf{r}, t)) + \cdots, \tag{3.5}$$

where the reference zeroth-order distribution function corresponds to the USF distribution function but taking into account the local dependence of the density and temperature and the change $\mathbf{V} \to \mathbf{c} = \mathbf{V} - \delta\mathbf{u}(\mathbf{r}, t)$. The new feature of this Chapman–Enskog expansion (in contrast with the conventional one) is that the successive approximations $f^{(k)}$ are of order k in the gradients of n, T and $\delta\mathbf{u}$ but retain all the orders in the shear rate a. More technical details on this Chapman–Enskog-like type of expansion can be found in appendix B.

The expansion (3.5) yields the corresponding expansion for the fluxes and the cooling rate when one substitutes (3.5) into their definitions (2.11)–(2.13):

$$\mathsf{P} = \mathsf{P}^{(0)} + \mathsf{P}^{(1)} + \cdots, \qquad \mathbf{q} = \mathbf{q}^{(0)} + \mathbf{q}^{(1)} + \cdots, \qquad \zeta = \zeta^{(0)} + \zeta^{(1)} + \cdots. \tag{3.6}$$

Finally, as in the usual Chapman–Enskog method, the time derivative is also expanded as

$$\partial_t = \partial_t^{(0)} + \partial_t^{(1)} + \partial_t^{(2)} + \cdots, \tag{3.7}$$

where the action of each operator $\partial_t^{(k)}$ is also defined in appendix B. In this paper, only the zeroth- and first-order approximations will be considered.

3.1. Zeroth-order approximation

Substituting the expansions (3.5) and (3.7) into equation (3.2), the kinetic equation for $f^{(0)}$ is given by

$$\partial_t^{(0)} f^{(0)} - a V_y \frac{\partial}{\partial V_x} f^{(0)} = J[\mathbf{V}|f^{(0)}, f^{(0)}].$$ (3.8)

To lowest order in the expansion the conservation laws give

$$\partial_t^{(0)} n = 0, \qquad \partial_t^{(0)} T = -\frac{2}{dn} a P_{xy}^{(0)} - T\zeta,$$ (3.9)

$$\partial_t^{(0)} \delta u_i + a_{ij} \delta u_j = 0.$$ (3.10)

Upon writing the second identity in equation (3.9) I have taken into account that the effective collision frequency $\omega \propto n T^q$ is assumed to be a functional of f only through the density and temperature. Consequently, $\omega^{(0)} = \omega, \omega^{(1)} = \omega^{(2)} = \cdots = 0$ and, using equation (2.14), $\zeta^{(0)} = \zeta, \zeta^{(1)} = \zeta^{(2)} = \cdots = 0$. It must be noticed that, in the case of IHS, $\zeta^{(1)}$ is different from zero but quite small [12].

Since $f^{(0)}$ is a normal solution, its dependence on time only occurs through n, $\delta\mathbf{u}$ and T:

$$\begin{aligned}
\partial_t^{(0)} f^{(0)} &= \frac{\partial f^{(0)}}{\partial n} \partial_t^{(0)} n + \frac{\partial f^{(0)}}{\partial T} \partial_t^{(0)} T + \frac{\partial f^{(0)}}{\partial \delta u_i} \partial_t^{(0)} \delta u_i \\
&= -\left(\frac{2}{dn} a P_{xy}^{(0)} + T\zeta\right) \frac{\partial}{\partial T} f^{(0)} - a_{ij} \delta u_j \frac{\partial}{\partial \delta u_i} f^{(0)} \\
&= -\left(\frac{2}{dn} a P_{xy}^{(0)} + T\zeta\right) \frac{\partial}{\partial T} f^{(0)} + a_{ij} \delta u_j \frac{\partial}{\partial c_i} f^{(0)},
\end{aligned}$$ (3.11)

where I have taken into account again that $f^{(0)}$ depends on $\delta\mathbf{u}$ through $\mathbf{c}$. Substitution of equation (3.11) into equation (3.8) yields the following kinetic equation for $f^{(0)}$:

$$-\left(\frac{2}{dn} a P_{xy}^{(0)} + T\zeta\right) \frac{\partial}{\partial T} f^{(0)} - a c_y \frac{\partial}{\partial c_x} f^{(0)} = J[\mathbf{V}|f^{(0)}, f^{(0)}].$$ (3.12)

The zeroth-order solution leads to $\mathbf{q}^{(0)} = \mathbf{0}$. In the case of model A, the zeroth-order pressure tensor is given by equations (2.22) and (2.23) while the expressions of the fourth-degree moments are displayed in appendix A. As shown in [12, 20, 30], for given values of a and α, the steady state condition (2.17) establishes a mapping between the density and temperature so that every density corresponds to one and only one temperature. Since the density $n(\mathbf{r}, t)$ and temperature $T(\mathbf{r}, t)$ are specified separately in the *local* USF state, the viscous heating only partially compensates for the collisional cooling and so, $\partial_t^{(0)} T \neq 0$ [20]. Consequently, the zeroth-order distribution $f^{(0)}$ depends on time through its dependence on temperature and the quantities a^* and α must be considered as independent parameters for general infinitesimal perturbations around the USF state. This fact gives rise to new and conceptual practical difficulties not present in the previous analysis made for elastic thermostatted gases [29].

The set of equations for $P_{ij}^{(0)}$ follows from equation (3.12) as

$$-\left(\frac{2}{dn} a P_{xy}^{(0)} + T\zeta\right) \frac{\partial}{\partial T} P_{ij}^{(0)} + a_{i\ell} P_{j\ell}^{(0)} + a_{j\ell} P_{i\ell}^{(0)} = -\nu_{0|2}\left(P_{ij}^{(0)} - p\delta_{ij}\right) - \zeta p\delta_{ij}.$$ (3.13)

The dependence of $P_{ij}^{(0)}$ on temperature occurs explicitly through the hydrostatic pressure $p = nT$ and through its dependence on a^*. Consequently,

$$T \frac{\partial}{\partial T} P_{ij}^{(0)} = T \frac{\partial}{\partial T} p P_{ij}^*(a^*) = p\left(1 - qa^* \frac{\partial}{\partial a^*}\right) P_{ij}^*(a^*),$$ (3.14)

where here $P_{ij}^* = P_{ij}^{(0)}/p$. Thus, in dimensionless form equation (3.13) becomes

$$-\left(\frac{2}{d}a^* P_{xy}^* + \zeta^*\right)\left(1 - qa^* \frac{\partial}{\partial a^*}\right) P_{ij}^* + a_{i\ell}^* P_{j\ell}^* + a_{j\ell}^* P_{i\ell}^* = -\nu_{0|2}^*(P_{ij}^* - \delta_{ij}) - \zeta^* \delta_{ij}, \quad (3.15)$$

where $\nu_{0|2}^* = \nu_{0|2}/\nu_0$. For $q = 0$ (model A), the solution to equation (3.15) is given by equations (2.22) and (2.23). For $q \neq 0$ (model B), equation (3.15) must be solved numerically to obtain the dependence of P_{ij}^* on a^*. In the case of IHS $\left(q = \frac{1}{2}\right)$, a detailed study of the dependence of P_{ij}^* on a^* has been carried out in [23]. For elastic gases ($\alpha = 1$), a comparison between the results derived for Maxwell molecules ($q = 0$) and hard spheres $\left(q = \frac{1}{2}\right)$ at the level of rheological properties has shown that both results are very close [7] so that there is a weak influence of the interaction model on transport properties. This suggests to expand the transport properties in powers of the interaction parameter q as an alternative to obtain accurate analytical results for non-Maxwell molecules. For inelastic collisions, I expect that such a good agreement is also kept and so the following formal expansion in q is considered:

$$P_{ij}^*(a^*) = P_{ij,A}^*(a^*) + q \Delta P_{ij}^*(a^*) + \cdots, \quad (3.16)$$

where $P_{ij,A}^*(a^*)$ is the known result for model A. Inserting equation (3.16) into equation (3.15) and neglecting terms nonlinear in q, one obtains

$$\Delta P_{yy}^* = -\frac{6\lambda^*}{\omega_{0|2}^*} \frac{\gamma}{(1 + 2\gamma)(1 + 6\gamma)^2}, \quad (3.17)$$

$$\Delta P_{xy}^* = -\frac{\lambda^* \widetilde{a}}{\omega_{0|2}^*} \frac{1 - 6\gamma}{(1 + 2\gamma)^2(1 + 6\gamma)^2}, \quad (3.18)$$

where $\omega_{0|2}^* = \omega_{0|2}/\nu_0$ and $\lambda^* = \lambda/\nu_0$, λ being given by equation (2.28). Upon writing equations (3.17) and (3.18), the following relations have been used:

$$a^* \frac{\partial}{\partial a^*} P_{yy,A}^* = -\frac{4\gamma}{(1 + 2\gamma)(1 + 6\gamma)}, \quad (3.19)$$

$$a^* \frac{\partial}{\partial a^*} P_{xy,A}^* = -\widetilde{a} \frac{1 - 2\gamma}{(1 + 2\gamma)^2(1 + 6\gamma)}. \quad (3.20)$$

From equation (3.16) and taking into account equations (2.22) and (2.23), the expressions for the elements of the pressure tensor for model B can be written as

$$P_{xx}^* = \frac{1}{1 + 2\gamma}\left[1 + 2d\gamma - q\frac{6\gamma\lambda^*}{\omega_{0|2}^*}\frac{1 - d}{(1 + 6\gamma)^2}\right], \quad (3.21)$$

$$P_{yy}^* = P_{zz}^* = \frac{1}{1 + 2\gamma}\left[1 - q\frac{6\gamma\lambda^*}{\omega_{0|2}^*(1 + 6\gamma)^2}\right], \quad (3.22)$$

$$P_{xy}^* = -\frac{\widetilde{a}}{(1 + 2\gamma)^2}\left[1 + q\frac{\lambda^*}{\omega_{0|2}^*}\frac{1 - 6\gamma}{(1 + 6\gamma)^2}\right]. \quad (3.23)$$

As expected, when the steady state condition (2.4) applies locally, then $\partial_t^{(0)} T = 0$, $\lambda^* = 0$ so that, according to equations (3.17) and (3.18), the results for the pressure tensor are independent of the model interaction considered. Note that the same type of approximation (3.16) can be used to estimate the fourth-degree moments for $q \neq 0$, although this calculation will be omitted here for simplicity.
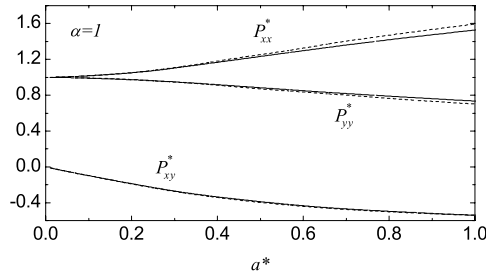
Figure 1. Shear-rate dependence of the elements of the reduced pressure tensor $P_{ij}^* = P_{ij}^{(0)}/p$ for $\alpha = 1$ in the three-dimensional case. The solid lines correspond to the results obtained for model A ($q = 0$) while the dashed lines refer to the results derived for model B for $q = \frac{1}{2}$ by using the approximation (3.16).
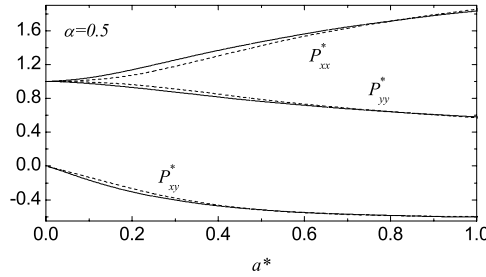

Figure 2. Shear-rate dependence of the elements of the reduced pressure tensor $P_{ij}^* = P_{ij}^{(0)}/p$ for $\alpha = 0.5$ in the three-dimensional case. The solid lines correspond to the results obtained for model A ($q = 0$) while the dashed lines refer to the results derived for model B for $q = \frac{1}{2}$ by using the approximation (3.16).

