Mass transport of an impurity in a strongly sheared granular gas

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Abstract. Transport coefficients associated with the mass flux of an impurity immersed in a granular gas under simple shear flow are determined from the inelastic Boltzmann equation. A normal solution is obtained via a Chapman–Enskog-like expansion around a local shear flow distribution that retains all the hydrodynamic orders in the shear rate. Due to the anisotropy induced by the shear flow, tensorial quantities are required to describe the diffusion process instead of the conventional scalar coefficients. The mass flux is determined to first order in the deviations of the hydrodynamic fields from their values in the reference state. The corresponding transport coefficients are given in terms of the solutions of a set of coupled linear integral equations, which are approximately solved by considering the leading terms in a Sonine polynomial expansion. The results show that the deviation of these generalized coefficients from their elastic forms is in general quite important, even for moderate dissipation.

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Contents

1. Introduction

The study of transport properties of an impurity or intruder immersed in a granular gas described by the inelastic Boltzmann equation is quite an interesting problem. In particular, when the gas is in homogeneous cooling state (HCS), the mass flux $\mathbf{j}_0$ for the impurity can be obtained by solving the Boltzmann equation by means of the Chapman–Enskog expansion [1] around the local version of the HCS. In the first order of the expansion, the mass flux $\mathbf{j}_0$ is linear in the gradients of mole fraction, pressure and temperature and in the external applied force. The corresponding transport coefficients are the diffusion coefficient D, the pressure diffusion coefficient D_p, the thermal diffusion coefficient D_T and the mobility χ. As in the elastic case, these coefficients are the solutions of a set of coupled integral equations [2,3] which can be approximately solved by considering the leading terms in a Sonine polynomial expansion. Explicit expressions for these transport coefficients have been obtained [2,3] in terms of the coefficients of restitution and the parameters of the mixture (masses and sizes). These analytical results show in general a good agreement with those obtained from computer simulations [4]–[7], even for strong dissipation.

However, the above HCS is not accessible experimentally. In order to keep a granular system fluidized, an external energy supply is required. In some experimental situations, the gas is driven into rapid flow by the presence of a shear field. In this case, a steady state is reached when viscous heating is exactly balanced by the energy dissipation in collisions. This unbounded shear flow problem is usually referred to as the simple or uniform shear flow (USF) and its study has received a great deal of attention in the

past years, especially in the case of monocomponent systems [8, 9]. Nevertheless, much less is known on transport in sheared granular mixtures [10]–[13]. In particular, the understanding of mass transport in granular shear flows is of practical interest, since, for instance, powders must frequently be mixed together before any sort of process can begin. Due to the complexity of the general problem (binary sheared granular mixture), to gain some insight one can first consider the special case of impurity or tracer particles, namely, a binary mixture where the mole fraction of one of the components is negligible. The tracer or impurity problem is more amenable to analytical treatment since tracer particles are directly enslaved to the granular gas and there are fewer parameters than in a general binary mixture problem. Even in this limit case, the analysis of mass transport in a strongly shearing granular gas is an intricate problem basically due to the anisotropy induced in the system by the shear flow. For this reason, tensorial quantities $\{D_{ij}, D_{p,ij}, D_{T,ij}, \chi_{ij}\}$ are required to describe the mass transport process instead of scalar coefficients $\{D, D_p, D_T, \chi\}$. The aim of this paper is to get the above tensors in the framework of the inelastic Boltzmann equation.

Some previous attempts have been carried out earlier in the case of the diffusion tensor D_{ij}, especially in the self-diffusion problem [14]. However, in all these studies the diffusion was observed in only one direction, usually the direction parallel to the velocity gradient. The full diffusion tensor has been obtained in granular gases using kinetic theory [15], measured in simulations of rapid granular shear [16], and even in some experiments of dense, granular shear flows in a two-dimensional Couette geometry [17]. Substantial work on granular diffusivity has been carried out by Hsiau and co-workers [18], who have measured self-diffusion coefficients in a variety of granular systems. All these results clearly show that diffusion is anisotropic and is significantly affected by inelasticity.

