Nonlinear transport for a dilute gas in steady Couette flow

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Transport properties of a dilute gas subjected to arbitrarily large velocity and temperature gradients (steady planar Couette flow) are determined. The results are obtained from the so-called ellipsoidal statistical (ES) kinetic model, which is an extension of the well-known BGK kinetic model to account for the correct Prandtl number. At a hydrodynamic level, the solution is characterized by constant pressure, and linear velocity and parabolic temperature profiles with respect to a scaled variable. The transport coefficients are explicitly evaluated as nonlinear functions of the shear rate. A comparison with previous results derived from a perturbative solution of the Boltzmann equation as well as from other kinetic models is carried out. Such a comparison shows that the ES predictions are in better agreement with the Boltzmann results than those of the other approximations. In addition, the velocity distribution function is also computed. Although the shear rates required for observing non-Newtonian effects are experimentally unrealizable, the conclusions obtained here may be relevant for analyzing computer results. © *1997 American Institute of Physics.* [S1070-6631(97)02203-4]

I. INTRODUCTION

One of the most interesting problems in which some insight into the behavior of nonequilibrium systems can be gained is that of steady planar Couette flow. It corresponds to a fluid between parallel plates maintained at different temperatures and in relative motion. These boundary conditions lead to combined heat and momentum transport. If x and y denote the coordinate parallel to the flow and normal to the plates, respectively, then the corresponding hydrodynamic balance equations read as

$$\frac{\partial}{\partial y}P_{xy}=\frac{\partial}{\partial y}P_{yy}=0\,, \tag{1}$$

$$\frac{\partial}{\partial y}q_y=-P_{xy}\frac{\partial}{\partial y}u_x\,, \tag{2}$$

where u_x is the x component of the flow velocity $\mathbf{u}$, P is the pressure tensor and $\mathbf{q}$ is the heat flux. Equation (2) shows that a thermal gradient $\partial T/\partial y$ is present due to the existence of a velocity gradient, even if both plates are at the same temperature. The above equations are not closed unless the pressure tensor and the heat flux are known functions of the hydrodynamic fields. In the Navier–Stokes regime, the constitutive equations are

$$P_{xy}=-\eta_0(n,T)\frac{\partial}{\partial y}u_x\,, \tag{3}$$

$$q_y=-\kappa_0(n,T)\frac{\partial}{\partial y}T, \tag{4}$$

where n and T are the local density and temperature while η_0 and κ_0 are the shear viscosity and the thermal conductivity, respectively. Here, n and T are related through the equation of state $p(n,T)\equiv\frac{1}{3}tr\mathsf{P}$ and the condition

$$p=\text{const.} \tag{5}$$

Even in this regime the problem remains of knowing the spatial dependence of the transport coefficients if an exact solution is to be determined. One possibility is to consider a dilute gas for which the state of the system is completely specified by the velocity distribution function $f(\mathbf{r},\mathbf{v};t)$ satisfying the Boltzmann equation.[1] Since in this case the ratio η_0/κ_0 is constant, it is easy to solve Eqs. (1) and (2) and obtain a linear velocity profile and a parabolic temperature profile.

A natural question is whether beyond the Navier–Stokes regime one may also find a solution valid for arbitrary velocity and temperature gradients. Very recently, Tijs and Santos[2] have shown that the Boltzmann equation admits a consistent solution for Maxwell molecules (particles interacting via an r^{-4} repulsive potential) characterized by similar profiles as those of the linear regime, but replacing in the constitutive Eqs. (3) and (4) η_0 and κ_0 by a generalized shear viscosity $\eta(a)=\eta_0F_\eta(a)$ and a generalized thermal conductivity $\kappa(a)=\kappa_0F_\kappa(a)$, respectively. Here, a is the constant (dimensionless) shear rate and F_η and F_κ are nonlinear functions of a. Unfortunately, explicit expressions for both functions may not be given in a closed form since they obey an algebraic hierarchy that cannot be solved in a recursive way. Therefore, they use a perturbation expansion in powers of the shear rate and derive results up to the super-Burnett approximation. If one wants to get the transport coefficients for arbitrary values of a, either one performs computer simulations or on the analytical side one considers kinetic models. Here, we take the second route.

An exact solution for the steady Couette flow, using the well-known BGK kinetic model,[3] has been available for some time.[4] As a matter of fact, the results of Ref. 2 were inspired in the BGK solution. However, the drawback inherent in the BGK equation, namely that it does not lead to the correct Prandtl number Pr, served as a motivation to reexamine[5] the same problem by using the Liu kinetic

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model[6] (an extension of the BGK model giving Pr correctly). The results indicate that the main difference between the BGK and Liu descriptions was in the generalized thermal conductivity but it was also found that, apart from other difficulties, the Liu model could yield negative values for the velocity distribution function, which is physically meaningless. Another kinetic model in which neither of these inconsistencies is present is the ellipsoidal statistical (ES) model.[1,7] In the ES model the local Maxwellian of the BGK collision term is replaced by an anisotropic three-dimensional Gaussian involving the pressure tensor and the Prandtl number. When Pr=1, the ES model reduces to the BGK equation. Since the ES model avoids the two difficulties mentioned above, in this paper we will use it for the analysis of the steady planar Couette flow. As described below, the comparison with the perturbative solution given by Tijs and Santos[2] shows that the results from the ES model are in much better agreement than those from either the BGK or the Liu models. This suggests that in general, the shear-rate dependence of the transport coefficients predicted by the ES approximation will be close to the one that might be obtained from the exact Boltzmann equation.

The paper is organized as follows. In Sec. II we give a brief description of the ES model. In Sec. III we construct a consistent solution of the ES model for steady planar Couette flow. The main transport properties of the problem are evaluated in Sec. IV. In Sec. V, the velocity distribution function corresponding to the bulk of the system is explicitly obtained in terms of the nonequilibrium parameters of the problem. Finally, we close the paper in Sec. VI with some concluding remarks.

II. THE ELLIPSOIDAL STATISTICAL (ES) KINETIC MODEL

The general idea behind the formulation of a kinetic model is to replace the complicated Boltzmann collision operator $J[f,f]$ by a simpler expression but retaining the main physical properties of the former. The usual choice is a relaxation term of the form

$$J[f,f]=-\nu(f-f_0), \tag{6}$$

where $f_0(\mathbf{r},\mathbf{v};t)$ is a certain reference function whose explicit form must account for the conservation laws but it may be chosen in different ways. The influence of the interaction potential considered is included through an effective collision frequency ν, which is velocity independent but it can depend on the density and temperature. For instance, for $r^{-\ell}$ potentials, $\nu\propto nT^{\mu}$ where $\mu=(1/2)-(2/\ell)$. The simplest selection for f_0 corresponds to the model proposed by Bhatnagar, Gross, and Krook (BGK),[3] where f_0 is the local equilibrium distribution function. In spite of its simplicity, the BGK model has been shown to be reliable in several far from equilibrium states, such as the uniform shear flow[8] and the steady Fourier flow.[9] Nevertheless, it has some deficiencies like the prediction Pr=1 instead of the correct Prandtl number. This is particularly important in situations where combined heat and momentum transport occur, as it is the case of steady Couette flow. In order to have a correct Prandtl number, a different f_0 can be considered. In the ES model the choice is[1,7]

$$f_0(\mathbf{v})=n\pi^{-3/2}(\det\boldsymbol{\alpha})^{1/2}\exp(-\alpha_{ij}V_iV_j), \tag{7}$$

where $\boldsymbol{\alpha}=\boldsymbol{\lambda}^{-1}$ and $\lambda_{ij}=A\delta_{ij}-(B/\rho)P_{ij}$. Here, $\rho=mn$, m being the mass of a particle,

$$n=\int d\mathbf{v}f, \tag{8}$$

is the local number density, $\mathbf{V}=\mathbf{v}-\mathbf{u}$,

$$\mathbf{u}=\frac{1}{n}\int d\mathbf{v}\mathbf{v}f, \tag{9}$$

is the local flow velocity, $A=(2k_BT/m)\mathrm{Pr}^{-1}$, k_B being the Boltzmann constant,