Figures 1 and 2 show the dependence of the reduced pressure tensor P_{ij}^* on the reduced shear rate a^* for a three-dimensional system ($d = 3$) and two different values of the coefficient of restitution: $\alpha = 1$ and $\alpha = 0.5$. I have considered the exact results obtained for model A and the results for model B (when $q = \frac{1}{2}$) by using approximation (3.16). It is quite apparent that, for a given interaction model, there is a weak influence of the inelasticity on the shear-rate dependence of the pressure tensor elements. Furthermore, for a given value of α, the differences between the results obtained for models A and B are very small in the range of shear rates considered. This means that the rheological properties are quite insensitive to the interaction model considered, even for strong dissipation. The dependence of the pressure tensor on dissipation in the steady state (i.e., when the (reduced) shear rate and the coefficient of restitution α are not independent parameters but they are related through equation (2.30)) is plotted in figure 3 along with the results obtained for IHS [23, 27]. It can be observed that the IMM results reproduce very well the IHS predictions over the range of values of the coefficient of restitution analysed.

3.2. First-order approximation

The analysis to first order in the gradients is worked out in appendix B. Only the final results are presented in this section. The distribution function $f^{(1)}$ is of the form

$$f^{(1)} = \mathbf{X}_n \cdot \nabla n + \mathbf{X}_T \cdot \nabla T + \mathsf{X}_u : \nabla \delta \mathbf{u}, \tag{3.24}$$
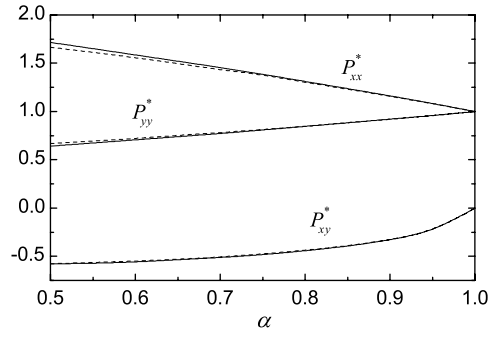
Figure 3. Plot of the reduced elements of the pressure tensor P* as functions of the coefficient of restitution α in the steady USF state. The solid lines are the results derived here for IMM while the dashed lines correspond to the results obtained for IHS in [27].

where the vectors $\mathbf{X}_n$ and $\mathbf{X}_T$ and the tensor X_u are functions of the true peculiar velocity $\mathbf{c}$. They are the solutions of the following linear integral equations:

$$-\left[\left(\frac{2}{dn}aP_{xy}^{(0)}+T\zeta\right)\partial_T+ac_y\frac{\partial}{\partial c_x}-\mathcal{L}\right]X_{n,i}+\frac{T}{n}\left[\frac{2a}{dp}(1-n\partial_n)P_{xy}^{(0)}-\zeta\right]X_{T,i}=Y_{n,i},$$

(3.25)

$$-\left[\left(\frac{2}{dn}aP_{xy}^{(0)}+T\zeta\right)\partial_T+\frac{2a}{dp}T\left(\partial_T P_{xy}^{(0)}\right)+(q+1)\zeta+ac_y\frac{\partial}{\partial c_x}-\mathcal{L}\right]X_{T,i}=Y_{T,i},$$

(3.26)

$$-\left[\left(\frac{2}{dn}aP_{xy}^{(0)}+T\zeta\right)\partial_T+ac_y\frac{\partial}{\partial c_x}-\mathcal{L}\right]X_{u,k\ell}-a\delta_{ky}X_{u,x\ell}=Y_{u,k\ell},$$

(3.27)

where $\mathbf{Y}_n(\mathbf{c})$, $\mathbf{Y}_T(\mathbf{c})$ and $\mathsf{Y}_u(\mathbf{c})$ are defined by equations (B.10)–(B.12), respectively. In addition, $\mathcal{L}$ is the linearized Boltzmann collision operator around the reference USF state:

$$\mathcal{L}X \equiv -(J[f^{(0)}, X] + J[X, f^{(0)}]).$$

(3.28)

It is worth noting that for $q = \frac{1}{2}$, equations (3.25)–(3.27) have the same structure as that of the Boltzmann equation for IHS [12]. The only difference between both models lies in the explicit form of the linearized operator $\mathcal{L}$.

With the distribution function $f^{(1)}$ determined by equation (3.24), the first-order corrections to the fluxes are

$$P_{ij}^{(1)} = -\eta_{ijk\ell}\frac{\partial\delta u_k}{\partial r_\ell},$$

(3.29)

$$q_i^{(1)} = -\kappa_{ij}\frac{\partial T}{\partial r_j} - \mu_{ij}\frac{\partial n}{\partial r_j},$$

(3.30)

where

$$\eta_{ijk\ell} = -\int d\mathbf{c}\, mc_i c_j X_{u,k\ell}(\mathbf{c}),$$

(3.31)

$$\kappa_{ij} = -\int d\mathbf{c}\frac{m}{2}c^2 c_i X_{T,j}(\mathbf{c}),$$

(3.32)

$$\mu_{ij} = -\int d\mathbf{c}\frac{m}{2}c^2 c_i X_{n,j}(\mathbf{c}).$$

(3.33)

Upon writing equations (3.29)–(3.33), the symmetry properties of $X_{n,i}$, $X_{T,i}$ and $X_{u,ij}$ have been used. In general, the set of *generalized* transport coefficients $\eta_{ijk\ell}$, κ_{ij} and μ_{ij} are nonlinear functions of the coefficient of restitution α and the reduced shear rate a^*. The anisotropy induced in the system by the presence of shear flow gives rise to new transport coefficients, reflecting broken symmetry. The momentum flux is expressed in terms of a viscosity tensor $\eta_{ijk\ell}(a^*, \alpha)$ of rank 4 which is symmetric and traceless in ij due to the properties of the pressure tensor $P_{ij}^{(1)}$. The heat flux is expressed in terms of a thermal conductivity tensor $\kappa_{ij}(a^*, \alpha)$ and a new tensor $\mu_{ij}(a^*, \alpha)$. Of course, for $a^* = 0$ and $\alpha = 1$, the usual Navier–Stokes constitutive equations for ordinary gases are recovered and the transport coefficients become

$$\eta_{ijk\ell} \rightarrow \eta_0 \left(\delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell} - \frac{2}{d}\delta_{ij}\delta_{k\ell} \right), \qquad \kappa_{ij} \rightarrow \kappa_0 \delta_{ij}, \quad \mu_{ij} \rightarrow 0, \tag{3.34}$$

where $\eta_0 = p/\nu_0$ and $\kappa_0 = d(d+2)\eta_0/2(d-1)m$ are the shear viscosity and thermal conductivity coefficients for elastic collisions [4].

4. Shear-rate-dependent transport coefficients

This section is devoted to the determination of the generalized transport coefficients $\eta_{ijk\ell}$, κ_{ij} and μ_{ij} associated with the momentum and heat fluxes. Let us consider each flux separately.

4.1. Momentum flux

To first order in the hydrodynamic gradients, the momentum flux is given by equation (3.29). To obtain the coefficient $\eta_{ijk\ell}$, I multiply both sides of equation (3.27) by mc_ic_j and integrate over **c**. The result is

$$\left(\frac{2}{dn}aP_{xy}^{(0)} + T\zeta \right) \partial_T \eta_{ijk\ell} - a(\delta_{ix}\eta_{jyk\ell} + \delta_{jx}\eta_{iyk\ell} - \delta_{ky}\eta_{ijx\ell}) - \nu_{0|2}\eta_{ijk\ell}$$
$$= -\delta_{k\ell}(1 - n\partial_n)P_{ij}^{(0)} - \left(\delta_{ik}P_{j\ell}^{(0)} + \delta_{jk}P_{i\ell}^{(0)} \right) + \frac{2}{dn}\left(P_{k\ell}^{(0)} - a\eta_{xyk\ell} \right)\partial_T P_{ij}^{(0)}. \tag{4.1}$$

Upon writing equation (4.1), the following results have been used [17, 15]:

$$\int d\mathbf{c}\, mc_ic_j \mathcal{L}X_{u,k\ell} = -\nu_{0|2}\eta_{ijk\ell}, \tag{4.2}$$

$$\int d\mathbf{c}\, mc_ic_j Y_{u,k\ell} = -\delta_{k\ell}(1 - n\partial_n)P_{ij}^{(0)} - \left(\delta_{ik}P_{j\ell}^{(0)} + \delta_{jk}P_{i\ell}^{(0)} \right)$$
$$+ \frac{2}{dn}\left(P_{k\ell}^{(0)} - a\eta_{xyk\ell} \right)\partial_T P_{ij}^{(0)}. \tag{4.3}$$

The generalized shear viscosity can be written as $\eta_{ijk\ell} = \eta_0 \eta_{ijk\ell}^*(a^*)$ where $\eta_{ijk\ell}^*(a^*)$ is a dimensionless function of the reduced shear rate a^* and the coefficient of restitution α. The dependence of $\eta_{ijk\ell}^*$ on temperature is through the reduced shear rate a^* and so

$$T\partial_T \eta_{ijk\ell} = T\partial_T \eta_0 \eta_{ijk\ell}^*(a^*) = (1-q)\eta_{ijk\ell} - q\eta_{ijk\ell}a^*\partial_{a^*}\ln\eta_{ijk\ell}^*(a^*). \tag{4.4}$$

Consequently, in dimensionless form, equation (4.1) yields

$$\left(\frac{2}{d}a^*P_{xy}^* + \zeta^* \right)[1 - q(1 + a^*\partial_{a^*})]\eta_{ijk\ell}^* - a^*(\delta_{ix}\eta_{jyk\ell}^* + \delta_{jx}\eta_{iyk\ell}^* - \delta_{ky}\eta_{ijx\ell}^*) - \nu_{0|2}^*\eta_{ijk\ell}^*$$
$$= -\delta_{k\ell}a^*\partial_{a^*}P_{ij}^* - (\delta_{ik}P_{j\ell}^* + \delta_{jk}P_{i\ell}^*) + \frac{2}{d}(P_{k\ell}^* - a^*\eta_{xyk\ell}^*)(1 - qa^*\partial_{a^*})P_{ij}^*, \tag{4.5}$$

where the following identity has been used:

$$n\frac{\partial}{\partial n}P_{ij}^{(0)} = n\frac{\partial}{\partial n}pP_{ij}^*(a^*) = p\left(1 - a^*\frac{\partial}{\partial a^*}\right)P_{ij}^*(a^*). \tag{4.6}$$

In the absence of shear field ($a^* = 0$), $P_{ij}^* = \delta_{ij}$, and so equation (4.5) has the solution

$$\eta_{ijk\ell}^* = [\nu_{0|2}^* - \zeta^*(1 - q)]^{-1}\left(\delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell} - \frac{2}{d}\delta_{ij}\delta_{k\ell}\right). \tag{4.7}$$

This expression coincides with the one derived by Santos [17] for IMM for vanishing shear rates. Beyond this limit case, in general equation (4.5) is a nonlinear differential equation that must be solved with the appropriate boundary conditions to obtain the hydrodynamic solution. The simplest model is that of model A, in which case the set (4.5) becomes a set of coupled algebraic equations.