At a kinetic level, in the tracer limit one can assume that the velocity distribution function $f(\mathbf{r}, \mathbf{v}, t)$ of the granular gas (excess component) obeys a closed (non-linear) Boltzmann equation, while the velocity distribution function $f_0(\mathbf{r}, \mathbf{v}, t)$ of the impurity satisfies a (linear) Boltzmann–Lorentz equation. Both kinetic equations have been solved in the (pure) USF problem to get the rheological properties of the mixture in the steady state [12, 15]. Let us assume now that the system (granular gas plus impurity) is in a state that deviates from the USF by *small* spatial gradients. In addition, we also assume that the impurity and gas particles are subjected to the action of external forces. Since the gas is slightly perturbed from the USF, the Boltzmann equation can be solved by expanding in small gradients around the (local) shear flow distribution instead of the (local) HCS. This Chapman–Enskog-like expansion has been very recently used [19, 20] to determine the heat and momentum fluxes to first order in the deviations of the hydrodynamic field gradients from their values in the reference shear flow state. Once the state of the excess component is well characterized, the goal here is to determine the mass transport of impurity by solving the Boltzmann–Lorentz equation by means of a similar perturbation scheme, namely, a gradient expansion around the corresponding shear flow distribution $f_0^{(0)}$ which applies for arbitrary values of the shear rate. In the first order of the expansion, the tensors D_{ij}, $D_{p,ij}$, $D_{T,ij}$, and χ_{ij} are identified from the mass flux $\mathbf{j}_0$. As was already pointed out in [19, 20], an important point is that, for general small deviations from the shear flow state, the zeroth-order distribution $f_0^{(0)}$ is not a stationary distribution since the collisional cooling cannot be compensated *locally* for viscous heating. This fact gives rise to new conceptual and practical difficulties not present in the results based on the

conventional Chapman–Enskog expansion. Due to these difficulties, the results will be restricted here to particular perturbations for which steady state conditions apply and so the (reduced) shear rate (which is the relevant non-equilibrium parameter of the problem) is coupled to dissipation.

The plan of the paper is as follows. In section 2 we describe the problem we are interested in. The USF state is analysed and the non-zero elements of the pressure tensor for the gas and the impurity are obtained by means of Grad's method [12, 15, 21]. Section 3 deals with the perturbation scheme used to solve the Boltzmann–Lorentz equation for the impurity to first order in the spatial gradients. The results show that the generalized transport coefficients D_{ij}, $D_{p,ij}$, $D_{T,ij}$, and χ_{ij} are the solutions of a set of coupled linear integral equations. A Sonine polynomial approximation taking the shear flow distribution $f_0^{(0)}$ as the weight function is applied in section 4 to solve these integral equations and get explicit expressions for these transport coefficients. The details of the calculations are displayed along several appendices. The dependence of some of these transport coefficients on dissipation is illustrated in the three-dimensional case, showing again that the influence of inelasticity on mass transport is quite significant. Finally, the paper is closed in section 5 with some concluding remarks.

2. Description of the problem

As already said, the aim of this work is to analyse mass transport of a dilute granular mixture subjected to uniform shear flow (USF) in the tracer limit, i.e., when the mole fraction of one of the species is negligible. In this case, the state of the granular gas (excess component) is not affected by the presence of the tracer particles. At a kinetic theory level, this implies that the velocity distribution function of the excess component obeys a closed non-linear Boltzmann equation. Moreover, the mole fraction of the tracer particles is so small that their mutual interactions can be neglected as compared with their interactions with the particles of the excess component. As a consequence, the velocity distribution function of the tracer particles satisfies a linear Boltzmann–Lorentz equation. This is formally equivalent to study an impurity or intruder in a dilute granular gas, and this will be the terminology used here. Let us start by offering a short review on some basic aspects of the inelastic Boltzmann equation and its solution in the USF state.