$$T=\frac{m}{3nk_B}\int d\mathbf{v}V^2f, \tag{10}$$

is the local temperature, $B=2(\mathrm{Pr}^{-1}-1)$, and

$$P_{ij}=\int d\mathbf{v}mV_iV_jf, \tag{11}$$

is the pressure tensor. The Prandtl number Pr in this model can be seen as an adjustable parameter to get the same shear viscosity and thermal conductivity coefficients as those of the Boltzmann equation. If one sets Pr=1, f_0 reduces to a local Maxwellian which amounts to saying the BGK model is recovered in this limit. Notice that the reference function involves not only the conserved hydrodynamic fields as in the BGK approximation, but also the dissipative momentum flux. In a dilute gas, the other relevant dissipative flux is the heat flux $\mathbf{q}$ defined as

$$\mathbf{q}=\int d\mathbf{v}\frac{m}{2}V^2\mathbf{V}f. \tag{12}$$

In this model, it is straightforward to evaluate the transport coefficients η_0 and κ_0 appearing in Eqs. (3) and (4). The result is[7]

$$\eta_0=\frac{nk_BT}{\nu Pr^{-1}}, \tag{13}$$

$$\kappa_0=\frac{5}{2}\frac{nk_B^2T}{m\nu}. \tag{14}$$

If we identify ν with a given eigenvalue of the linearized Boltzmann collision operator and take Pr=2/3, then the expressions (13) and (14) coincide with those derived from the Boltzmann equation.[10] Strictly speaking, this identification is only exact for Maxwell molecules and for what is known as the first Enskog approximation to the transport coefficients of other interaction laws. Although the actual calculations will be carried out for arbitrary values of Pr, in the end we will only consider the cases of Pr=2/3 (ES results) and Pr=1 (BGK results).

III. CONSISTENT SOLUTION FOR THE HYDRODYNAMIC FIELDS

Let us consider a dilute gas enclosed between two parallel plates in relative motion and maintained at different temperatures. Let the x axis be parallel to the motion and the y axis be normal to the walls. We are interested in analyzing a steady state under arbitrary velocity and temperature gradients along the y direction. In this case, the ES model yields

$$v_y \frac{\partial}{\partial y} f = -\nu(f - f_0), \tag{15}$$

where f_0 is given by Eq. (7) with

$$\boldsymbol{\alpha} = \frac{\rho}{\Delta} \begin{pmatrix} A\rho - BP_{yy} & BP_{xy} & 0 \\ BP_{xy} & A\rho - BP_{xx} & 0 \\ 0 & 0 & \dfrac{\Delta}{A\rho - BP_{zz}} \end{pmatrix}. \tag{16}$$

Here, $\Delta = A^2\rho^2 - AB\rho(P_{xx}+P_{yy}) + B^2(P_{xx}P_{yy} - P_{xy}^2)$ and the structure of $\boldsymbol{\alpha}$ reflects the symmetry of the problem. Since our interest lies in obtaining the hydrodynamic profiles in the bulk of the system far away from the plates, rather than introducing the appropriate boundary conditions for Eq. (15), we first assume a given form for the profiles and later we verify their consistency. Thus, we expect to describe transport phenomena in the bulk region by looking for a consistent solution regardless of the actual characteristics of the boundaries. This idea has already been followed in previous works.

In the same way as in the BGK description,[4] we assume that Eq. (15) admits a solution consistent with the profiles

$$p = nk_B T = \text{const}, \tag{17}$$

$$\frac{\partial}{\partial s} u_x = a = \text{const}, \tag{18}$$

$$\frac{\partial^2}{\partial s^2} T = -\frac{2m}{k_B}\gamma(a) = \text{const}. \tag{19}$$

Here, we have introduced the scaled space variable s defined through the relation $ds = \nu(y)dy$. The variable s measures the distance in units of mean-free paths, and also it scales all the influence of the interaction potential. Therefore, in terms of s the results are independent of the law of interaction considered. Of course, one can rewrite the profiles as functions of y when one knows the dependence of ν on n and T. Further, a is the reduced shear rate and $\gamma(a)$ is a dimensionless function to be determined by consistency. This function measures the curvature of the temperature profile.

To verify the consistency of the assumed solution it is necessary to prove that

$$\int d\mathbf{v}\{1, \mathbf{V}, V^2\} f = \left\{ n, 0, \frac{3p}{m} \right\}. \tag{20}$$

In order to show that relations (20) are fulfilled, an explicit expression for f should be required. Such an expression will be provided in Sec. V. However, at this stage it is sufficient to use the formal series representation,

$$f = \left(1 + v_y \frac{\partial}{\partial s}\right)^{-1} f_0 = \sum_{k=0}^{\infty} (-v_y)^k \frac{\partial^k}{\partial s^k} f_0. \tag{21}$$

The fulfillment of the conditions for the density n and the velocity $\mathbf{V}$ readily follows after substitution of Eq. (21) in relations (20). On the other hand, in contrast with what happens in both the BGK and the Liu models, the condition for the temperature gives γ in terms of a and the nonzero elements of the pressure tensor whose shear-rate dependence is not known. Consequently, in order to close the problem, the evaluation of these nonzero elements is also needed. This leads, in principle (see Appendix A), to the four coupled nonlinear implicit Eqs. (A14), (A15), (A16), and (A18) for the set $\{\gamma, P_{xy}, P_{yy}, P_{zz}\}$ as functions of the shear rate a. Here, we have taken into account that $P_{xx} = 3p - P_{yy} - P_{zz}$. If we now set Pr=1, then $\gamma(a)$ obeys a closed equation (not involving the pressure tensor) and one recovers the BGK results.[4]

For Pr≠1, although the natural parameter is a, from a practical point of view it is convenient to take β [defined by Eq. (A13)] as an independent variable. This allows us to express explicitly all the unknowns as functions of β. In particular,

$$\gamma = \frac{p}{A\rho}(1 + B[1 - 2\beta(F_1 + 2F_2)])\beta, \tag{22}$$

and

$$a^2 = \frac{(2 - BC_1)^2}{4A\rho} \times \frac{A\rho(2C_3 - 3BC_1C_4) + 3p(2 - BC_1)(2 - BC_4)}{B^2(C_2F_0^2 + C_1^2F_1) - 2B(F_0^2 + 2C_1F_1) + 4F_1}\beta. \tag{23}$$

Here,

$$C_1(\beta) \equiv 2\beta(F_1 + 2F_2) - 1, \tag{24}$$

$$C_2(\beta) \equiv 2\beta(F_1 + F_2) - 1, \tag{25}$$

$$C_3(\beta) \equiv 2\beta(3F_1 + 2F_2) - 3, \tag{26}$$

$$C_4(\beta) \equiv 2\beta F_1 - 1, \tag{27}$$

and the functions $F_r \equiv F_r(\beta)$ are defined in Eq. (A11). In two instances, namely, for small and large shear rates it is possible to directly relate γ and a. In the former case,

$$\gamma(a) = \frac{\text{Pr}}{5} a^2 \left[1 - \frac{4}{25}(7\text{Pr}^2 - 35\text{Pr} + 10)a^2\right] + \cdots, \tag{28}$$

while in the latter case,

$$\gamma(a) \sim \tfrac{1}{3} a^2. \tag{29}$$

Notice that whereas for small shear rates the ES and BGK equations predict different behaviors for γ, the asymptotic value for the ratio γ/a^2 is the same for both models. This last result contrasts with the one found in the Liu model[5] where no physical solutions consistent with the profiles (17)–(19) exist for values of γ larger than a certain critical value.

IV. TRANSPORT PROPERTIES

The transport properties of the steady Couette flow are related to the momentum and heat fluxes. We begin with the nonzero elements of the pressure tensor. In terms of β, they are given by

$$P_{yy}=A\rho\frac{C_1(\beta)}{BC_1(\beta)-2}, \tag{30}$$

$$P_{zz}=A\rho\frac{C_4(\beta)}{BC_4(\beta)-2}, \tag{31}$$

$$P_{xy}=-2A\rho\frac{F_0}{(2-BC_1(\beta))^2}a. \tag{32}$$

Notice that the pressure tensor does not explicitly depend on the thermal gradient, as it also happens in the solution of the Boltzmann equation.[2] The xy element defines the nonlinear shear viscosity $\eta(a)=\eta_0F_\eta(a)$ with

$$F_\eta(a)=-\frac{\mathrm{Pr}}{pa}P_{xy}. \tag{33}$$

Normal stress effects are measured by the viscometric functions $\Psi_1(a)$ and $\Psi_2(a)$. In reduced form, they are defined as

$$\Psi_1(a)=\frac{P_{yy}-P_{xx}}{p\mathrm{Pr}^2a^2}, \tag{34}$$

$$\Psi_2(a)=\frac{P_{zz}-P_{yy}}{p\mathrm{Pr}^2a^2}. \tag{35}$$

The explicit expressions of F_η, Ψ_1, and Ψ_2 can be easily obtained by substituting Eqs. (30)–(32) into Eqs. (33)–(35).