4.1.1. Model A. When $q = 0$, then the shear viscosity obeys the equation

$$\left(\frac{2}{d}a^*P_{xy}^* - \omega_{0|2}^*\right)\eta_{ijk\ell}^* - a^*(\delta_{ix}\eta_{jyk\ell}^* + \delta_{jx}\eta_{iyk\ell}^* - \delta_{ky}\eta_{ijx\ell}^*)$$

$$= -\delta_{k\ell}a^*\partial_{a^*}P_{ij}^* - (\delta_{ik}P_{j\ell}^* + \delta_{jk}P_{i\ell}^*) + \frac{2}{d}(P_{k\ell}^* - a^*\eta_{xyk\ell}^*)P_{ij}^*. \tag{4.8}$$

The explicit dependence of $\eta_{ijk\ell}^*$ on a^* and α can be obtained by solving the set of algebraic equations (4.8). As an example, the coefficients of the form η_{ijxy}^* are obtained in detail in appendix C. The nonzero elements of η_{ijxy}^* are given by

$$\eta_{xyxy}^* = \frac{1}{\omega_{0|2}^*}\frac{1 - 2\gamma}{(1 + 2\gamma)^2(1 + 6\gamma)}, \tag{4.9}$$

$$\eta_{xxxy}^* = \frac{4}{\omega_{0|2}^*}\frac{1 - d}{d}\frac{\widetilde{a}}{(1 + 2\gamma)^3(1 + 6\gamma)}, \tag{4.10}$$

$$\eta_{yyxy}^* = \eta_{zzxy}^* = \frac{4}{d\omega_{0|2}^*}\frac{\widetilde{a}}{(1 + 2\gamma)^3(1 + 6\gamma)}. \tag{4.11}$$

When $\alpha = 1$, equation (4.9) reduces to the one previously derived for elastic gases [29].

4.1.2. Model B. When $q \neq 0$, the coefficients $\eta_{ijk\ell}^*$ must be obtained numerically. However, assuming that the dependence of $\eta_{ijk\ell}^*$ on a^* and α for $q \neq 0$ does not differ significantly from that obtained for model A ($q = 0$), one can consider again the approximation

$$\eta_{ijk\ell}^* = \eta_{ijk\ell,A}^* + q\Delta\eta_{ijk\ell}^* + \cdots, \tag{4.12}$$

to obtain analytical results for $\eta_{ijk\ell}^*$. In equation (4.12), $\eta_{ijk\ell,A}^*$ refers to the known result for model A. The unknown $\Delta\eta_{ijk\ell}^*$ can be determined by inserting (4.12) into equation (4.5) and retaining only linear terms in q. As an illustration, the quantities $\Delta\eta_{ijxy}^*$ are also evaluated in appendix C. Their explicit expressions are given by equations (C.6)–(C.10).

4.2. Heat flux

The heat flux is defined by equation (3.30) in terms of the coefficients κ_{ij}, equation (3.32) and μ_{ij}, equation (3.33). To obtain these generalized Navier–Stokes coefficients, let us introduce the tensors

$$\kappa_{ijk\ell} = -\int d\mathbf{c}\frac{m}{2}c_i c_j c_k X_{T,\ell}(\mathbf{c}), \tag{4.13}$$

$$\mu_{ijk\ell} = -\int d\mathbf{c}\frac{m}{2}c_i c_j c_k X_{n,\ell}(\mathbf{c}). \tag{4.14}$$

The generalized thermal conductivity κ_{ij} and the new coefficient μ_{ij} are given by

$$\kappa_{ij} = \kappa_{kkij}, \qquad \mu_{ij} = \mu_{kkij}. \tag{4.15}$$

Let us consider first the tensor $\kappa_{ijk\ell}$. To obtain it, I multiply both sides of equation (3.26) by $\frac{m}{2}c_i c_j c_k$ and integrate over velocity. After some algebra, one obtains

$$
\begin{aligned}
&\left(\frac{2}{dn}a P_{xy}^{(0)} + T\zeta\right)\partial_T \kappa_{ijk\ell} + \left(\frac{2a}{dp}\left(T\partial_T P_{xy}^{(0)}\right) + (q+1)\zeta - \nu_{0|3}\right)\kappa_{ijk\ell} \\
&\quad - a(\delta_{ix}\kappa_{jky\ell} + \delta_{jx}\kappa_{iky\ell} + \delta_{kx}\kappa_{ijy\ell}) \\
&\quad - \frac{\nu_{2|1} - \nu_{0|3}}{d+2}(\delta_{jk}\kappa_{i\ell} + \delta_{ik}\kappa_{j\ell} + \delta_{ij}\kappa_{k\ell}) = -\frac{m}{2}\partial_T N_{ijk\ell}^{(0)} \\
&\quad + \frac{1}{2mn}\left(P_{kj}^{(0)}\partial_T P_{i\ell}^{(0)} + P_{ik}^{(0)}\partial_T P_{j\ell}^{(0)} + P_{ij}^{(0)}\partial_T P_{k\ell}^{(0)}\right),
\end{aligned} \tag{4.16}
$$

where [15]

$$\nu_{2|1} = \frac{(1+\alpha)[5d + 4 - (d+8)\alpha]}{8d}\nu_0, \qquad \nu_{0|3} = \frac{3}{2}\nu_{0|2}, \tag{4.17}$$

$$N_{ijk\ell}^{(0)} = \int d\mathbf{c}\, c_i c_j c_k c_\ell f^{(0)}(\mathbf{c}). \tag{4.18}$$

Upon writing equation (4.16), the following results have been used:

$$\int d\mathbf{c}\frac{m}{2}c_i c_j c_k \mathcal{L}X_{T,\ell} = -\nu_{0|3}\kappa_{ijk\ell} - \frac{\nu_{2|1} - \nu_{0|3}}{d+2}(\delta_{jk}\kappa_{i\ell} + \delta_{ik}\kappa_{j\ell} + \delta_{ij}\kappa_{k\ell}), \tag{4.19}$$

$$\int d\mathbf{c}\frac{m}{2}c_i c_j c_k Y_{T,\ell} = -\frac{m}{2}\partial_T N_{ijk\ell}^{(0)} + \frac{1}{2mn}\left(P_{kj}^{(0)}\partial_T P_{i\ell}^{(0)} + P_{ik}^{(0)}\partial_T P_{j\ell}^{(0)} + P_{ij}^{(0)}\partial_T P_{k\ell}^{(0)}\right). \tag{4.20}$$

As in the case of the shear viscosity, the thermal conductivity tensor κ_{ij} can also be written as $\kappa_{ij} = \kappa_0 \kappa_{ij}^*(\alpha, a^*)$, where κ_{ij}^* is a dimensionless function of a^* and α. Thus,

$$T\partial_T \kappa_{ijk\ell} = T\partial_T \kappa_0 \kappa_{ijk\ell}^*(a^*) = (1-q)\kappa_{ijk\ell} - q\kappa_{ijk\ell}a^*\partial_{a^*}\ln\kappa_{ijk\ell}^*(a^*). \tag{4.21}$$

In addition, the derivative $\partial_T N_{ijk\ell}^{(0)}$ of the fourth-degree moments of the zeroth-order distribution can be evaluated as

$$T\partial_T N_{ijk\ell}^{(0)} = 4\frac{nT^2}{m^2}(2 - qa^*\partial_{a^*})N_{ijk\ell}^*(a^*), \tag{4.22}$$

where I have introduced the reduced fourth-degree moments

$$N_{ijk\ell}^*(a^*) = \frac{1}{4}\frac{m^2}{nT^2}N_{ijk\ell}^{(0)}. \tag{4.23}$$

Finally, in dimensionless form, equation (4.16) becomes

$$
\begin{aligned}
&\left(\frac{2}{d}a^* P_{xy}^* + \zeta^*\right)[1 - q(1 + a^*\partial_{a^*})]\kappa_{ijk\ell}^* + \left[\frac{2a^*}{d}(1 - qa^*\partial_{a^*})P_{xy}^* + (q+1)\zeta^* - \nu_{0|3}^*\right]\kappa_{ijk\ell}^* \\
&\quad - a^*(\delta_{ix}\kappa_{jky\ell}^* + \delta_{jx}\kappa_{iky\ell}^* + \delta_{kx}\kappa_{ijy\ell}^*) - \frac{\nu_{2|1}^* - \nu_{0|3}^*}{d+2}(\delta_{jk}\kappa_{i\ell}^* + \delta_{ik}\kappa_{j\ell}^* + \delta_{ij}\kappa_{k\ell}^*) \\
&= -\frac{4(d-1)}{d(d+2)}(2 - qa^*\partial_{a^*})N_{ijk\ell}^* + \frac{d-1}{d(d+2)}[P_{kj}^*(1 - qa^*\partial_{a^*})P_{i\ell}^* \\
&\quad + P_{ik}^*(1 - qa^*\partial_{a^*})P_{j\ell}^* + P_{ij}^*(1 - qa^*\partial_{a^*})P_{k\ell}^*],
\end{aligned} \tag{4.24}
$$

where $\nu_{0|3}^* = \nu_{0|3}/\nu_0$ and $\nu_{2|1}^* = \nu_{2|1}/\nu_0$.