We consider a granular gas composed by smooth inelastic disks ($d = 2$) or spheres ($d = 3$) of mass m and diameter σ. The inelasticity of collisions among all pairs is accounted for by a *constant* coefficient of restitution α ($0 \leq \alpha \leq 1$) that only affects the translational degrees of freedom of grains. We also assume that the particles feel the presence of an external conservative force $\mathbf{F}$ (such as a gravity field). At low density, the time evolution of the one-particle velocity distribution function of the gas $f(\mathbf{r}, \mathbf{v}, t)$ is given by the inelastic Boltzmann equation [22, 23]

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{\mathbf{F}}{m} \cdot \frac{\partial}{\partial \mathbf{v}} \right) f(\mathbf{r}, \mathbf{v}, t) = J[\mathbf{v}|f(t), f(t)], \tag{1}$$

where the Boltzmann collision operator $J[\mathbf{v}|f, f]$ is

$$J[\mathbf{v}_1|f, f] = \sigma^{d-1} \int d\mathbf{v}_2 \int d\widehat{\boldsymbol{\sigma}}\, \Theta(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})$$
$$\times \left[\alpha^{-2} f(\mathbf{r}, \mathbf{v}_1') f(\mathbf{r}, \mathbf{v}_2', t) - f(\mathbf{r}, \mathbf{v}_1, t) f(\mathbf{r}, \mathbf{v}_2, t) \right]. \tag{2}$$

Here, $\widehat{\boldsymbol{\sigma}}$ is a unit vector along their line of centres, Θ is the Heaviside step function, and $\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity. The primes on the velocities denote the initial values $\{\mathbf{v}_1', \mathbf{v}_2'\}$ that lead to $\{\mathbf{v}_1, \mathbf{v}_2\}$ following a binary collision:

$$\mathbf{v}_1' = \mathbf{v}_1 - \tfrac{1}{2}\left(1 + \alpha^{-1}\right)(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})\widehat{\boldsymbol{\sigma}}, \qquad \mathbf{v}_2' = \mathbf{v}_2 + \tfrac{1}{2}\left(1 + \alpha^{-1}\right)(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})\widehat{\boldsymbol{\sigma}}. \tag{3}$$

The first five velocity moments of f define the number density

$$n(\mathbf{r}, t) = \int \mathrm{d}\mathbf{v}\, f(\mathbf{r}, \mathbf{v}, t), \tag{4}$$

the flow velocity

$$\mathbf{u}(\mathbf{r}, t) = \frac{1}{n(\mathbf{r}, t)} \int \mathrm{d}\mathbf{v}\, \mathbf{v} f(\mathbf{r}, \mathbf{v}, t), \tag{5}$$

and the *granular* temperature

$$T(\mathbf{r}, t) = \frac{m}{\mathrm{d}n(\mathbf{r}, t)} \int \mathrm{d}\mathbf{v}\, V^2 f(\mathbf{r}, \mathbf{v}, t), \tag{6}$$

where $\mathbf{V}(\mathbf{r}, t) \equiv \mathbf{v} - \mathbf{u}(\mathbf{r}, t)$ is the peculiar velocity. The macroscopic balance equations for density n, momentum $m\mathbf{u}$, and energy $(d/2)nT$ follow directly from equation (1) by multiplying with 1, $m\mathbf{v}$, and $\frac{1}{2}mv^2$ and integrating over $\mathbf{v}$:

$$D_t n + n\nabla \cdot \mathbf{u} = 0, \tag{7}$$

$$D_t \mathbf{u} + (mn)^{-1}\left(\nabla \cdot \mathsf{P} - n\mathbf{F}\right) = 0, \tag{8}$$

$$D_t T + \frac{2}{\mathrm{d}n}\left(\nabla \cdot \mathbf{q} + P_{ij}\nabla_j u_i\right) = -\zeta T, \tag{9}$$

where $D_t = \partial_t + \mathbf{u} \cdot \nabla$ is the material time derivative. The microscopic expressions for the pressure tensor P, the heat flux $\mathbf{q}$, and the cooling rate ζ are given, respectively, by

$$\mathsf{P}(\mathbf{r}, t) = \int \mathrm{d}\mathbf{v}\, m\mathbf{V}\mathbf{V} f(\mathbf{r}, \mathbf{v}, t), \tag{10}$$

$$\mathbf{q}(\mathbf{r}, t) = \int \mathrm{d}\mathbf{v}\, \tfrac{1}{2}mV^2\mathbf{V} f(\mathbf{r}, \mathbf{v}, t), \tag{11}$$

$$\zeta(\mathbf{r}, t) = -\frac{1}{\mathrm{d}n(\mathbf{r}, t)T(\mathbf{r}, t)} \int \mathrm{d}\mathbf{v}\, mV^2 J[\mathbf{r}, \mathbf{v}|f(t)]. \tag{12}$$