Now, we turn to the heat flux vector. Although the temperature gradient is only directed along the y axis (so that there is a response in this direction), the presence of the shear flow induces a nonzero x component of the heat flux. These components, which are calculated in Appendix B, have the form

$$q_y=-\kappa_0\frac{\mathrm{Pr}}{5}\frac{a^2}{\gamma}F_\eta(a)\frac{\partial}{\partial y}T, \tag{36}$$

$$q_x=\frac{pk_B}{m\nu}\Phi(a)\frac{\partial}{\partial y}T, \tag{37}$$

where $\Phi(a)$ is given by Eq. (B31). From Eq. (36), which can be regarded as a generalization of Fourier's law, one may define a generalized thermal conductivity coefficient $\kappa(a)=\kappa_0F_\kappa(a)$ in which

$$F_\kappa(a)=\frac{\mathrm{Pr}}{5}\frac{a^2}{\gamma}F_\eta(a). \tag{38}$$

Equation (37) provides information about the anisotropy induced by the shear flow. It gives the transport of energy along the x axis due to a thermal gradient parallel to the y axis. This effect is absent in the Navier–Stokes regime and in fact, the corresponding transport coefficient, is a generalization of a Burnett coefficient. To the best of our knowledge, this is the first time that the shear-rate dependence of $\Phi(a)$ has been explicitly provided.

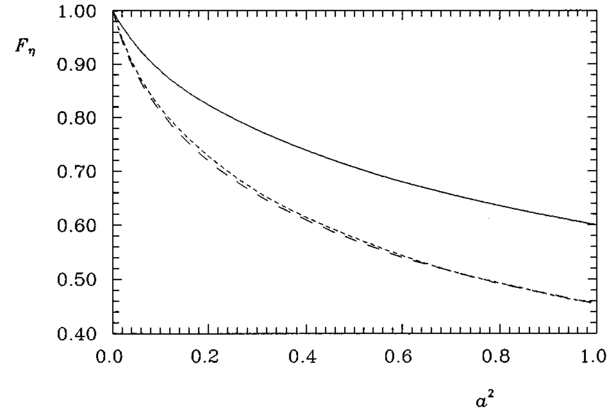

FIG. 1. Shear-rate dependence of the reduced shear viscosity $F_\eta(a)=\eta(a)/\eta_0$ for the ES model (——), the BGK model (– – –), and the Liu model (- - -).

Before we analyze the general shear-rate dependence of the transport coefficients, it is interesting to consider the small shear rate limit in order to make a comparison with the perturbative solution of the Boltzmann equation.[2] In this limit, one gets

$$F_\eta(a)=1-\tfrac{6}{5}\mathrm{Pr}(1+2\mathrm{Pr})a^2+\cdots, \tag{39}$$

$$\Psi_1(a)=-\tfrac{14}{5}+\tfrac{8}{125}(149\mathrm{Pr}^2+150\mathrm{Pr}+70)a^2+\cdots, \tag{40}$$

$$\Psi_2(a)=\tfrac{4}{5}-\tfrac{8}{125}(49\mathrm{Pr}^2+75\mathrm{Pr}+20)a^2+\cdots, \tag{41}$$

$$F_\kappa(a)=1-\tfrac{2}{25}(16\mathrm{Pr}^2+85\mathrm{Pr}-20)a^2+\cdots, \tag{42}$$

$$\Phi(a)=\tfrac{7}{2}(\mathrm{Pr}+1)a+\cdots. \tag{43}$$

In the lowest order (Burnett order), both viscometric functions coincide with the Boltzmann equation values irrespective of the choice of the Prandtl number. This is also true for the Liu model if an adequate choice of the ratio of the two collision frequencies introduced in the model is made.[11] On the other hand, the numerical coefficient corresponding to the super-Burnett contribution to F_η is different for the ES model (Pr=2/3), the BGK model (Pr=1), and the Liu model. Comparing with the Boltzmann result,[2] the relative error for the ES model is around 15%, while for the other two models is larger than one hundred per cent. Concerning the transport of heat, F_κ is again different for the three models. The comparison with the Boltzmann prediction[2] indicates that the relative error in this case is around 8% for the ES model, 100% for the BGK model, and 22% for the Liu model. As for $\Phi(a)$, the ES model gives the exact Boltzmann value while the relative error in the BGK model is around 20%. The above comparison clearly indicates the superiority of the ES model versus both the BGK and Liu models.

To gain some more insight into the performance of these kinetic models, in Figs. 1–5 we display F_η, F_κ, Ψ_1, Ψ_2, and $\Omega\equiv\Phi/((7/2)(\mathrm{Pr}+1)a)$ as functions of the shear rate. Except for Ω, we observe that the qualitative trends are similar for all three models, namely, the corresponding transport property decreases as the shear rate increases. According to

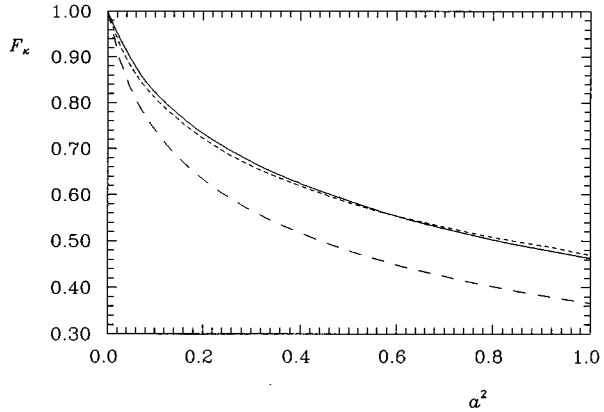

FIG. 2. Shear-rate dependence of the reduced thermal conductivity $F_\kappa(a) = \kappa(a)/\kappa_0$ for the ES model (—–), the BGK model (– – –), and the Liu model (- - -).

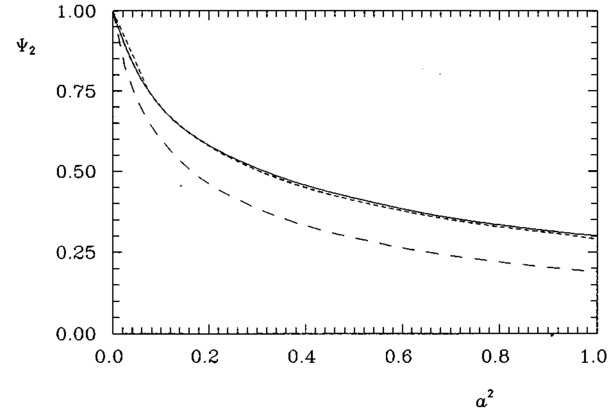

FIG. 4. Shear-rate dependence of the reduced second viscometric function $\Psi_2(a)$ for the ES model (—–), the BGK model (– – –), and the Liu model (- - -).

the small a results discussed above, one expects that the Boltzmann equation will also present the same behavior. Nevertheless, the quantitative discrepancies, especially between the ES and BGK results, become rather significant with increasing shear rate. In general, the BGK values are lower than those of the ES model and it is somewhat surprising that the Liu model is numerically very close to the ES model for the generalized thermal conductivity and the second viscometric function. The most notable difference between the ES and BGK models shows in the case of Ω. Not only there is a huge quantitative discrepancy, but for not too large shear rates while the ES approximation predicts an increase of the transport of energy along the x direction, the opposite trend is present in the BGK approximation.