The set of equations defining the elements of the tensor $\mu_{ijk\ell}$ can be easily obtained by following similar mathematical steps as those just made for the thermal conductivity tensor. Thus, the reduced tensor $\mu_{ijk\ell}^* = (n/T\kappa_0)\mu_{ijk\ell}$ is determined from

$$
\left(\frac{2}{d}a^* P_{xy}^* + \zeta^*\right)[2 - q(1 + a^*\partial_{a^*})]\mu_{ijk\ell}^* - \left(\frac{2a^{*2}}{d}(\partial_{a^*}P_{xy}^*) - \zeta^*\right)\kappa_{ijk\ell}^*
$$
$$
- \nu_{0|3}^*\mu_{ijk\ell}^* - a^*(\delta_{ix}\mu_{jky\ell}^* + \delta_{jx}\mu_{iky\ell}^* + \delta_{kx}\mu_{ijy\ell}^*)
$$
$$
- \frac{\nu_{2|1}^* - \nu_{0|3}^*}{d+2}(\delta_{jk}\mu_{i\ell}^* + \delta_{ik}\mu_{j\ell}^* + \delta_{ij}\mu_{k\ell}^*)
$$
$$
= -\frac{4(d-1)}{d(d+2)}(1 - a^*\partial_{a^*})N_{ijk\ell}^* + \frac{d-1}{d(d+2)}[P_{kj}^*(1 - a^*\partial_{a^*})P_{i\ell}^*
$$
$$
+ P_{ik}^*(1 - a^*\partial_{a^*})P_{j\ell}^* + P_{ij}^*(1 - a^*\partial_{a^*})P_{k\ell}^*]. \tag{4.25}
$$

In the absence of shear rate ($a^* = 0$), the solutions to equations (4.24) and (4.25) give the explicit forms for the thermal conductivity tensor κ_{ij}^* and the tensor μ_{ij}^*. They can be written as

$$
\kappa_{ij}^* = \delta_{ij}\frac{d-1}{d}(1+c)(\nu_{2|1}^* - 2\zeta^*)^{-1}, \tag{4.26}
$$

$$
\mu_{ij}^* = \frac{\kappa_{ij}^*}{1+c}\frac{\zeta^* + \frac{1}{2}c\nu_{2|1}^*}{\nu_{2|1}^* - (2-q)\zeta^*}, \tag{4.27}
$$

where the fourth cumulant c is [17]

$$
c(\alpha) = \frac{12(1-\alpha)^2}{4d - 7 + 3\alpha(2-\alpha)}. \tag{4.28}
$$

Equations (4.26) and (4.27) coincide with the previous expressions derived by Santos [17] for the Navier–Stokes transport coefficients of IMM associated with the heat flux.

As in the case of the shear viscosity, equations (4.24) and (4.25) become algebraic for model A ($q = 0$). Even for this model, although the solution to equations (4.24) and (4.25) is simple, it involves a quite tedious algebra due to the presence of the fourth-degree moments $N_{ijk\ell}^*$ whose expressions for hard spheres ($d = 3$) are displayed in appendix A. As an illustration, the detailed form of the elements κ_{xy}^* and κ_{yy}^* of the thermal conductivity tensor is obtained in appendix C. A more complete list of the coefficients κ_{ij}^* and μ_{ij}^* can be obtained from the author upon request.

5. Comparison with the transport coefficients for IHS

As said in the introduction, the expressions for the generalized transport coefficients of IHS described by the Boltzmann equation have been recently derived [12, 20] when the steady state conditions (2.30) apply. These expressions have been obtained by using a BGK-like kinetic model of the Boltzmann equation for a three-dimensional system [12]. In this section, some of the coefficients obtained here for IMM will be compared with those presented for IHS in the steady state. Beyond the steady state conditions, the dependence of the generalized transport coefficients $\eta_{ijk\ell}^*$ and κ_{ij}^* on both the reduced shear rate a^* and the coefficient of restitution α will be also analysed for several situations. In this case and to the best of my knowledge, there are no available IHS results and so a comparison between IMM and IHS cannot be carried out.

Let us study first the coefficients η_{xyxy}^* and η_{yyxy}^* in the steady USF state. As said before, in this situation a^* is related to α through equation (2.30) and so, the coefficient of restitution
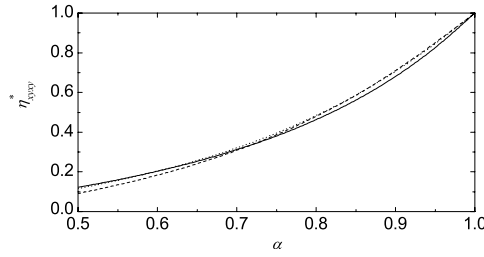
Figure 4. Plot of the reduced element η^*_{xyxy} as a function of the coefficient of restitution α in the steady USF state for a three-dimensional system. The solid line is the result derived here for model A ($q = 0$), the dashed line refers to the result for model B for $q = \frac{1}{2}$ by using approximation (4.12), and the dotted line corresponds to the result obtained for IHS in [12].
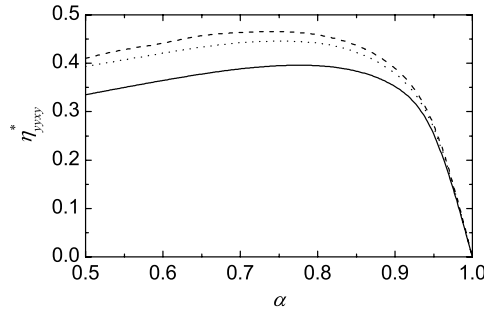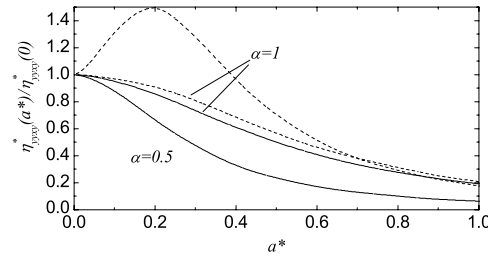

Figure 5. Plot of the reduced element η^*_{yyxy} as a function of the coefficient of restitution α in the steady USF state for a three-dimensional system. The solid line is the result derived here for model A ($q = 0$), the dashed line refers to the result for model B for $q = \frac{1}{2}$ by using approximation (4.12), and the dotted line corresponds to the results obtained for IHS in [12].

is the only control parameter of the system. Figures 4 and 5 show the α-dependence of η^*_{xyxy} and η^*_{yyxy}, respectively, as given by models A, B with $q = \frac{1}{2}$ by using approximation (4.12), and IHS [12] in the steady USF state. It is apparent that the agreement between the predictions of model B for IMM (which tries to mimic the true IHS model) and IHS is excellent in the whole range of values of α analysed. This confirms the reliability of IMM to reproduce the main trends observed for the coefficients $\eta^*_{ijk\ell}$. Regarding the results for models A and B, we observe quantitative differences between both models for the coefficient η^*_{yyxy} for strong dissipation. In the case of general perturbations, the shear-rate dependence of the ratios $\eta^*_{yyxy}(a^*)/\eta^*_{yyxy}(0)$ and $\eta^*_{xyxy}(a^*)/\eta^*_{xyxy}(0)$ is plotted in figures 6 and 7, respectively, for two values of the coefficient of restitution: $\alpha = 1$ and $\alpha = 0.5$. Here, $\eta^*_{ijxy}(0)$ is the value of η^*_{ijxy} for vanishing shear rates given by equations (C.3) for model A and equations (C.11) and (C.12) for model B. We observe that, in general, the ratios $\eta^*_{ijxy}(a^*)/\eta^*_{ijxy}(0)$ are monotonically decreasing functions of the shear rate (shear-thinning effect), except in the region of small shear rates in the case of model B for $\eta^*_{xyxy}(a^*)/\eta^*_{xyxy}(0)$ when $\alpha = 0.5$. Figures 6 and 7 also show that, at a given value of a^*, the value of $\eta^*_{ijxy}(a^*)/\eta^*_{ijxy}(0)$ decrease with dissipation. This means that the inelasticity produces an inhibition of the momentum transport since the value of $\left| P^{(1)}_{xy} \right|$ for inelastic collisions is smaller than that obtained in the elastic case.
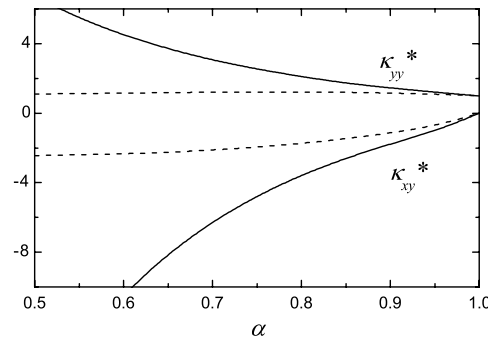
Next, the thermal conductivity tensor κ^*_{ij} is considered when the steady conditions (2.30) hold. The dependence of the elements κ^*_{yy} and κ^*_{xy} on α is plotted in figure 8 for IMM (in the case of model A) and for IHS [12]. As happens for the usual thermal conductivity
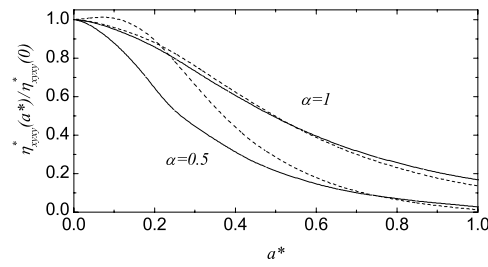
Figure 6. Shear-rate dependence of the ratio $\eta_{xyxy}^*(a^*)/\eta_{xyxy}^*(0)$ for two values of the coefficient of restitution α in the three-dimensional case. The solid lines correspond to the results obtained for model A ($q = 0$), while the dashed lines refer to the results derived for model B for $q = \frac{1}{2}$ by using approximation (4.12).
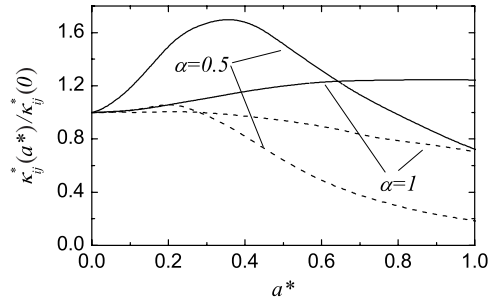

Figure 7. Shear-rate dependence of the ratio $\eta_{yyxy}^*(a^*)/\eta_{yyxy}^*(0)$ for two values of the coefficient of restitution α in the three-dimensional case. The solid lines correspond to the results obtained for model A ($q = 0$), while the dashed lines refer to the results derived for model B for $q = \frac{1}{2}$ by using approximation (4.12).