Let us suppose now that an impurity or intruder of mass m_0 and diameter σ_0 is added to the gas. As said before, the presence of the intruder does not perturb the state of the gas, so that its velocity distribution function is still determined by the Boltzmann equation (1). In addition, the macroscopic flow velocity and temperature for the mixture composed by the gas plus the impurity are the same as those for the gas, namely they are given by equations (5) and (6), respectively. The velocity distribution function $f_0(\mathbf{r}, \mathbf{v}, t)$ of the impurity satisfies the linear Boltzmann–Lorentz equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla + \frac{\mathbf{F}_0}{m_0} \cdot \frac{\partial}{\partial \mathbf{v}}\right) f_0(\mathbf{r}, \mathbf{v}, t) = J_0[\mathbf{v}|f_0(t), f(t)], \tag{13}$$

where the collision operator $J_0[\mathbf{v}|f_0, f]$ is now

$$
\begin{aligned}
J_0\left[\mathbf{v}_1|f_0, f\right] = \overline{\sigma}^{d-1} \int \mathrm{d}\mathbf{v}_2 \int \mathrm{d}\widehat{\boldsymbol{\sigma}}\, \Theta(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g})(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g}) \\
\times\, \left[\alpha_0^{-2} f_0(\mathbf{r}, \mathbf{v}_1') f(\mathbf{r}, \mathbf{v}_2', t) - f_0(\mathbf{r}, \mathbf{v}_1, t) f(\mathbf{r}, \mathbf{v}_2, t)\right].
\end{aligned}
\tag{14}
$$

Here, $\mathbf{F}_0$ denotes an external force acting on impurity, $\overline{\sigma} = (\sigma + \sigma_0)/2$ and α_0 $(0 \leq \alpha_0 \leq 1)$ is the coefficient of restitution for impurity–gas collisions. The precollisional velocities are given by

$$
\begin{aligned}
\mathbf{v}_1' &= \mathbf{v}_1 - \frac{m}{m + m_0}\left(1 + \alpha_0^{-1}\right)(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})\widehat{\boldsymbol{\sigma}}, \\
\mathbf{v}_2' &= \mathbf{v}_2 + \frac{m_0}{m + m_0}\left(1 + \alpha_0^{-1}\right)(\widehat{\boldsymbol{\sigma}} \cdot \mathbf{g}_{12})\widehat{\boldsymbol{\sigma}}.
\end{aligned}
\tag{15}
$$

As shown in [24], the operator $J_0[\mathbf{v}|f_0, f]$ is the same as that of an *elastic* impurity $(\alpha_0 = 1)$ with an effective mass $m_0^* = m_0 + (m_0 + m)(1 - \alpha_0)/(1 + \alpha_0)$.

The number density for the impurity is

$$
n_0(\mathbf{r}, t) = \int \mathrm{d}\mathbf{v}\, f_0(\mathbf{r}, \mathbf{v}, t).
\tag{16}
$$

The impurity may freely exchange momentum and energy with the particles of the gas and, therefore, these are not invariants of the collision operator $J_0[\mathbf{v}|f_0, f]$. Only the number density n_0 is conserved, whose continuity equation is directly obtained from equation (13)

$$
D_t n_0 + n_0 \nabla \cdot \mathbf{u} + \frac{\nabla \cdot \mathbf{j}_0}{m_0} = 0,
\tag{17}
$$

where $\mathbf{j}_0$ is the mass flux for the impurity, relative to the local flow $\mathbf{u}$,

$$
\mathbf{j}_0 = m_0 \int \mathrm{d}\mathbf{v}\, \mathbf{V}\, f_0(\mathbf{r}, \mathbf{v}, t).
\tag{18}
$$

At a kinetic level, an interesting quantity is the local temperature of the impurity defined as

$$
T_0(\mathbf{r}, t) = \frac{m_0}{d n_0(\mathbf{r}, t)} \int \mathrm{d}\mathbf{v}\, V^2 f_0(\mathbf{r}, \mathbf{v}, t).
\tag{19}
$$

This quantity measures the mean kinetic energy of the impurity. As will be shown later, the global temperature T and the temperature of the impurity T_0 are in general different, so that the granular energy per particle is not equally distributed between the two components of the system [10, 12].