V. VELOCITY DISTRIBUTION FUNCTION

It is evident that the general description of transport processes in a dilute gas requires the explicit knowledge of the velocity distribution function $f(\mathbf{r},\mathbf{v};t)$. Unfortunately, all the known exact solutions of the Boltzmann equation for inhomogeneous situations are given in terms of a finite number of moments. These moments constitute the only indirect information on the distribution function, but the latter cannot be explicitly determined. Perhaps, one of the main advantages of considering kinetic models is that the velocity distribution function may be obtained. In the cases of the uniform shear flow[8] and the Fourier flow,[9] the BGK solutions have been shown to present a good agreement with Monte Carlo numerical solutions of the Boltzmann equation, especially in the region of thermal velocities.[12] This fact suggests the reliability of kinetic models for describing the ''real'' distribution.

In this section our aim is to get an explicit solution of the stationary ES equation (15) incorporating the specified boundary conditions. Since we are interested in describing the bulk of the system (outside of the boundary layer), we will consider idealized boundary conditions under which the contribution to the distribution function arising from the boundary term vanishes. One possibility is to choose moving plates with zero temperature at the walls. These are conditions describing an infinite system (in the sense that the separation between the plates is infinite) corresponding to planar Couette flow between very cold walls. In this special case the boundary layers vanish as the Knudsen number vanishes at the walls. Consequently, neither velocity slip nor tempera-

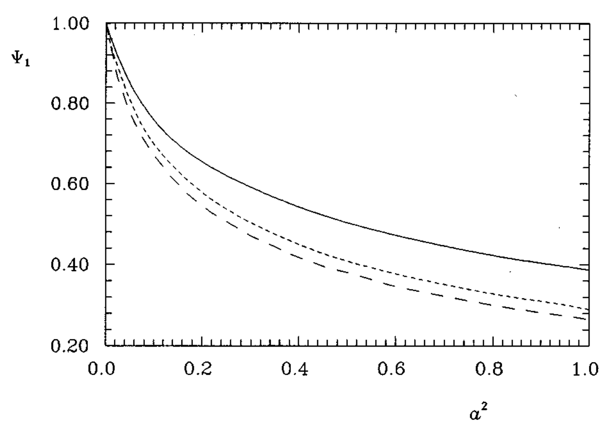

FIG. 3. Shear-rate dependence of the reduced first viscometric function $\Psi_1(a)$ for the ES model (—–), the BGK model (– – –), and the Liu model (- - -).

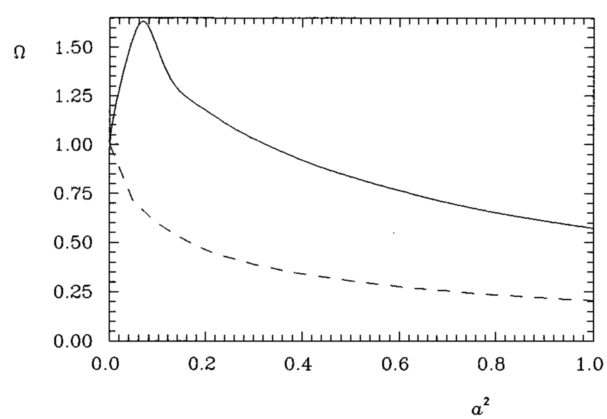

FIG. 5. Shear-rate dependence of the cross coefficient $\Omega(a) \equiv \Phi(a)/((7/2)(\mathrm{Pr}+1)a)$ for the ES model (—–), and the BGK model (– – –).

ture jumps at the walls can be predicted from this type of solution. Such a solution can be considered as a *normal* solution since the effect of the boundary conditions appears *only* indirectly through the explicit space dependence of the hydrodynamic fields. In the context of the BGK model, the appropriateness of such a solution has been thoroughly examined in Ref. 13. For the present kinetic model, the same type of arguments also apply.

In terms of the scaled variable s, the formal solution to Eq. (15) is

$$f(s,\mathbf{v})=\exp\left(-\frac{s-s_0}{v_y}\right)f(s_0,\mathbf{v})+\int_{s_0}^{s}ds'\times\exp\left(-\frac{s-s'}{v_y}\right)\frac{1}{v_y}f_0(s',\mathbf{v}), \quad (44)$$

where s_0 is an arbitrary point. According to the symmetry of the planar Couette flow, it is sufficient to get the half distribution $f_+(s,\mathbf{v})=\Theta(v_y)f(s,\mathbf{v})$, Θ being the Heaviside step function. As said before, we want to get rid of boundary effects which can be achieved by choosing s_0 such that $T(s_0)=0$ and imposing the boundary condition $f_+(s_0,\mathbf{v})=0$. After having found in Sec. III a consistent solution characterized by the profiles (17)–(19), Eq. (44) is no longer a formal expression. Substituting these hydrodynamic profiles into (44), and after some manipulations, one obtains the following expression for f_+:

$$f_+(s,\mathbf{v})=f^{\mathrm{LE}}(s,\mathbf{v})e^{\xi^2}\frac{2\zeta(1+\zeta)^{3/2}}{\epsilon\xi_y}(\det\widetilde{\alpha})^{1/2}\times\int_0^1 dt[2t-(1-\zeta)t^2]^{-5/2}\exp\left(-\frac{2\zeta}{1+\zeta}\frac{1-t}{\epsilon\xi_y}\right)\times\exp\left\{-\frac{1+\zeta}{2t-(1-\zeta)t^2}\left[\widetilde{\alpha}_{xx}\left(\xi_x+\frac{2a\zeta}{1+\zeta}\frac{1-t}{\epsilon}\right)^2+\widetilde{\alpha}_{yy}\xi_y^2+\widetilde{\alpha}_{zz}\xi_z^2+2\widetilde{\alpha}_{xy}\xi_y\left(\xi_x+\frac{2a\zeta}{1+\zeta}\frac{1-t}{\epsilon}\right)\right]\right\}. \quad (45)$$

Here, we have introduced the local equilibrium distribution function,

$$f^{\mathrm{LE}}=n\left(\frac{m}{2\pi k_BT}\right)^{3/2}\exp\left(-\frac{m}{2k_BT}V^2\right), \quad (46)$$

$\boldsymbol{\xi}\equiv(m/2k_BT(s))^{1/2}(\mathbf{v}-\mathbf{u}(s))$ is the velocity reduced with the thermal local velocity, and

$$\zeta=\frac{\epsilon}{(\epsilon^2+8\gamma)^{1/2}}, \quad (47)$$

where

$$\epsilon=\left(\frac{2k_B}{mT(s)}\right)^{1/2}\frac{\partial T}{\partial s}, \quad (48)$$

is the reduced local thermal gradient. Further, the dimensionless elements $\widetilde{\alpha}_{ij}$ have been defined in Appendix A.

The symmetry of the problem implies that the other half-distribution f_- can be obtained from Eq. (45) by changing ϵ to $-\epsilon$ and $\boldsymbol{\xi}$ to $-\boldsymbol{\xi}$:

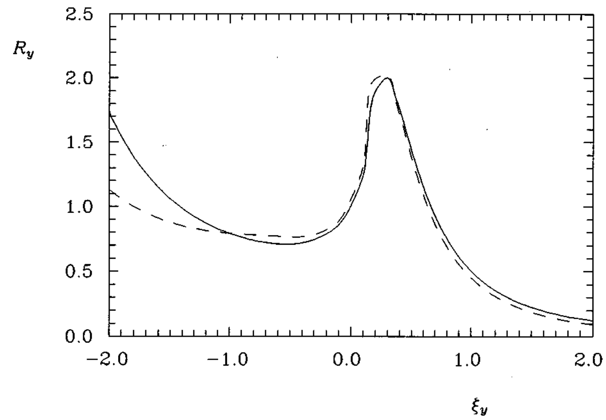

FIG. 6. Reduced distribution function $R_y(\xi_y)$ versus ξ_y for $a=1$ and $\epsilon=1$. The solid line refers to the ES model while the dashed line corresponds to the BGK model.

$$f_-(s,\mathbf{v})=f^{\mathrm{LE}}(s,\mathbf{v})e^{\xi^2}\frac{2\zeta(1+\zeta)^{3/2}}{\epsilon|\xi_y|}(\det\widetilde{\alpha})^{1/2}\times\int_1^{2/(1-\zeta)}dt[2t-(1-\zeta)t^2]^{-5/2}\exp\left(-\frac{2\zeta}{1+\zeta}\frac{1-t}{\epsilon\xi_y}\right)\times\exp\left\{-\frac{1+\zeta}{2t-(1-\zeta)t^2}\left[\widetilde{\alpha}_{xx}\left(\xi_x+\frac{2a\zeta}{1+\zeta}\frac{1-t}{\epsilon}\right)^2+\widetilde{\alpha}_{yy}\xi_y^2+\widetilde{\alpha}_{zz}\xi_z^2+2\widetilde{\alpha}_{xy}\xi_y\left(\xi_x+\frac{2a\zeta}{1+\zeta}\frac{1-t}{\epsilon}\right)\right]\right\}, \quad (49)$$

where the change of the dummy integration variable $t\rightarrow[2-(1+\zeta)t]/(1-\zeta)$ has been performed.