Figure 8. Plot of the reduced elements κ_{yy}^* and κ_{xy}^* of the thermal conductivity tensor as functions of the coefficient of restitution α in the steady USF state for a three-dimensional system. The solid lines correspond to IMM for model A ($q = 0$), while the dashed lines refer to the results for IHS derived in [12].

coefficient [19], the trends observed for IHS are strongly exaggerated by IMM, especially at high inelasticity. It is possible that the disagreement between both interaction potentials at the level of the heat flux might be mitigated in part if one considered model B instead of model A for IMM. However, this would require to estimate the fourth-degree moments of the reference state by using the approximation (3.16) for $q = \frac{1}{2}$, which is quite an intricate problem. The shear-rate dependence of κ_{ij}^* is illustrated in figure 9 for the ratios $\kappa_{xy}^*(a^*)/\kappa_{xy}^*(0)$ and $\kappa_{yy}^*(a^*)/\kappa_{yy}^*(0)$ for two values of the coefficient of restitution: $\alpha = 0.5$ and $\alpha = 1$. Here, $\kappa_{xy}^*(0)$ is a Burnett coefficient given by equation (C.28) and $\kappa_{yy}^*(0)$ is the Navier–Stokes

Figure 9. Shear-rate dependence of the ratios $\kappa_{yy}^*(a^*)/\kappa_{yy}^*(0)$ (solid lines) and $\kappa_{xy}^*(a^*)/\kappa_{xy}^*(0)$ (dashed lines) for two values of the coefficient of restitution α in the three-dimensional case. All the results have been derived from model A ($q = 0$).

thermal conductivity coefficient given by equation (4.26) (or, equivalently, equation (C.29)). We see that κ_{yy}^* increases with a^* in the region of shear rates considered for elastic collisions, while it does not present a monotonic dependence on a^* in the inelastic case since it reaches a maximum and then decreases with respect to its Navier–Stokes value. Consequently, while for elastic gases the shear flow enhances the transport of energy along the direction of the gradient of the flow velocity (y axis), this is not the case for inelastic collisions for large shear rates, where the heat flux is inhibited by the shearing motion. The off-diagonal element κ_{xy}^* measures cross effects in the thermal conduction since it gives the transport of energy along the x axis due to a thermal gradient parallel to the y axis. This cross coupling does not appear in the linear regime as the heat flux $q_x^{(1)}$ is at least of Burnett order (proportional to $a^* \partial T/\partial y$). The element κ_{xy}^* is negative and its absolute value decreases with respect to its Burnett value $\kappa_{xy}^*(0)$ as the shear rate increases, regardless of the value of dissipation.

6. Discussion

In this paper, the transport properties of d-dimensional IMM in inhomogeneous states close to the USF state have been analysed. The physical situation is such that the granular gas is in a state that deviates from the USF by small spatial gradients. Given that the system is subjected to a strong shear flow, the corresponding transport coefficients associated with the irreversible heat and momentum fluxes are nonlinear functions of the shear rate. The explicit evaluation of these coefficients has been the primary objective of this work. The search for such expressions has been prompted by recent results [12] derived from a simple kinetic model of the inelastic Boltzmann equation, which provides explicit expressions for the above coefficients in the case of IHS. Since this kinetic model is a simplified version of the nonlinear (inelastic) Boltzmann equation, a still open problem is to obtain the explicit shear-rate dependence of these transport coefficients when the true Boltzmann collision operator is considered. For this reason, due to technical difficulties associated with the mathematical structure of the Boltzmann equation for IHS, the problem studied in [12] has been revisited here for IMM. For this interaction model, the collision rate becomes independent of the relative velocity of the two colliding particles so that the velocity moments of order k of the Boltzmann collision operator can be *exactly* written in terms of moments of order $k' \leqslant k$ [15]. Thanks to this feature, the second- and fourth-degree velocity moments corresponding to the (pure) USF state have been recently obtained [16]. The knowledge of the latter moments provides the proper basis to explicitly evaluate transport properties around USF.

As said before, I have been interested in a situation where *weak* spatial gradients of density, velocity and temperature coexist with a *strong* shear rate. Under these conditions, the Boltzmann equation is solved by means of an extension of the Chapman–Enskog method to arbitrary reference states [20]. In the case of the USF state, due to the anisotropy induced by the shear field, tensorial quantities are required to describe the momentum and heat fluxes instead of the usual Navier–Stokes transport coefficients [3, 17]. In the first order of the expansion the momentum and heat fluxes are given by equations (3.29) and (3.30), respectively, where the components of the set of generalized transport coefficient $\eta_{ijk\ell}$, κ_{ij} and μ_{ij} are the solutions of equations (4.5), (4.24) and (4.25), respectively. Such coefficients are *nonlinear* functions of both the shear rate and the coefficient of restitution. As expected, there are many new transport coefficients in comparison to the case of states near equilibrium or near the cooling state. These coefficients provide all the information on the physical mechanisms involved in the transport of momentum and energy under shear flow. Of course, the usual form of the Navier–Stokes coefficients for IMM is recovered in the absence of shear flow [17].

The purpose of this work has been two-fold. First, the evaluation of the shear-rate-dependent transport coefficients of IMM is worthwhile studying by itself as a tractable model to gain some insight into the combined effect of shear flow and dissipation on the transport properties of the system. Second, the comparison between the exact results derived here for IMM with those previously obtained for IHS by using a simple kinetic model allows us to assess the relevance of IMM to reproduce the trends observed in the case of IHS. Recent comparisons [17–19] carried out between both interaction models show good agreement, especially in the USF problem [18].

The results derived here are given in terms of an effective collision frequency ω, which depends on space and time through its dependence on the density and temperature. Here, I have considered the general dependence $\omega \propto nT^q$, where $q = 0$ is closer to the original Maxwell molecules for elastic collisions and $q = \frac{1}{2}$ is closer to hard spheres. To make some contact with the results obtained for IHS [12], ω is chosen to reproduce the cooling rate ζ of IHS evaluated at the local equilibrium approximation. This yields the relation (2.4). While model A ($q = 0$) lends itself to exact analytical results for $\eta_{ijk\ell}$, κ_{ij} and μ_{ij}, that is not the case for model B ($q \neq 0$) since the latter requires to be solved numerically to determine the dependence of the above generalized coefficients on both the shear rate and dissipation. However, some analytical approximate expressions can be obtained by considering an expansion of the transport properties in powers of the interaction parameter q. In this case, the results obtained for $q = \frac{1}{2}$ have been compared with those obtained for IHS. Figures 4, 5, 8 and 9 illustrate this comparison when the steady state conditions (2.17) apply locally (and so, a^* and α are related through equation (2.30)). It is apparent that the IMM predictions compare quite well with IHS in the case of the viscosity tensor $\eta_{ijk\ell}$, although the discrepancies between both interaction models increase in the case of the thermal conductivity tensor κ_{ij}. In this latter case, the IMM results tend to exaggerate the trends observed for IHS.

An application of the results obtained in this paper would be to perform a linear stability analysis of the hydrodynamic equations with respect to the steady simple shear flow state. This analysis allows one to determine the hydrodynamic modes for states near USF as well as the conditions for instabilities at long wavelengths. Previous results for IHS [12] in the three-dimensional case indicate that the USF is linearly stable when the perturbations are along the velocity gradient (y direction) only, while it becomes unstable when the perturbations are along the vorticity direction (z direction) only. The question now is whether these conclusions are similar in the case of IMM. Another possible direction of study is to extend the analysis made here for a single gas to the important subject of granular mixtures. Previous works carried out by the author and co-workers [18, 19] for mixtures of IMM have shown the tractability of

the Maxwell kinetic theory for these complex systems and stimulate the performance of this study in the near future.

Acknowledgments

Appendix A. Fourth-degree moments in the USF state for model A

In this Appendix, the explicit expressions of the set of fourth-degree moments of the zeroth-order distribution $f^{(0)}$ are given in the case of model A ($q = 0$) for a three-dimensional system ($d = 3$). As the set of independent moments, it is convenient to take the (dimensionless) Ikenberry moments [31]

$$\{M^*_{4|0}, M^*_{2|xx}, M^*_{2|yy}, M^*_{0|yyyy}, M^*_{0|zzzz}, M^*_{2|xy}, M^*_{0|xxxy}, M^*_{0|xyyy}\}, \tag{A.1}$$

where

$$M^*_{4|0} = \frac{1}{n}\left(\frac{m}{2T}\right)^2 \int d\mathbf{c}\, c^4 f^{(0)}(\mathbf{c}), \tag{A.2}$$

$$M^*_{2|ij} = \frac{1}{n}\left(\frac{m}{2T}\right)^2 \int d\mathbf{c}\, c^2 \left(c_i c_j - \frac{1}{3}c^2\delta_{ij}\right) f^{(0)}(\mathbf{c}), \tag{A.3}$$

$$M^*_{0|ijk\ell} = \frac{1}{n}\left(\frac{m}{2T}\right)^2 \int d\mathbf{c}\, \Bigg[c_i c_j c_k c_\ell - \frac{1}{7}c^2(c_i c_j \delta_{k\ell} + c_i c_k \delta_{j\ell} + c_i c_\ell \delta_{jk}$$

$$+ c_j c_k \delta_{i\ell} + c_j c_\ell \delta_{ik} + c_k c_\ell \delta_{ij}) + \frac{1}{35}c^4(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\Bigg] f^{(0)}(\mathbf{c}). \tag{A.4}$$

By using matrix form, the moments (A.1) are given by [16]

$$\mathcal{M} = \mathcal{L}^{-1} \cdot \mathcal{C}, \tag{A.5}$$

where $\mathcal{M}$ is the column matrix defined by the set (A.1)

$$\mathcal{M} = \begin{pmatrix} M^*_{4|0} \\ M^*_{2|xx} \\ M^*_{2|yy} \\ M^*_{0|yyyy} \\ M^*_{0|zzzz} \\ M^*_{2|xy} \\ M^*_{0|xxxy} \\ M^*_{0|xyyy} \end{pmatrix}, \tag{A.6}$$

and $\mathcal{L}$ is the square matrix

$$\mathcal{L} = 4\omega^*_{0|2}\gamma\mathcal{I} + \mathcal{L}', \tag{A.7}$$

where $\mathcal{I}$ is the 8×8 identity matrix and

$$\mathcal{L}' = \begin{pmatrix} \omega_{4|0}^* & 0 & 0 & 0 & 0 & 4a^* & 0 & 0 \\ 0 & \omega_{2|2}^* & 0 & 0 & 0 & \frac{32}{21}a^* & 2a^* & 0 \\ 0 & 0 & \omega_{2|2}^* & 0 & 0 & -\frac{10}{21}a^* & 0 & 2a^* \\ 0 & 0 & 0 & \omega_{0|4}^* & 0 & -\frac{150}{245}a^* & 0 & -\frac{12}{7}a^* \\ 0 & 0 & 0 & 0 & \omega_{0|4}^* & \frac{24}{245}a^* & \frac{12}{7}a^* & \frac{12}{7}a^* \\ \frac{7}{15}a^* & \frac{2}{7}a^* & \frac{9}{7}a^* & -\frac{7}{3}a^* & -\frac{1}{3}a^* & \omega_{2|2}^* & 0 & 0 \\ 0 & \frac{15}{49}a^* & -\frac{6}{49}a^* & -\frac{5}{2}a^* & -\frac{5}{14}a^* & 0 & \omega_{0|4}^* & 0 \\ 0 & -\frac{6}{49}a^* & \frac{15}{49}a^* & 2a^* & \frac{1}{7}a^* & 0 & 0 & \omega_{0|4}^* \end{pmatrix}, \tag{A.8}$$

with [15]

$$\omega_{4|0}^* = \frac{(1+\alpha)^2(5+6\alpha-3\alpha^2)}{48}, \tag{A.9}$$

$$\omega_{2|2}^* = \frac{(1+\alpha)^2(34+21\alpha-6\alpha^2)}{168}, \tag{A.10}$$

$$\omega_{0|4}^* = \frac{(1+\alpha)^2(150+21\alpha-3\alpha^2)}{378}. \tag{A.11}$$