Let us start by describing the state of the system (gas plus impurity) in the (pure) USF. This idealized macroscopic state is characterized by the absence of external forces $(\mathbf{F} = \mathbf{F}_0 = \mathbf{0})$, constant densities n and n_0, a uniform temperature, and a linear velocity profile $\mathbf{u} = \mathbf{a} \cdot \mathbf{r}$, where the elements of the tensor $\mathbf{a}$ are $a_{ij} = a\delta_{ix}\delta_{jy}$, a being the constant shear rate. This linear velocity profile assumes no boundary layer near the walls and is generated by the Lee–Edwards boundary conditions [25], which are simply periodic boundary conditions in the local Lagrangian frame moving with the flow velocity. For elastic gases, the temperature grows in time due to viscous heating and so a steady state is not possible unless an external (artificial) force is introduced [26]. However, for

inelastic gases, the temperature changes in time due to the competition between two (opposite) mechanisms: on the one hand, viscous (shear) heating and, on the other hand, energy dissipation in collisions. A steady state is achieved when the two mechanisms cancel each other and the fluid autonomously seeks the temperature at which the above balance occurs. Under these conditions, in the steady state the balance equation (9) becomes

$$aP_{xy} = -\frac{d}{2}\zeta p,$$ (20)

where $p = nT$ is the hydrostatic pressure. The balance equation (20) shows the intrinsic connection between the shear field and dissipation in the system. As a consequence, the shear flow state associated with (20) is inherently beyond the scope of the Navier–Stokes or Newtonian hydrodynamic equations [21] since the collisional cooling (which is fixed by the mechanical properties of the particles) sets the strength of the velocity gradient in the steady state. Furthermore, note that for given values of the shear rate a and the coefficient of restitution α, the relation (20) gives the temperature T in the steady state as a *unique* function of the density n. In fact, the reduced shear rate $a^* = a/\nu(n,T)$ is *only* a function of the coefficient of restitution α in the steady state. Here, $\nu(n,T)$ is a characteristic collision frequency given by

$$\nu = \frac{\pi^{(d-1)/2}}{\Gamma(d/2)}\frac{8}{d+2}n\sigma^{d-1}\sqrt{\frac{T}{m}}.$$ (21)

The relevant transport properties of the system in the steady USF are related to the pressure tensor P since $\mathbf{q} = \mathbf{j}_0 = \mathbf{0}$. Furthermore, one can introduce the partial pressure tensor P_0 defined as

$$\mathsf{P}_0 = \int \mathrm{d}\mathbf{v}\, m_0 \mathbf{V}\mathbf{V}\, f_0(\mathbf{V}).$$ (22)

Explicit expressions for the non-zero elements of P and P_0 have been recently obtained from the Boltzmann equation by means of Grad's method [15]. A brief summary of these results is given in appendix A. The accuracy of this Grad's solution has been confirmed by comparison with Monte Carlo simulations of the Boltzmann equation [12, 27, 28], even for strong dissipation.

3. Chapman–Enskog-like expansion: transport properties around USF

The main goal of this paper is to determine the mass flux for the impurity in the presence of USF. In that case, let us assume that the USF state is disturbed by small spatial perturbations. The response of the system to those perturbations gives rise to contributions to the mass flux, which can be characterized by generalized transport coefficients. This section is devoted to the evaluation of these coefficients.

In order to analyse this problem we have to start from the Boltzmann–Lorentz equation (13) with a general time and space dependence. Let $\mathbf{u}_s = \mathsf{a} \cdot \mathbf{r}$ be the flow velocity of the *undisturbed* USF state. In the *disturbed* state, however the true velocity $\mathbf{u}$ is in general different from $\mathbf{u}_s$ since $\mathbf{u} = \mathbf{u}_s + \delta\mathbf{u}$, $\delta\mathbf{u}$ being a small perturbation to $\mathbf{u}_s$. As a consequence, the true peculiar velocity is now $\mathbf{c} \equiv \mathbf{v} - \mathbf{u} = \mathbf{V} - \delta\mathbf{u}$, where $\mathbf{V} = \mathbf{v} - \mathbf{u}_s$.