Equations (45) and (49) clearly show the complicated dependence of the distribution function on the two nonequilibrium parameters a and ϵ. Notice that the dependence on the shear rate appears through ζ and the nonzero elements of $\boldsymbol{\alpha}$ (which is related to the pressure tensor). In the BGK case (Pr=1), $\widetilde{\alpha}_{xx}=\widetilde{\alpha}_{yy}=\widetilde{\alpha}_{zz}=1$ and $\widetilde{\alpha}_{xy}=0$, and one recovers previous results.[13] In order to investigate the distortion of the distribution function with respect to the local equilibrium, it is convenient to define the marginal distributions,

$$R_y(\xi_y)\equiv\frac{\int_{-\infty}^{\infty}d\xi_x\int_{-\infty}^{\infty}d\xi_z f}{\int_{-\infty}^{\infty}d\xi_x\int_{-\infty}^{\infty}d\xi_z f^{\mathrm{LE}}} \quad (50)$$

and

$$R_x(\xi_x)\equiv\frac{\int_{-\infty}^{\infty}d\xi_y\int_{-\infty}^{\infty}d\xi_z f}{\int_{-\infty}^{\infty}d\xi_y\int_{-\infty}^{\infty}d\xi_z f^{\mathrm{LE}}}. \quad (51)$$

In Figs. 6 and 7 we plot R_y and R_x, respectively, for $a=\epsilon=1$ and for the ES model (Pr=2/3) and the BGK model (Pr=1). Since the system is far from equilibrium, the distortion from local equilibrium ($R_y=R_x=1$) as well as the asymmetry are quite evident. Although the dependence of f on a and ϵ in both models is very different, the shape of the marginal distributions is rather similar at least in the range of thermal velocities. This is consistent with the general shear-rate dependence of the main transport properties. Neverthe-

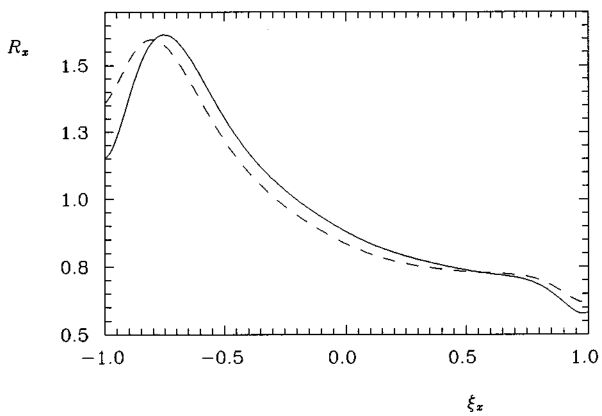

FIG. 7. The same as in Fig. 6, but for $R_x(\xi_x)$ as a function of ξ_x.

less, there exist significant discrepancies especially for large and negative ξ_y and/or ξ_x. This may probably explain the differences observed for q_x where large velocity tails play an important role.

VI. DISCUSSION

In this paper we have examined a steady nonequilibrium problem involving combined heat and momentum transport. The system consists of a dilute gas enclosed between two relatively moving parallel plates which are kept in general at different temperatures. An appealing aspect of this state, which is known as planar Couette flow, is that it is generated by means of realistic boundary conditions and hence, it can be in principle implemented experimentally. Nevertheless, because of the way the system is sheared, the density, flow velocity, and temperature are nonhomogeneous, thus making the analysis of this problem more complicated than say the case of uniform shear flow.[8]

In planar Couette flow, there are two parameters measuring the departure of the system from equilibrium: the (reduced) shear rate a and the (reduced) thermal gradient ϵ. The main motivation of our study has been to obtain the transport properties, which are related to the pressure tensor and the heat flux vector, for arbitrary values of a and ϵ. Of course, the natural framework to pursue such a study is the exact Boltzmann equation. However, due to the complexity of the problem, so far it has only been possible to get a perturbative solution[2] up to super-Burnett order. Consequently, the other alternative if one wants to derive analytical results is to use simple kinetic models. This had already been done for planar Couette flow with the BGK[4] and the Liu[5] models. The results of these two models present drawbacks: in the BGK model the Prandtl number (which plays a key role in this situation) is not given correctly, while in the Liu case, which does not share this difficulty, unphysical predictions (such as negative values for the distribution function) restrict its range of validity. In view of these limitations, it seems natural to consider another kinetic equation that avoids such drawbacks. A good candidate is the so-called ellipsoidal statistical (ES) model,[1,7] that can be seen as an extension of the BGK approximation since it reduces to this latter if one sets the Prandtl number (which explicitly enters as an adjustable parameter in the collision term of the ES approximation) Pr=1.

We have found a consistent solution of the ES model characterized by a uniform pressure, and linear velocity and quadratic temperature profiles with respect to a given scaled variable. The scaling allows us to have a *universal* description, namely the results are independent of the interaction potential considered. The ES solution is a close relative to the ones derived earlier for the BGK and Liu models, and without this previous insight the formulation of the problem with the ES model would have been an almost insuperable task. Nevertheless, it must be pointed out that the presence of the pressure tensor in the collision term gives rise in principle to four coupled transcendental equations that cater for the consistency of the sought solution. Concerning the transport properties, we have determined the generalized shear viscosity $F_\eta(a)$, the generalized thermal conductivity $F_\kappa(a)$, the viscometric functions $\Psi_{1,2}(a)$, and a generalized cross coefficient $\Phi(a)$ measuring the transport of energy along the direction normal to the thermal gradient. This last coefficient had not been calculated previously. All coefficients exhibit a monotonous decrease with the shear rate, although $\Phi(a)$ only does so after attaining a maximum. The comparison of the ES model results with those obtained from the Boltzmann equation in the limit of small shear rates[2] indicates that there is a better agreement than using either the BGK or Liu predictions. This fact suggests that, for finite values of the shear rate, the exact Boltzmann results will not be too far from the ES results. In order to test the value of this conjecture, it would be very interesting to perform computer simulations, a task that we are currently considering. It should be pointed out that in the two limiting cases of pure uniform shear flow[14] and pure planar Fourier flow,[15] where computer simulations are indeed already available,[12] the forms of the relevant transport properties allow us to conclude that the agreement found when comparing the BGK model and the Monte Carlo results will also hold for the ES model.

Apart from evaluating the transport properties, we have also explicitly obtained the velocity distribution function. This quantity provides all the complete information on the nonequilibrium state of the system. Our solution applies only in the bulk of the system, i.e., far away from the boundaries. In the same way as in the BGK model,[13] in order to get such a solution we have considered idealized boundary conditions of zero wall temperatures. The velocity distribution function displays a strong dependence on the nonequilibrium parameters and so a great distortion from local equilibrium is observed. Once again, the availability of simulation results for the distribution function of the Boltzmann equation would be welcome to assess the reliability of our result.

It could be argued that since planar Couette flow involves realistic boundary conditions, a comparison of our results with actual experiments would be in order. While this would of course be highly desirable, the following argument suffices to show that the goal is so far unreachable at least for a low-density gas. The main goal of this work has been to consider non-Newtonian effects as given for instance by the reduced shear viscosity $F_{\eta}(a)$. In order to compute the actual velocity gradient, an expression for the collision frequency ν can be obtained from the Navier–Stokes shear viscosity η_0 through $\nu=(Prp)/\eta_0$. Consequently, $\partial u_x/\partial y=a\nu=(pPra)/\eta_0$. Now, from Fig. 1 it follows that a 10% decrease in the nonlinear shear viscosity with respect to its Navier–Stokes value occurs for $a\simeq 0.316$. If we consider argon at $p=1$ atm and $T=273$ K, the experimental value of η_0 (Ref. 10) is $\eta_0\simeq 2117\times 10^{-7}$ g cm^{-1} s^{-1}, so that for such a change to be noticeable (about 10%) one needs $\partial u_x/\partial y\simeq 3.29\times 10^{9}$ s^{-1}. Clearly such a high shear rate is presently unattainable in the laboratory and this fact precludes a direct comparison with experimental results.[16] In this sense, the derivation of explicit expressions for the transport properties in far from equilibrium states may prove to be useful for analyzing computer simulation results.