The column matrix $\mathcal{C}$ is

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \\ \mathcal{C}_3 \\ \mathcal{C}_4 \\ \mathcal{C}_5 \\ \mathcal{C}_6 \\ \mathcal{C}_7 \\ \mathcal{C}_8 \end{pmatrix}, \tag{A.12}$$

where

$$\mathcal{C}_1 = \frac{9}{4}\lambda_1^* - \lambda_2^*\left(6M_{0|yy}^{*~2} + 2M_{0|xy}^{*~2}\right), \tag{A.13}$$

$$\mathcal{C}_2 = \frac{3}{2}\lambda_3^* M_{0|xx}^* - \frac{1}{3}\lambda_4^*\left(6M_{0|yy}^{*~2} + M_{0|xy}^{*~2}\right), \tag{A.14}$$

$$\mathcal{C}_3 = \frac{3}{2}\lambda_3^* M_{0|yy}^* - \frac{1}{3}\lambda_4^*\left(M_{0|xy}^{*~2} - 3M_{0|yy}^{*~2}\right), \tag{A.15}$$

$$\mathcal{C}_4 = \frac{1}{35}\lambda_5^*\left(81M_{0|yy}^{*~2} - 48M_{0|xy}^{*~2}\right), \tag{A.16}$$

$$\mathcal{C}_5 = \frac{1}{35}\lambda_5^*\left(81M_{0|yy}^{*~2} + 12M_{0|xy}^{*~2}\right), \tag{A.17}$$

$$\mathcal{C}_6 = \frac{3}{2}\lambda_3^* M_{0|xy}^* + \lambda_4^* M_{0|xy}^* M_{0|yy}^*, \tag{A.18}$$

$$\mathcal{C}_7 = -\frac{36}{7}\lambda_5^* M_{0|xy}^* M_{0|yy}^*, \tag{A.19}$$

$$\mathcal{C}_8 = \frac{27}{7}\lambda_5^* M_{0|xy}^* M_{0|yy}^*. \tag{A.20}$$

In equations (A.13)–(A.20), I have introduced the second-degree moments

$$M_{0|ij}^* = \tfrac{1}{2}(P_{ij}^* - \delta_{ij}), \tag{A.21}$$

where P_{ij}^* is given by equations (2.22), (2.23) and the quantities

$$\lambda_1^* = \frac{5}{144}(1+\alpha)^2(11 - 6\alpha + 3\alpha^2), \tag{A.22}$$

$$\lambda_2^* = \frac{(1+\alpha)^2(1 + 6\alpha - 3\alpha^2)}{24}, \tag{A.23}$$

$$\lambda_3^* = \frac{(1+\alpha)^2(22 - 21\alpha + 6\alpha^2)}{72}, \tag{A.24}$$

$$\lambda_4^* = \frac{(1+\alpha)^2(21\alpha - 1 - 6\alpha^2)}{84}, \tag{A.25}$$

$$\lambda_5^* = \frac{(1+\alpha)^2(39 - 21\alpha + 3\alpha^2)}{378}. \tag{A.26}$$

The explicit dependence of the fourth-degree moments (A.1) on a^* and α can be determined from equation (A.5). These expressions are very long and not very illuminating so that they will be omitted here for the sake of clarity.

The knowledge of the fourth-degree moments of the reference USF state is needed to compute the thermal conductivity tensor κ_{ij} and the tensor μ_{ij} (see for instance, equations (4.24) and (4.25)). The relationship between the (canonical) moments defined in (A.1) and the moments $N_{ijk\ell}^*$ defined by equations (4.18) and (4.23) is given by

$$N_{xxxx}^* = \tfrac{1}{5}M_{4|0}^* + \tfrac{6}{7}M_{2|xx}^* + \tfrac{4}{3}(M_{0|yyyy}^* + M_{0|zzzz}^*), \tag{A.27}$$

$$N_{yyyy}^* = \tfrac{1}{5}M_{4|0}^* + \tfrac{6}{7}M_{2|yy}^* + M_{0|yyyy}^*, \tag{A.28}$$

$$N_{xxyy}^* = \tfrac{1}{15}M_{4|0}^* + \tfrac{1}{7}(M_{2|xx}^* + M_{2|yy}^*) - \tfrac{7}{6}M_{0|yyyy}^* - \tfrac{1}{6}M_{0|zzzz}^*, \tag{A.29}$$

$$N_{xxzz}^* = \tfrac{1}{15}M_{4|0}^* - \tfrac{1}{7}M_{2|yy}^* - \tfrac{1}{6}M_{0|yyyy}^* - \tfrac{7}{6}M_{0|zzzz}^*, \tag{A.30}$$

$$N_{yyzz}^* = \tfrac{1}{15}M_{4|0}^* - \tfrac{1}{7}M_{2|xx}^* + \tfrac{1}{6}M_{0|yyyy}^* + \tfrac{1}{6}M_{0|zzzz}^*, \tag{A.31}$$

$$N_{xxxy}^* = \tfrac{3}{7}M_{2|xy}^* + M_{0|xxxy}^*, \tag{A.32}$$

$$N_{xyyy}^* = \tfrac{3}{7}M_{2|xy}^* + M_{0|xyyy}^*, \tag{A.33}$$

$$N_{xyzz}^* = \tfrac{1}{7}M_{2|xy}^* - (M_{0|xxxy}^* + M_{0|xyyy}^*). \tag{A.34}$$

Appendix B. Chapman–Enskog-like expansion

In this Appendix, some technical details of the Chapman–Enskog-like type of expansion used in the main text are described. First, the action of the operators $\partial_t^{(k)}$ on the hydrodynamic

fields can be obtained from the corresponding balance equations associated with the Boltzmann equation (3.2). They are given by

$$\partial_t n + \mathbf{u}_0 \cdot \nabla n = -\nabla \cdot (n\delta\mathbf{u}), \tag{B.1}$$

$$\partial_t \delta\mathbf{u} + \mathsf{a} \cdot \delta\mathbf{u} + (\mathbf{u}_0 + \delta\mathbf{u}) \cdot \nabla\delta\mathbf{u} = -(mn)^{-1}\nabla \cdot \mathsf{P}, \tag{B.2}$$

$$\frac{d}{2}n\partial_t T + \frac{d}{2}n(\mathbf{u}_0 + \delta\mathbf{u}) \cdot \nabla T + aP_{xy} + \nabla \cdot \mathbf{q} + \mathsf{P} : \nabla\delta\mathbf{u} = -\frac{d}{2}p\zeta, \tag{B.3}$$

where the pressure tensor P, the heat flux $\mathbf{q}$ and the cooling rate ζ are defined by equations (2.11)–(2.13), respectively, with the replacement $\mathbf{V} \to \mathbf{c}$. From equations (B.1)–(B.3), it is easy to see that the action of the operator $\partial_t^{(0)}$ is given by equations (3.9) and (3.10), while in the case of the operator $\partial_t^{(1)}$ the result is

$$\partial_t^{(1)}n + \mathbf{u}_0 \cdot \nabla n = -\nabla \cdot (n\delta\mathbf{u}), \tag{B.4}$$

$$\partial_t^{(1)}\delta\mathbf{u} + (\mathbf{u}_0 + \delta\mathbf{u}) \cdot \nabla\delta\mathbf{u} = -\frac{1}{mn}\nabla \cdot \mathsf{P}^{(0)}, \tag{B.5}$$

$$\frac{d}{2}n\partial_t^{(1)}T + \frac{d}{2}n(\mathbf{u}_0 + \delta\mathbf{u}) \cdot \nabla T = -aP_{xy}^{(1)} - \mathsf{P}^{(0)} : \nabla\delta\mathbf{u}, \tag{B.6}$$

where

$$P_{ij}^{(1)} = \int d\mathbf{c}\, mc_i c_j f^{(1)}(\mathbf{c}). \tag{B.7}$$

Substituting expansions (3.5) and (3.6) into equation (3.2), one obtains the kinetic equation for the velocity distribution $f^{(1)}$

$$\left(\partial_t^{(0)} - aV_y\frac{\partial}{\partial V_x} + \mathcal{L}\right)f^{(1)} = -\left[\partial_t^{(1)} + (\mathbf{V} + \mathbf{u}_0) \cdot \nabla\right]f^{(0)}, \tag{B.8}$$

where $\mathcal{L}$ is the linearized Boltzmann collision operator around the USF defined by equation (3.28). Substituting equations (B.4)–(B.6) into equation (B.8) yields

$$\left(\partial_t^{(0)} - aV_y\frac{\partial}{\partial V_x} + \mathcal{L}\right)f^{(1)} = \mathbf{Y}_n \cdot \nabla n + \mathbf{Y}_T \cdot \nabla T + \mathsf{Y}_u : \nabla\delta\mathbf{u}, \tag{B.9}$$

where

$$Y_{n,i} = -\frac{\partial f^{(0)}}{\partial n}c_i + \frac{1}{\rho}\frac{\partial f^{(0)}}{\partial \delta u_j}\frac{\partial P_{ij}^{(0)}}{\partial n}, \tag{B.10}$$

$$Y_{T,i} = -\frac{\partial f^{(0)}}{\partial T}c_i + \frac{1}{\rho}\frac{\partial f^{(0)}}{\partial \delta u_j}\frac{\partial P_{ij}^{(0)}}{\partial T}, \tag{B.11}$$

$$Y_{u,ij} = n\frac{\partial f^{(0)}}{\partial n}\delta_{ij} - \frac{\partial f^{(0)}}{\partial \delta u_i}c_j + \frac{2}{dn}\frac{\partial f^{(0)}}{\partial T}\left(P_{ij}^{(0)} - a\eta_{xyij}\right). \tag{B.12}$$

Equations (B.9)–(B.12) are similar to those obtained for IHS [12], except that $\zeta^{(1)} = 0$ for IMM. The solution to equation (B.9) is of the form given by equation (3.24). To obtain the corresponding integral equations verifying the unknowns $X_{n,i}$, $X_{T,i}$ and $X_{u,ij}$, one has to take into account the action of the time derivative $\partial_t^{(0)}$ on the temperature and density gradients,

$$\partial_t^{(0)}\nabla_i T = \nabla_i\partial_t^{(0)}T$$
$$= \left(\frac{2a}{dn^2}(1 - n\partial_n)P_{xy}^{(0)} - \frac{\zeta T}{n}\right)\nabla_i n - \left(\frac{2a}{dn}\partial_T P_{xy}^{(0)} + (q+1)\zeta\right)\nabla_i T, \tag{B.13}$$

$$\partial_t^{(0)} \nabla_i \delta u_j = \nabla_i \partial_t^{(0)} \delta u_j = -a_{jk} \nabla_i \delta u_k. \tag{B.14}$$

Substituting the expression of $f^{(1)}$ given by (3.24) into (B.8), the integral equations for $X_{n,i}$, $X_{T,i}$ and $X_{u,ij}$ are identified as the coefficients of the independent gradients. This leads to equations (3.25)–(3.27).