In the Lagrangian frame moving with $\mathbf{u}_s$, the Boltzmann–Lorentz equation can be written as

$$\frac{\partial}{\partial t}f_0 - aV_y\frac{\partial}{\partial V_x}f_0 + (\mathbf{V} + \mathbf{u}_s)\cdot\nabla f_0 + \frac{\mathbf{F}_0}{m_0}\cdot\frac{\partial}{\partial \mathbf{V}}f_0 = J_0[\mathbf{V}|f_0, f], \qquad (23)$$

where here the derivative ∇f_0 is taken at constant $\mathbf{V}$. The macroscopic balance equations associated with this disturbed USF state follows from the general equations (8), (9), and (17) when one takes into account that $\mathbf{u} = \mathbf{u}_s + \delta\mathbf{u}$. The result is

$$\partial_t n_0 + \mathbf{u}_s\cdot\nabla n_0 = -\nabla\cdot(n_0\delta\mathbf{u}) - \frac{\nabla\cdot\mathbf{j}_0}{m_0}, \qquad (24)$$

$$\partial_t\delta\mathbf{u} + \mathsf{a}\cdot\delta\mathbf{u} + (\mathbf{u}_s + \delta\mathbf{u})\cdot\nabla\delta\mathbf{u} = -(mn)^{-1}\left(\nabla\cdot\mathsf{P} - n\mathbf{F}\right), \qquad (25)$$

$$\frac{d}{2}n\partial_t T + \frac{d}{2}n(\mathbf{u}_s + \delta\mathbf{u})\cdot\nabla T + aP_{xy} + \nabla\cdot\mathbf{q} + \mathsf{P}:\nabla\delta\mathbf{u} = -\frac{d}{2}p\zeta, \qquad (26)$$

where the pressure tensor P, the heat flux $\mathbf{q}$, the cooling rate ζ, and the mass flux $\mathbf{j}_0$ are defined by equations (10), (11), (12), and (18), respectively, with the replacement $\mathbf{V}\to\mathbf{c}$.

We assume that the deviations from the USF state are small. This means that the spatial gradients of the hydrodynamic fields $A(\mathbf{r}, t)$ are small. As noted in [2], there is some flexibility in the case of granular mixtures in the representation of the heat and mass fluxes since they can be defined in a variety of equivalent ways depending on the choice of hydrodynamic gradients used. In fact, some care is required in comparing transport coefficients in different representations using different independent gradients for the driving forces. Here, as in previous works [2, 3], I take the concentration of impurity $x_0 = n_0/n$, the pressure p, the temperature T, and the local flow velocity $\delta\mathbf{u}$ as hydrodynamic fields, i.e.,

$$A(\mathbf{r}, t) \equiv \{x_0(\mathbf{r}, t), p(\mathbf{r}, t), T(\mathbf{r}, t), \delta\mathbf{u}(\mathbf{r}, t)\}. \qquad (27)$$

Since the system is strongly sheared, a solution to the Boltzmann–Lorentz equation (23) can be obtained by means of a generalization of the conventional Chapman–Enskog method [1] where the velocity distribution function is expanded about a *local* shear flow reference state in terms of the small spatial gradients of the hydrodynamic fields relative to those of USF. This is the main new ingredient of the expansion. This type of Chapman–Enskog-like expansion has been considered in the case of elastic gases to get the set of shear rate dependent transport coefficients [26, 29] in a thermostated shear flow problem and it has also been recently considered [19, 20] for inelastic gases.

In the context of the Chapman–Enskog method [1], we look for a *normal* solution of the form

$$f_0(\mathbf{r}, \mathbf{V}, t) \equiv f_0(A(\mathbf{r}, t), \mathbf{V}). \qquad (28)$$