As a final point, we want to remark that in spite of the apparent simplicity of kinetic models, their relevance in dealing with nonequilibrium problems cannot be overlooked. In fact, they have been shown in the past to be very reliable for evaluating nonlinear transport properties in several problems.[8,9] In this context, our results may be regarded as a further confirmation of the usefulness of the ES model.

ACKNOWLEDGMENTS

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APPENDIX A: CONSISTENCY OF THE SOLUTION

In this appendix we show that the kinetic equation (15) admits a consistent solution given by the profiles (17)–(19). For that, it is necessary to prove Eq. (20), namely, f reproduces the five conserved hydrodynamic moments. In order to compute the above relations, it is convenient to rewrite the exponential term $\exp(-\alpha_{ij}V_iV_j)$ appearing in f_0 in the form

$$\exp(-\alpha_{ij}V_iV_j)=\exp\left(cV_y\frac{\partial}{\partial V_x}\right)\exp\left(-\frac{m}{2k_BT}\widetilde{b}_iV_i^2\right), \tag{A1}$$

where $c=\widetilde{\alpha}_{xy}/\widetilde{\alpha}_{xx}$, $\widetilde{b}_x=\widetilde{\alpha}_{xx}$, $\widetilde{b}_y=\widetilde{\alpha}_{yy}-(\widetilde{\alpha}_{xy}^2/\widetilde{\alpha}_{xx})$, and $\widetilde{b}_z=\widetilde{\alpha}_{zz}$. Here, $\widetilde{\alpha}_{ij}=(2k_BT/m)\alpha_{ij}$ and use has been made of the identity

$$\exp\left(cV_y\frac{\partial}{\partial V_x}\right)\Phi(V_x,V_y,V_z)=\Phi(V_x+cV_y,V_y,V_z). \tag{A2}$$

The elements $\widetilde{\alpha}_{ij}$ as well as $\widetilde{b}_i$ do not depend on s since the pressure tensor is uniform in the steady planar Couette flow. Further, it is easy to show that $\det\widetilde{\boldsymbol{\alpha}}=\widetilde{b}_x\widetilde{b}_y\widetilde{b}_z$ and in the following, we will make use of the general property

$$\int d\mathbf{V}G(\mathbf{V})e^{cV_y\partial/\partial V_x}F(\mathbf{V})=\int d\mathbf{V}F(\mathbf{V})e^{-cV_y\partial/\partial V_x}G(\mathbf{V}). \tag{A3}$$

Now, we are in conditions to verify (20). Let us start with the density. Taking into account the series representation for f, one gets

$$\begin{aligned}\int d\mathbf{v}f&=\pi^{-3/2}\sum_{k=0}^{\infty}(-\partial_s)^k n(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}V_y^k e^{cV_y\partial/\partial V_x}\\&\quad\times\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]\\&=\pi^{-3/2}\sum_{k=0}^{\infty}\partial_s^{2k}n(\det\boldsymbol{\alpha})^{1/2}\\&\quad\times\int d\mathbf{v}V_y^{2k}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]\\&=n+\frac{p}{k_B}\sum_{k=1}^{\infty}(2k-1)!!\left(\frac{k_B}{m}\right)^k\widetilde{b}_y^{-k}\partial_s^{2k}T^{k-1}=n,\end{aligned} \tag{A4}$$

since according to the s dependence of the temperature, for $k\geqslant 1$, $\partial_s^{2k}T^{k-1}=0$. This automatically implies the verification of the first consistency condition. The condition for the x component of the velocity is

$$\begin{aligned}\int d\mathbf{v}V_xf&=\pi^{-3/2}\sum_{k=0}^{\infty}\int d\mathbf{v}V_x(-\partial_s)^k n(\det\boldsymbol{\alpha})^{1/2}V_y^k e^{cV_y\partial/\partial V_x}\\&\quad\times\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]\\&=\pi^{-3/2}\sum_{k=0}^{\infty}\partial_s^k n(\det\boldsymbol{\alpha})^{1/2}\\&\quad\times\int d\mathbf{v}V_x(-V_y)^k\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]\\&\quad+\pi^{-3/2}a\\&\quad\times\sum_{k=0}^{\infty}(k+1)\partial_s^k n(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}(-V_y)^{k+1}\\&\quad\times\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]\\&=\frac{p}{m}\sum_{k=0}^{\infty}(2k+1)!!\widetilde{b}_y^{-(k+1)}[c-(2k+1)a]\\&\quad\times\partial_s^{2k+1}\left(\frac{k_BT}{m}\right)^{k+1}=0,\end{aligned} \tag{A5}$$

where use has been made of the operator identity

$$\partial_s^kV_x=V_x\partial_s^k-ak\partial_s^{k-1}. \tag{A6}$$

The consistency conditions for the y and z components of $\mathbf{V}$ can be similarly verified.

The fact that the reference function f_0 is given in terms of the pressure tensor (whose shear-rate dependence is not known) implies that the condition for the temperature does not lead to a closed equation for γ as a function of a. It is thus necessary to compute the nonzero elements of the pressure tensor. The xx element is

$$P_{xx}=m\int d\mathbf{v}V_x^2f=m\pi^{-3/2}\sum_{k=0}^{\infty}\int d\mathbf{v}V_x^2(-\partial_s)^kn(\det\boldsymbol{\alpha})^{1/2}V_y^ke^{cV_y\partial/\partial V_x}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]$$

$$=m\pi^{-3/2}\sum_{k=0}^{\infty}\partial_s^kn(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}V_x^2(-V_y)^ke^{cV_y\partial/\partial V_x}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]+2am\pi^{-3/2}\sum_{k=1}^{\infty}k\partial_s^{k-1}n(\det\boldsymbol{\alpha})^{1/2}$$

$$\times\int d\mathbf{v}V_x(-V_y)^ke^{cV_y\partial/\partial V_x}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]+a^2m\pi^{-3/2}\sum_{k=2}^{\infty}k(k-1)\partial_s^{k-2}n(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}(-V_y)^ke^{cV_y\partial/\partial V_x}$$

$$\times\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]. \quad \text{(A7)}$$

Here, since V_x^2 is a quadratic function of the variable s, we have used the identity

$$\partial_s^kV_x^2=V_x^2\partial_s^k-2ak\partial_s^{k-1}V_x-a^2k(k-1)\partial_s^{k-2}. \quad \text{(A8)}$$

Taking into account the property (A3), one gets

$$P_{xx}=m\pi^{-3/2}\sum_{k=0}^{\infty}\partial_s^kn(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}(V_x-cV_y)^2(-V_y)^k\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]+2ma\pi^{-3/2}$$

$$\times\sum_{k=0}^{\infty}(k+1)\partial_s^kn(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}(V_x-cV_y)(-V_y)^{k+1}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]+ma^2\pi^{-3/2}$$

$$\times\sum_{k=0}^{\infty}(k+2)(k+1)\partial_s^kn(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}(-V_y)^{k+2}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]$$

$$=p(\widetilde{b}_x^{-1}+c^2\widetilde{b}_y^{-1})+p\sum_{k=1}^{\infty}(2k-1)!!\widetilde{b}_y^{-k}[\widetilde{b}_x^{-1}+(2k+1)c^2\widetilde{b}_y^{-1}]\partial_s^{2k}\left(\frac{k_BT}{m}\right)^k+2pa$$

$$\times\sum_{k=0}^{\infty}(2k+1)(2k+1)!![c+(k+1)a]\partial_s^{2k}\left(\frac{k_BT}{m}\right)^k$$

$$=p(\widetilde{b}_x^{-1}+c^2\widetilde{b}_y^{-1})+p\sum_{k=1}^{\infty}(2k)!(2k-1)!![\widetilde{b}_x^{-1}+(2k+1)c^2\widetilde{b}_y^{-1}](-\gamma/\widetilde{b}_y)^k+2pa\widetilde{b}_y^{-1}\sum_{k=0}^{\infty}(2k+1)!(2k+1)!!$$

$$\times[c+(k+1)a]\left(\frac{-\gamma}{\widetilde{b}_y}\right)^k. \quad \text{(A9)}$$