Appendix C. Transport coefficients

In this Appendix, some of the reduced transport coefficients $\eta_{ijk\ell}^*$, κ_{ij}^* and μ_{ij}^* are explicitly computed. Let us consider first the coefficients of the form η_{ijxy}^* for model A. According to equations (4.8), these coefficients verify the equations

$$\left(\frac{4a^*}{d} P_{xy}^* - \omega_{0|2}^*\right) \eta_{xyxy}^* - a^* \eta_{yyxy}^* = \frac{2}{d} P_{xy}^{*2} - P_{yy}^*, \tag{C.1}$$

$$\left(\frac{2a^*}{d} P_{xy}^* - \omega_{0|2}^*\right) \eta_{yyxy}^* + \frac{2a^*}{d} P_{yy}^* \eta_{xyxy}^* = \frac{2}{d} P_{xy}^* P_{yy}^*, \tag{C.2}$$

where the expressions of the reduced elements P_{ij}^* are given by equations (2.22) and (2.23). Since $\eta_{yyxy}^* = \eta_{zzxy}^*$ and $\eta_{ijk\ell}^*$ is a traceless tensor, then $\eta_{xxxy}^* = -(d-1)\eta_{yyxy}^*$. The solution to the set of equations (C.1) and (C.2) leads to equations (4.9) and (4.11) when one takes into account relations (2.25) and (2.26). For small shear rates ($a^* \ll 1$), the coefficients η_{ijxy}^* behave as

$$\eta_{xyxy}^* \approx \frac{1}{\omega_{02}^*}\left(1 - \frac{12}{d}\tilde{a}^2\right), \qquad \eta_{jjxy}^* \approx \frac{4}{\omega_{02}^*}\frac{1 - \delta_{jx}d}{d}\tilde{a}. \tag{C.3}$$

For model B, one can estimate η_{ijxy}^* by using approximation (4.12). Neglecting nonlinear terms in q, it is easy to see that the quantities $\Delta\eta_{ijxy}^*$ are determined from the set of coupled equations

$$\left(\frac{4a^*}{d} P_{xy,A}^* - \omega_{0|2}^*\right) \Delta\eta_{xyxy}^* - a^* \Delta\eta_{yyxy}^* = \frac{4}{d}(\Delta P_{xy}^* P_{xy,A}^* - a^* \eta_{xyxy,A}^* \Delta P_{xy}^*) - \Delta P_{yy}^*$$
$$+ \frac{2}{d}(a^* P_{xy,A}^* + \zeta^*)(1 + a^* \partial_{a^*})\eta_{xyxy,A}^* - \frac{2}{d}(P_{xy,A}^* - a^* \eta_{xyxy,A}^*)a^* \partial_{a^*} P_{xy,A}^*, \tag{C.4}$$

$$\left(\frac{2a^*}{d} P_{xy,A}^* - \omega_{0|2}^*\right) \Delta\eta_{yyxy}^* + \frac{2a^*}{d} P_{yy,A}^* \Delta\eta_{xyxy}^* = \frac{2}{d}\Delta P_{yy}^*(P_{xy,A}^* - a^* \eta_{xyxy,A}^*)$$
$$+ \frac{2}{d}\Delta P_{xy}^*(P_{yy,A}^* - a^* \eta_{yyxy,A}^*) + \frac{2}{d}(a^* P_{xy,A}^* + \zeta^*)(1 + a^* \partial_{a^*})\eta_{yyxy,A}^*$$
$$- \frac{2}{d}(P_{xy,A}^* - a^* \eta_{xyxy,A}^*)a^* \partial_{a^*} P_{yy,A}^*, \tag{C.5}$$

where the elements ΔP_{ij}^* are given by equations (3.17) and (3.18) and the viscosities $\eta_{xyxy,A}^*$ and $\eta_{yyxy,A}^*$ for Model A are given by equations (4.9) and (4.11), respectively. The solution to equations (C.4) and (C.5), along with the constraint $\Delta\eta_{xxxy}^* = -(d-1)\Delta\eta_{yyxy}^*$, provides the explicit form for the terms $\Delta\eta_{jjxy}^*$. Their nonzero elements can be written as

$$\Delta\eta_{xyxy}^* = \frac{2\Lambda_{xy}}{d\omega_{0|2}^{*2}(1 + 2\gamma)^2(1 + 6\gamma)^4}, \tag{C.6}$$

$$\Delta\eta^*_{xxxy} = -\frac{2\widetilde{a}(1-d)\Lambda_{yy}}{d^2\omega^{*2}_{0|2}(1+2\gamma)^3(1+6\gamma)^4}, \tag{C.7}$$

$$\Delta\eta^*_{yyxy} = \Delta\eta^*_{zzxy} = -\frac{2\widetilde{a}\Lambda_{yy}}{d^2\omega^{*2}_{0|2}(1+2\gamma)^3(1+6\gamma)^4}, \tag{C.8}$$

where

$$\Lambda_{xy} = 72d\omega^*_{0|2}\gamma^4 - 12\gamma^3[9d\omega^*_{0|2} + 2(5+2d)\zeta^*] + 2\gamma^2[2(13-2d)\zeta^* - 17d\omega^*_{0|2}]$$
$$+ \gamma[3d\omega^*_{0|2} + 2(11+4d)\zeta^*] - \zeta^*, \tag{C.9}$$

$$\Lambda_{yy} = 144d\omega^*_{0|2}\gamma^3 - 12\gamma^2[2d\omega^*_{0|2} + (14+5d)\zeta^*] - 4\gamma[6d\omega^*_{0|2} + (4+7d)\zeta^*]$$
$$+ (d+10)\zeta^*. \tag{C.10}$$

For small shear rates, the coefficients η^*_{ijxy} have the expansions

$$\eta^*_{xyxy} \approx \frac{1}{\omega^*_{02}}\left(1 - 2q\frac{\zeta^*}{d\omega^*_{02}}\right) + \frac{\widetilde{a}^2}{d\omega^*_{02}}\left[2q\frac{2(25+4d)\zeta^* + 3d\omega^*_{02}}{d\omega^*_{02}} - 12\right], \tag{C.11}$$

$$\eta^*_{jjxy} \approx \frac{4}{d\omega^*_{02}}(1-\delta_{jx}d)\left(1 - q\zeta^*\frac{d+10}{2d\omega^*_{02}}\right)\widetilde{a}. \tag{C.12}$$

The evaluation of the coefficients κ^*_{ij} and μ^*_{ij} is more intricate than that of $\eta^*_{ijk\ell}$ since it involves the fourth-degree moments of the zeroth-order distribution. Here, only the coefficients κ^*_{yy} and κ^*_{xy} for Model A in the case of a three-dimensional system ($d = 3$) will be considered. These coefficients are given by

$$\kappa^*_{xy} = \kappa^*_{xxxy} + \kappa^*_{yyxy} + \kappa^*_{zzxy}, \tag{C.13}$$

$$\kappa^*_{yy} = \kappa^*_{xxyy} + \kappa^*_{yyyy} + \kappa^*_{zzyy}. \tag{C.14}$$

In the case of Model A ($q = 0$), the coefficients $\kappa^*_{ijk\ell}$ obey the set of coupled equations

$$\left[2\left(\frac{2}{3}a^*P^*_{xy} + \zeta^*\right) - \nu^*_{0|3}\right]\kappa^*_{ijk\ell} - \frac{\nu^*_{2|1} - \nu^*_{0|3}}{5}(\delta_{jk}\kappa^*_{i\ell} + \delta_{ik}\kappa^*_{j\ell} + \delta_{ij}\kappa^*_{k\ell})$$
$$- a^*(\delta_{ix}\kappa^*_{jky\ell} + \delta_{jx}\kappa^*_{iky\ell} + \delta_{kx}\kappa^*_{ijy\ell})$$
$$= -\frac{2}{15}(8N^*_{ijk\ell} - P^*_{kj}P^*_{i\ell} - P^*_{ik}P^*_{j\ell} - P^*_{ij}P^*_{k\ell}). \tag{C.15}$$

Equation (C.15) shows that κ^*_{yy} and κ^*_{xy} are coupled, so that their calculation involves the set of coefficients

$$\{\kappa^*_{xxyy}, \kappa^*_{yyyy}, \kappa^*_{zzyy}, \kappa^*_{xyyy}, \kappa^*_{xxxy}, \kappa^*_{yyxy}, \kappa^*_{zzxy}\}. \tag{C.16}$$

By using matrix notation, the set of seven coupled algebraic equations for the unknowns (C.16) can be written as

$$\mathcal{P}_{\sigma\sigma'}\mathcal{Q}_{\sigma'} = \mathcal{R}_{\sigma}, \qquad \sigma = 1, \ldots, 7. \tag{C.17}$$

Here, $\mathcal{Q}$ is the column matrix defined by the set (C.16), $\mathcal{P}$ is the square matrix given by

$$\mathcal{P} = \begin{pmatrix} \beta + \phi & \phi & \phi & -2a^* & 0 & 0 & 0 \\ 3\phi & \beta + 3\phi & 3\phi & 0 & 0 & 0 & 0 \\ \phi & \phi & \beta + \phi & 0 & 0 & 0 & 0 \\ 0 & -a^* & 0 & \beta & \phi & \phi & \phi \\ -3a^* & 0 & 0 & 0 & \beta + 3\phi & 3\phi & 3\phi \\ 0 & -a^* & 0 & 0 & \phi & \beta + \phi & \phi \\ 0 & 0 & -a^* & 0 & \phi & \phi & \beta + \phi \end{pmatrix}, \quad \text{(C.18)}$$

where

$$\beta \equiv 2\left(\frac{2}{3}a^* P_{xy}^* + \zeta^*\right) - \nu_{0|3}^* = -\frac{1+\alpha}{12}\left[12\gamma(1+\alpha) + 7\alpha + 2\right], \quad \text{(C.19)}$$

$$\phi \equiv \frac{\nu_{0|3}^* - \nu_{2|1}^*}{5} = \frac{1}{24}(1+\alpha)^2. \quad \text{(C.20)}$$