This special solution expresses the fact that the space dependence of the reference shear flow is completely absorbed in the relative velocity $\mathbf{V}$ and all other space and time dependence occurs entirely through a *functional* dependence on the fields $A(\mathbf{r}, t)$. Moreover, in the presence of external forces it is necessary to characterize the magnitude of these forces relative to the gradients as well. Here, it is assumed that the magnitude of the external forces $\mathbf{F}$ and $\mathbf{F}_0$ is of first order in perturbation expansion. The functional

dependence (28) can be made local by an expansion of the distribution function in powers of the hydrodynamic gradients:

$$f_0(A(\mathbf{r}, t), \mathbf{V}) = f_0^{(0)}(\mathbf{V}) + f_0^{(1)}(\mathbf{V}) + \cdots, \tag{29}$$

where the reference zeroth-order distribution function corresponds to the USF distribution function but taking into account the local dependence of the concentration, pressure and temperature and the change $\mathbf{V} \to \mathbf{V} - \delta\mathbf{u}(\mathbf{r}, t) = \mathbf{c}$. The successive approximations $f^{(k)}$ are of order k in the strength of the external forces as well as in the gradients of x_0, p, T, and $\delta\mathbf{u}$ but retain all the orders in the shear rate a. Here, only the first-order approximation will be considered.

When the expansion (29) is substituted into the definitions (10)–(12), and (18), one gets the corresponding expansions for the fluxes and the cooling rate:

$$\mathsf{P} = \mathsf{P}^{(0)} + \mathsf{P}^{(1)} + \cdots, \qquad \mathbf{q} = \mathbf{q}^{(0)} + \mathbf{q}^{(1)} + \cdots, \tag{30a}$$

$$\zeta = \zeta^{(0)} + \zeta^{(1)} + \cdots, \qquad \mathbf{j}_0 = \mathbf{j}_0^{(0)} + \mathbf{j}_0^{(1)} + \cdots. \tag{30b}$$

Finally, as in the usual Chapman–Enskog method, the time derivative is also expanded as

$$\partial_t = \partial_t^{(0)} + \partial_t^{(1)} + \partial_t^{(2)} + \cdots, \tag{31}$$

where the action of each operator $\partial_t^{(k)}$ is obtained from the hydrodynamic equations (24)–(27). These results provide the basis for generating the Chapman–Enskog solution to the inelastic Boltzmann–Lorentz equation (23).

3.1. Zeroth-order approximation

Substituting the expansions (29)–(31) into equation (23), the kinetic equation for $f_0^{(0)}$ is given by

$$\partial_t^{(0)} f_0^{(0)} - aV_y \frac{\partial}{\partial V_x} f_0^{(0)} = J_0[\mathbf{V}|f_0^{(0)}, f^{(0}], \tag{32}$$

where use has been made of the fact that $\mathbf{F}_0$ is of first order in gradients. To lowest order in the expansion the conservation laws give

$$\partial_t^{(0)} x_0 = 0, \qquad T^{-1}\partial_t^{(0)} T = p^{-1}\partial_t^{(0)} p = -\frac{2}{\mathrm{d}p} a P_{xy}^{(0)} - \zeta^{(0)}, \tag{33}$$

$$\partial_t^{(0)} \delta u_i + a_{ij}\delta u_j = 0. \tag{34}$$

As shown in [19, 20], for given values of a and α, the steady state condition (20) establishes a mapping between the pressure p and temperature T so that every pressure corresponds to one and only one temperature. Since the pressure $p(\mathbf{r}, t)$ and temperature $T(\mathbf{r}, t)$ are specified separately in the *local* USF state, the viscous heating term $a|P_{xy}^{(0)}|$ only partially compensates for the collisional cooling and so, the pressure and temperature depend on time. This implies that the zeroth-order distributions for the gas $f^{(0)}$ and the impurity $f_0^{(0)}$ both depend on time through their dependence on the pressure and temperature. Consequently, in general the reduced shear rate $a^* = a/\nu(p, T)$ depends on space and time so that, a^* and α must be considered as independent parameters for general infinitesimal

perturbations around the USF state. This fact gives rise to conceptual and practical difficulties not present in the case of elastic collisions [29].