This equation may be represented in terms of the functions $F_r(x)\equiv[(d/dx)x]^rF_0(x)$, where

$$F_0(x)=\frac{2}{x}\int dt\ t\exp\left(\frac{-t^2}{2}\right)K_0(2x^{-1/4}t^{1/2}), \quad \text{(A10)}$$

K_0 being the zeroth-order modified Bessel function. The asymptotic series of F_r is[4]

$$F_r(x)=\sum_{k=0}^{\infty}(k+1)^r(2k+1)!(2k+1)!!(-x)^k. \quad \text{(A11)}$$

Consequently, P_{xx} can be cast in the form

$$P_{xx}=p\widetilde{b}_x^{-1}[1-2\beta F_1(\beta)]+pc^2\widetilde{b}_y^{-1}\{1-2\beta[F_1(\beta)$$
$$+2F_2(\beta)]\}+2pa\widetilde{b}_y^{-1}[cF_0(\beta)+aF_1(\beta)], \quad \text{(A12)}$$

where we have introduced the auxiliary variable

$$\beta=\frac{\gamma}{\widetilde{b}_y}. \quad \text{(A13)}$$

Similarly, the yy element of the pressure tensor is

$$P_{yy}=m\int d\mathbf{v}V_y^2 f$$

$$=m\pi^{-3/2}\sum_{k=0}^{\infty}(-\partial_s)^k n(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}V_y^{k+2}e^{cV_y\partial/\partial V_x}$$

$$\times\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]$$

$$=p\sum_{k=0}^{\infty}(2k+1)!!\widetilde{b}_y^{-(k+1)}\partial_s^{2k}(k_BT/m)^k$$

$$=p\widetilde{b}_y^{-1}\sum_{k=0}^{\infty}(2k)!(2k+1)!!(-\beta)^k$$

$$=p\widetilde{b}_y^{-1}\{1-2\beta[F_1(\beta)+2F_2(\beta)]\}. \qquad (A14)$$

The zz element of the pressure tensor is

$$P_{zz}=m\int d\mathbf{v}V_z^2 f$$

$$=m\pi^{-3/2}\sum_{k=0}^{\infty}(-\partial_s)^k n(\det\boldsymbol{\alpha})^{1/2}$$

$$\times\int d\mathbf{v}V_z^2V_y^k e^{cV_y\partial/\partial V_x}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]$$

$$=p\widetilde{b}_z^{-1}+p\widetilde{b}_z^{-1}\sum_{k=1}^{\infty}(2k-1)!!\widetilde{b}_y^{-k}\partial_s^{2k}\left(\frac{k_BT}{m}\right)^k$$

$$=p\widetilde{b}_z^{-1}+p\widetilde{b}_z^{-1}\sum_{k=1}^{\infty}(2k)!(2k-1)!!(-\beta)^k$$

$$=p\widetilde{b}_z^{-1}[1-2\beta F_1(\beta)]. \qquad (A15)$$

The consistency condition for the temperature follows from the fact that $P_{xx}+P_{yy}+P_{zz}=3p$. Use of Eqs. (A12), (A14), and (A15) leads to

$$\widetilde{b}_x^{-1}+(1+c^2)\widetilde{b}_y^{-1}+\widetilde{b}_z^{-1}-3+2a\widetilde{b}_y^{-1}[cF_0(\beta)+aF_1(\beta)]$$
$$=2\beta\{\widetilde{b}_x^{-1}F_1(\beta)+(1+c^2)\widetilde{b}_y^{-1}[F_1(\beta)+2F_2(\beta)]$$
$$+\widetilde{b}_z^{-1}F_1(\beta). \qquad (A16)$$

The only case in which Eq. (A16) provides a direct relationship (not involving the pressure tensor) between γ and a is $Pr=1$ (BGK approximation). In this case, $\widetilde{b}_i=1$ and $c=0$, so that

$$a^2=\gamma\frac{3F_1(\gamma)+2F_2(\gamma)}{F_1(\gamma)}. \qquad (A17)$$

Finally, the xy element of the pressure tensor is also needed. It is given by

$$P_{xy}=m\int d\mathbf{v}V_xV_yf=m\pi^{-3/2}\sum_{k=0}^{\infty}\int d\mathbf{v}V_x(-\partial_s)^kn(\det\boldsymbol{\alpha})^{1/2}V_y^{k+1}e^{cV_y\partial/\partial V_x}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]$$

$$=m\pi^{-3/2}\sum_{k=0}^{\infty}(-\partial_s)^kn(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}V_xV_y^{k+1}e^{cV_y\partial/\partial V_x}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]-ma\pi^{-3/2}$$

$$\times\sum_{k=0}^{\infty}(k+1)(-\partial_s)^kn(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}V_y^{k+2}e^{cV_y\partial/\partial V_x}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]$$

$$=m\pi^{-3/2}\sum_{k=0}^{\infty}(-\partial_s)^kn(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}(V_x-cV_y)V_y^{k+1}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]-ma\pi^{-3/2}$$

$$\times\sum_{k=0}^{\infty}(k+1)(-\partial_s)^kn(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}V_y^{k+2}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_iV_i^2\right]$$

$$=-p\sum_{k=0}^{\infty}(2k)!(2k+1)!!\widetilde{b}_y^{-1}[c+(2k+1)a](-\beta)^k=-pc\widetilde{b}_y^{-1}\{1-2\beta[F_1(\beta)+2F_2(\beta)]\}-pa\widetilde{b}_y^{-1}F_0(\beta). \qquad (A18)$$

From a mathematical point of view, the problem is now well posed in the sense that we have four (independent) coupled equations, for instance Eqs. (A14), (A15), (A16), and (A18) for the unknown set $\{\gamma,P_{xy},P_{yy},P_{zz}\}$ as functions of the shear rate a. These equations guarantee that our solution is self-consistent. As it stands, the solution to this system is very complicated *a priori*, since the coefficients $\widetilde{b}_i$ depend on the pressure tensor and this dependence involves the functions F_r. Nevertheless when, instead of a, we take β as independent variable it is straightforward to explicitly obtain γ, a^2, and the elements of the pressure tensor as functions of β [cf. Eqs. (22), (23), (30), (31), and (32)].

APPENDIX B: CALCULATION OF THE HEAT FLUX

In this appendix, we evaluate the two nonzero components of the heat flux vector, namely, q_x and q_y. In the latter case, rather than a direct computation, we will profit from the fact that the consistency of the solution necessarily implies that[4]

$$q_y = -\kappa_0 F_\kappa(a) \frac{\partial T}{\partial s}, \tag{B1}$$

where κ_0 is given by Eq. (14). By substituting Eq. (B1) into the energy balance equation (2), one gets the result quoted in Eq. (36).