The column matrix $\mathcal{R}$ is

$$\mathcal{R} = -\frac{2}{15}\begin{pmatrix} 8N_{xxyy}^* - 2P_{xy}^{*2} - P_{xx}^* P_{yy}^* \\ 8N_{yyyy}^* - 3P_{yy}^{*2} \\ 8N_{yyzz}^* - P_{zz}^{*2} \\ 8N_{xyyy}^* - 3P_{yy}^* P_{xy}^* \\ 8N_{xxxy}^* - 3P_{xx}^* P_{xy}^* \\ 8N_{xyyy}^* - 3P_{yy}^* P_{xy}^* \\ 8N_{xyzz}^* - P_{zz}^* P_{xy}^* \end{pmatrix}, \quad \text{(C.21)}$$

The solution to equation (C.17) is

$$\mathcal{Q}_\sigma = (\mathcal{P}^{-1})_{\sigma\sigma'}\mathcal{R}_{\sigma'}. \quad \text{(C.22)}$$

This relation provides an explicit expression for the coefficients of the form κ_{ijyy}^* and κ_{ijxy}^*. From these expressions one can obtain the elements κ_{xy}^* and κ_{yy}^*. They can be written as

$$\kappa_{xy}^* = \frac{\Psi_{xy}}{\Delta}, \qquad \kappa_{yy}^* = \frac{\Psi_{yy}}{\Delta}, \quad \text{(C.23)}$$

where

$$\Psi_{xy} = a^*\beta(3\beta + 8\phi)\mathcal{R}_1 + \frac{a^*}{\beta}\mathcal{R}_2[2a^{*2}(3\beta + 2\phi) + \beta^2(\beta - 2\phi)]$$

$$+ \frac{a^*}{\beta}\mathcal{R}_3[\beta^2(\beta - 2\phi) - 12a^{*2}\phi]$$

$$+ 2\mathcal{R}_4 a^{*2}(3\beta + 8\phi) + (\mathcal{R}_5 + \mathcal{R}_6 + \mathcal{R}_7)[\beta^2(\beta + 5\phi) + 6a^{*2}\phi], \quad \text{(C.24)}$$

$$\Psi_{yy} = \beta^2(\beta + 5\phi)\mathcal{R}_1 + \mathcal{R}_2[2a^{*2}(\beta + 7\phi) + \beta^2(\beta + 5\phi)] + \mathcal{R}_3[\beta^2(\beta + 5\phi) + 4a^{*2}\phi]$$

$$+ 2\mathcal{R}_4 a^*\beta(\beta + 5\phi) - 2(\mathcal{R}_5 + \mathcal{R}_6 + \mathcal{R}_7)a^*\beta\phi, \quad \text{(C.25)}$$

$$\Delta = 2a^{*2}\phi(6\beta + 23\phi) + \beta^2(\beta^2 + 10\beta\phi + 25\phi^2). \quad \text{(C.26)}$$

In the limit of small shear rates ($a^* \to 0$), the coefficients κ_{xy}^* and κ_{yy}^* behave as

$$\kappa_{xy}^* \approx \kappa_{xy}^{(1)}(\alpha)a^* + \mathcal{O}(a^{*3}), \qquad \kappa_{yy}^* \approx \kappa_{yy}^{(0)}(\alpha) + \mathcal{O}(a^{*2}), \tag{C.27}$$

where

$$\kappa_{xy}^{(1)}(\alpha) = \frac{112}{5} \frac{162\alpha^7 - 585\alpha^6 + 981\alpha^5 - 111136\alpha^4 + 7272\alpha^3 + 31267\alpha^2 - 49855\alpha - 9466}{(1-9\alpha)^2(1+\alpha)^3[3\alpha(\alpha-2)-5][3\alpha(2\alpha-7)-34]}, \tag{C.28}$$

$$\kappa_{yy}^{(0)}(\alpha) = -\frac{16}{1+\alpha} \frac{17 + 9\alpha(\alpha-2)}{(9\alpha-1)[3\alpha(\alpha-2)-5]}. \tag{C.29}$$

The coefficient $\kappa_{xy}^{(1)}(\alpha)$ is a Burnett coefficient while $\kappa_{yy}^{(0)}(\alpha)$ gives the Navier–Stokes thermal conductivity coefficient for a three-dimensional inelastic gas, equation (4.26). When $\alpha = 1$, one recovers well-known results [4] for elastic collisions, namely, $\kappa_{yy}^{(0)} = 1$ and $\kappa_{xy}^{(1)} = -\frac{7}{2}$.

References

[1] Goldshtein A and Shapiro M 1995 *J. Fluid Mech.* **282** 75
    Brey J J, Dufty J W and Santos A 1997 *J. Stat. Phys.* **87** 1051
    van Noije T P C, Ernst M H and Brito R 1998 *Physica* A **251** 266
[2] Brilliantov N and Pöschel T 2004 *Kinetic Theory of Granular Gases* (Oxford: Oxford University Press)
[3] Brey J J, Dufty J W, Kim C S and Santos A 1998 *Phys. Rev.* E **58** 4638
    Brey J J and Cubero D 2001 *Granular Gases* (*Lectures Notes in Physics* vol 564) ed T Pöschel and S Luding (Berlin: Springer) pp 59–78
    Garzó V and Montanero J M 2002 *Physica* A **313** 336
    Garzó V and Dufty J W 2002 *Phys. Fluids* **14** 1476
[4] Chapman S and Cowling T G 1970 *The Mathematical Theory of Nonuniform Gases* (Cambridge: Cambridge University Press)
[5] Brey J J, Ruiz-Montero M J and Cubero D 1999 *Europhys. Lett.* **48** 359
    Garzó V and Montanero J M 2002 *Physica* A **313** 336
    Montanero J M and Garzó V 2003 *Phys. Rev.* E **67** 021308
    Montanero J M, Santos A and Garzó V 2005 *Rarefied Gas Dynamics 24* vol 762 ed M Capitelli (New York: AIP) pp 797–802 (*Preprint* cond-mat/0411219)
    Brey J J and Ruiz-Montero M J 2004 *Phys. Rev.* E **70** 051301
    Brey J J, Ruiz-Montero M J, Maynar P and García de Soria M I 2005 *J. Phys.: Condens. Matter* **17** S2489
[6] Cercignani C 1988 *The Boltzmann Equation and its Applications* (New York: Springer)
[7] Garzó V and Santos A 2003 *Kinetic Theory of Gases in Shear Flows. Nonlinear Transport* (Dordrecht: Kluwer Academic)
[8] Bird G A 1994 *Molecular Gas Dynamics and the Direct Simulation Monte Carlo of Gas Flows* (Oxford: Clarendon)
[9] Brey J J, Dufty J W and Santos A 1999 *J. Stat. Phys.* **97** 281
[10] Tij M, Tahiri E E, Montanero J M, Garzó V, Santos A and Dufty J W 1999 *J. Stat. Phys.* **103** 1035
[11] Campbell C S 1990 *Annu. Rev. Fluid Mech.* **22** 57
     Goldhirsch I 2003 *Annu. Rev. Fluid Mech.* **35** 267
[12] Garzó V 2006 *Phys. Rev.* E **73** 021304
[13] See for instance, Bobylev A V, Carrillo J A and Gamba I M 2000 *J. Stat. Phys.* **98** 743
     Carrillo J A, Cercignani C and Gamba I M 2000 *Phys. Rev.* E **62** 7700
     Ben-Naim E and Krapivsky P L 2000 *Phys. Rev.* E **61** R5
     Cercignani C 2001 *J. Stat. Phys.* **102** 1407
     Ernst M H and Brito R 2002 *Europhys. Lett.* **58** 182
     Ernst M H and Brito R 2002 *J. Stat. Phys.* **109** 407
     Ernst M H and Brito R 2002 *Phys. Rev.* E **65** 040301
     Baldasarri A, Marconi U M B and Puglisi A 2002 *Europhys. Lett.* **58** 14
     Ben-Naim E and Krapivsky P L 2002 *Phys. Rev.* E **66** 011309
     Krapivsky P L and Ben-Naim E 2002 *J. Phys. A: Math. Gen.* **35** L147

Ben-Naim E and Krapivsky P L 2002 *Eur. Phys. J.* E **8** 507

Marconi U M B and Puglisi A 2002 *Phys. Rev.* E **65** 051305

Marconi U M B and Puglisi A 2002 *Phys. Rev.* E **66** 011301

Bobylev A V and Cercignani C 2003 *J. Stat. Phys.* **110** 333

Bobylev A V, Cercignani C and Toscani G 2003 *J. Stat. Phys.* **111** 403

Santos A and Ernst M H 2003 *Phys. Rev.* E **68** 011305

Bobylev A V and Gamba I M 2006 *J. Stat. Phys.* **124** 497

Ernst M H, Trizac E and Barrat A 2006 *J. Stat. Phys.* **124** 549

Ernst M H, Trizac E and Barrat A 2006 *Europhys. Lett.* **76** 56

Barrat A, Trizac E and Ernst M H 2007 *J. Phys. A: Math. Theor.* **40** 4057

[14] Kohlstedt K, Snezhko A, Sapozhnikov M V, Arnarson I S, Olafsen J S and Ben-Naim E 2005 *Phys. Rev. Lett.* **95** 068001

[15] Garzó V and Santos A 2007 *Preprint* cond-mat/0703704

[16] Santos A and Garzó V 2007 *Preprint* cond-mat/0706.0475

[17] Santos A 2003 *Physica* A **321** 442

[18] Garzó V 2003 *J. Stat. Phys.* **112** 657

[19] Garzó V and Astillero A 2005 *J. Stat. Phys.* **118** 935

[20] Lutsko J F 2006 *Phys. Rev.* E **73** 021302

[21] Bobylev A V, Carrillo J A and Gamba I M 2000 *J. Stat. Phys.* **98** 743

Carrillo J A, Cercignani C and Gamba I M 2000 *Phys. Rev.* E **62** 7700

Bobylev A V and Cercignani C 2002 *J. Stat. Phys.* **106** 547

[22] Astillero A and Santos A 2007 *Europhys. Lett.* **78** 24002

[23] Santos A, Garzó V and Dufty J W 2004 *Phys. Rev.* E **69** 061303

[24] Lees A W and Edwards S F 1973 *J. Phys. C: Solid State Phys.* **5** 1921

[25] Dufty J W, Santos A, Brey J J and Rodríguez R F 1986 *Phys. Rev.* A **33** 459

[26] Ikenberry E and Truesdell C 1956 *J. Ration. Mech. Anal.* **5** 1

Truesdell C 1956 *J. Ration. Mech. Anal.* **5** 55

[27] Brey J J, Ruiz-Montero M J and Moreno F 1997 *Phys. Rev.* E **55** 2846

[28] Santos A and Astillero A 2005 *Phys. Rev.* E **72** 031308

Astillero A and Santos A 2005 *Phys. Rev.* E **72** 031309

[29] Lee M and Dufty J W 1997 *Phys. Rev.* E **56** 1733

[30] Garzó V 2007 *J. Stat. Mech.* P02012

[31] Truesdell C and Muncaster R G 1980 *Fundamentals of Maxwell's Kinetic Theory of a Simple Monatomic Gas* (New York: Academic)