On the other hand, the computation of q_x is rather more involved. This component is defined as

$$q_x = \frac{m}{2}\int d\mathbf{v} V^2 V_x f = \frac{m}{2}(I_x + I_y + I_z), \tag{B2}$$

where

$$I_{x,y,z} \equiv \int d\mathbf{v} V_{x,y,z}^2 V_x f. \tag{B3}$$

We will start with I_x. This contribution is given by

$$\begin{aligned} I_x = {} & \pi^{-3/2}\sum_{k=0}^{\infty} \partial_s^k n(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v} V_x^3(-V_y)^k e^{cV_y\partial/\partial V_x}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_i V_i^2\right] + 3a\pi^{-3/2}\sum_{k=1}^{\infty} k\partial_s^{k-1} n(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v} V_x^2 \\ & \times(-V_y)^k e^{cV_y\partial/\partial V_x}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_i V_i^2\right] + 3a^2\pi^{-3/2}\sum_{k=2}^{\infty} k(k-1)\partial_s^{k-2} n(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v} V_x(-V_y)^k e^{cV_y\partial/\partial V_x} \\ & \times\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_i V_i^2\right] + a^3\pi^{-3/2}\sum_{k=2}^{\infty} k(k-1)(k-2)\partial_s^{k-3} n(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}(-V_y)^k e^{cV_y\partial/\partial V_x}\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_i V_i^2\right] \\ & \equiv I_x^{(1)} + I_x^{(2)} + I_x^{(3)} + I_x^{(4)}, \end{aligned} \tag{B4}$$

where the operator identity,

$$\begin{aligned} \partial_s^k V_x^3 = {} & V_x^3\partial_s^k - 3ak\partial_s^{k-1}V_x^2 - 3a^2k(k-1)\partial_s^{k-2}V_x \\ & - a^3k(k-1)(k-2)\partial_s^{k-3}, \end{aligned} \tag{B5}$$

has been used. The term $I_x^{(1)}$ is

$$\begin{aligned} I_x^{(1)} = {} & \pi^{-3/2}\sum_{k=0}^{\infty} \partial_s^k n(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}(V_x - cV_y)^3 \\ & \times(-V_y)^k\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_i V_i^2\right] \\ = {} & -\pi^{-3/2}\sum_{k=0}^{\infty} \partial_s^k n(\det\boldsymbol{\alpha})^{1/2}\int d\mathbf{v}(3cV_yV_x^2 + c^3V_y^3) \\ & \times(-V_y)^k\exp\left[-\left(\frac{2k_BT}{m}\right)\widetilde{b}_i V_i^2\right] \\ = {} & \frac{k_Bp}{m^2}\sum_{k=0}^{\infty}(k+1)(2k+1)!(2k+1)!! \\ & \times\widetilde{b}_y^{-1}[3c\widetilde{b}_x^{-1} + (2k+3)c^3\widetilde{b}_y^{-1}](-\beta)^k\partial_s T \\ = {} & \frac{k_Bp}{m^2}\widetilde{b}_y^{-1}\{3c\widetilde{b}_x^{-1}F_1(\beta) + 2c^3\widetilde{b}_y^{-1}[F_2(\beta) \\ & + \tfrac{1}{2}F_1(\beta)]\partial_s T. \end{aligned} \tag{B6}$$

In deriving the last line, we have accounted for the s dependence of the temperature so that

$$\partial_s^{2k+1}\left(\frac{k_BT}{m}\right)^{k+1} = \frac{k_B}{m}\frac{(2k+2)!}{2}(-\gamma)^k\partial_s T. \tag{B7}$$

The other contributions to I_x are calculated in a similar way. The results are

$$\begin{aligned} I_x^{(2)} = {} & 6\frac{k_Bp}{m^2}a\widetilde{b}_y^{-1}\{\widetilde{b}_x^{-1}F_2(\beta) + c^2\widetilde{b}_y^{-1}[F_3(\beta) \\ & + \tfrac{1}{2}F_2(\beta)]\}\partial_s T, \end{aligned} \tag{B8}$$

$$I_x^{(3)} = 6\frac{k_Bp}{m^2}a^2c\widetilde{b}_y^{-2}[4F_4(\beta) + 4F_3(\beta) + F_2(\beta)]\partial_s T, \tag{B9}$$

and

$$\begin{aligned} I_x^{(4)} = {} & 4\frac{k_Bp}{m^2}a^3\widetilde{b}_y^{-2}[4F_5(\beta) + 8F_4(\beta) + 5F_3(\beta) \\ & + F_2(\beta)]\partial_s T. \end{aligned} \tag{B10}$$

The second contribution I_y in Eq. (B20) is given by

$$
\begin{aligned}
I_y &= \int d\mathbf{v} V_x V_y^2 f \\
&= \pi^{-3/2} \sum_{k=0}^{\infty} \int d\mathbf{v} V_x V_y^2 \partial_s^k n (\det \boldsymbol{\alpha})^{1/2} \\
&\quad \times (-V_y)^k \exp\left[-\left(\frac{2k_BT}{m} \right) \widetilde{b}_i V_i^2 \right] \\
&= \pi^{-3/2} \sum_{k=0}^{\infty} \partial_s^k n (\det \boldsymbol{\alpha})^{1/2} \int d\mathbf{v} V_x V_y^2 (-V_y)^k \\
&\quad \times \exp\left[-\left(\frac{2k_BT}{m} \right) \widetilde{b}_i V_i^2 \right] + a\pi^{-3/2} \sum_{k=0}^{\infty} (k+1) \\
&\quad \times \partial_s^k n (\det \boldsymbol{\alpha})^{1/2} \int d\mathbf{v} V_y^2 (-V_y)^{k+1} \\
&\quad \times \exp\left[-\left(\frac{2k_BT}{m} \right) \widetilde{b}_i V_i^2 \right] \\
&= \frac{k_B p}{m^2} \sum_{k=0}^{\infty} (k+1)(2k+1)! \\
&\quad \times (2k+3)!! \widetilde{b}_y^{-2} [c + 2(k+1)a](-\beta)^k \partial_s T \\
&= 2\frac{k_B p}{m^2} \widetilde{b}_y^{-2} \{ c[F_2(\beta) + \tfrac{1}{2} F_1(\beta)] + 2a[F_3(\beta) \\
&\quad + \tfrac{1}{2} F_2(\beta)\} \partial_s T. \qquad \text{(B11)}
\end{aligned}
$$

The term I_z can be similarly evaluated. The result is

$$
I_z = \frac{k_B p}{m^2} \widetilde{b}_y^{-1} \widetilde{b}_z^{-1} [cF_1(\beta) + 2aF_2(\beta)] \partial_s T. \qquad \text{(B12)}
$$

Substitution of Eqs. (B6), (B8), (B9), (B10), (B11), and (B12) into Eq. (B2) yields the corresponding expression for q_x, which can be written in the form (37) where the function $\Phi(a)$ is given by

$$
\begin{aligned}
\Phi(a) = \tfrac{1}{2} \widetilde{b}_y^{-1} \{ & 4a^3 \widetilde{b}_y^{-1} (F_2 + 5F_3 + 8F_4 + 4F_5) \\
& + 6a^2 c \widetilde{b}_y^{-1} (F_2 + 4F_3 + 4F_4) + a[6\widetilde{b}_x^{-1} F_2 \\
& + \widetilde{b}_y^{-1} (F_2 + 2F_3)(2 + 3c^2) + 2\widetilde{b}_z^{-1} F_2] \\
& + c[3\widetilde{b}_x^{-1} F_1 + \widetilde{b}_y^{-1} (F_1 + 2F_2)(1 + c^2) \\
& + \widetilde{b}_z^{-1} F_1], \qquad \text{(B13)}
\end{aligned}
$$

where for simplicity we have omitted the explicit dependence on β of the functions F_r. In general, Φ exhibits a complicated dependence on the shear rate. A simple case corresponds to the BGK approximation (Pr=1), where

$$
\Phi(a) = [5F_2 + 2F_3 + 2a^2(F_2 + 5F_3 + 8F_4 + 4F_5)]a. \qquad \text{(B14)}
$$

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[14] In the uniform shear flow state, the system is sheared by applying Lees–Edwards periodic boundary conditions so that the density and temperature are homogeneous. As a consequence, the rheological properties derived in this problem are different to those obtained in the planar Couette state [see for instance, A. Santos, V. Garzó, and J. J. Brey, ''Comparison between the homogeneous-shear and the sliding-boundary methods to produce shear flow,'' Phys. Rev. A **46**, 8018 (1992)]. Anyway, recently one of the authors has proved that the ES and BGK models lead to the same transport coefficients in the uniform shear flow problem with a suitable scaling of the collision frequency (V. Garzó, unpublished).

[15] The case of pure Fourier flow (both walls at rest and no flow velocity exists) is obtained in the limit $a \to 0$ (which implies $\gamma \to 0$) with $\epsilon > 0$ fixed. In this case $P_{ij} = p\delta_{ij}$ so that $\lambda_{ij} = (2k_BT/m)\delta_{ij}$, and consequently the ES model reduces to the BGK model.

[16] It is worth pointing out that when one does not consider atomic particles but mesoscopic ones, such as colloids or micelles, the shear rates required to observe non-Newtonian effects can be attainable in laboratory conditions.