Since $f_0^{(0)}$ is a normal solution, the time derivative in equation (32) can be represented more usefully as

$$
\begin{aligned}
\partial_t^{(0)} f_0^{(0)} &= \frac{\partial f_0^{(0)}}{\partial x_0} \partial_t^{(0)} x_0 + \frac{\partial f_0^{(0)}}{\partial p} \partial_t^{(0)} p + \frac{\partial f_0^{(0)}}{\partial T} \partial_t^{(0)} T + \frac{\partial f_0^{(0)}}{\partial \delta u_i} \partial_t^{(0)} \delta u_i \\
&= -\left(\frac{2}{\mathrm{d}p} a P_{xy}^{(0)} + \zeta^{(0)} \right) \left(p\frac{\partial}{\partial p} + T\frac{\partial}{\partial T} \right) f_0^{(0)} - a_{ij}\delta u_j \frac{\partial}{\partial \delta u_i} f_0^{(0)} \\
&= -\left(\frac{2}{\mathrm{d}p} a P_{xy}^{(0)} + \zeta^{(0)} \right) \left(p\frac{\partial}{\partial p} + T\frac{\partial}{\partial T} \right) f_0^{(0)} + a_{ij}\delta u_j \frac{\partial}{\partial c_i} f_0^{(0)},
\end{aligned}
\tag{35}
$$

where in the last step we have taken into account that $f_0^{(0)}$ depends on $\delta \mathbf{u}$ only through the peculiar velocity $\mathbf{c}$. Substituting (35) into (32) yields the following kinetic equation for $f_0^{(0)}$:

$$
-\left(\frac{2}{\mathrm{d}p} a P_{xy}^{(0)} + \zeta^{(0)} \right) \left(p\frac{\partial}{\partial p} + T\frac{\partial}{\partial T} \right) f_0^{(0)} - a c_y \frac{\partial}{\partial c_x} f_0^{(0)} = J_0[\mathbf{V}|f_0^{(0)}, f^{(0)}].
\tag{36}
$$

To solve equation (36) one needs to know the dependence of the momentum flux $P_{xy}^{(0)}$ on the pressure p and temperature T. A detailed study of this problem has been carried out in [21]. The first non-trivial velocity moment of the distribution $f_0^{(0)}$ corresponds to the partial pressure tensor $\mathsf{P}_0^{(0)}$ defined as

$$
\mathsf{P}_0^{(0)} = m_0 \int \mathrm{d}\mathbf{c}\, \mathbf{cc}\, f_0^{(0)}(\mathbf{c}).
\tag{37}
$$

The temperature of the impurity $T_0(t)$ can be determined from the trace of $\mathsf{P}_0^{(0)}$. From equation (36), one gets

$$
-\left(\frac{2}{\mathrm{d}p} a P_{xy}^{(0)} + \zeta^{(0)} \right) \left(p\frac{\partial}{\partial p} + T\frac{\partial}{\partial T} \right) P_{0,ij}^{(0)} + a_{i\ell} P_{0,j\ell}^{(0)} + a_{j\ell} P_{0,i\ell}^{(0)} = B_{ij},
\tag{38}
$$

where $a_{ij} = a\delta_{ix}\delta_{jy}$ and

$$
B_{ij} = m_0 \int \mathrm{d}\mathbf{c}\, c_i c_j\, J_0[f_0^{(0)}, f^{(0)}].
\tag{39}
$$

A good estimate of the collisional moment B_{ij} can be made by considering Grads's approximation [15]. In this case, B_{ij} is given by equation (B.6). The steady state solution of equation (38) is also displayed in appendix A. However, in general the equations (38) must be solved numerically to get the dependence of the zeroth-order pressure tensor $P_{0,ij}^{(0)}(p,T)$ on pressure and temperature. The behaviour of the pressure tensors $P_{ij}^{(0)}$ and $P_{0,ij}^{(0)}$ near the (steady) USF state is studied in appendix B. In what follows, $P_{ij}^{(0)}$ and $P_{0,ij}^{(0)}$ will be considered as known functions of p and T.

3.2. First-order approximation

The analysis to first order in the gradients is similar to the one carried out in [20] for the one-component case. Some details of this derivation are given in appendix C. The distribution function $f_0^{(1)}$ is of the form

$$f_0^{(1)} = \boldsymbol{\mathcal{A}}_0 \cdot \nabla x_0 + \boldsymbol{\mathcal{B}}_0 \cdot \nabla p + \boldsymbol{\mathcal{C}}_0 \cdot \nabla T + \mathcal{D}_0 : \nabla \delta \mathbf{u} + \boldsymbol{\mathcal{E}}_0 \cdot \boldsymbol{\mathcal{F}}, \tag{40}$$

where

$$\boldsymbol{\mathcal{F}} = \mathbf{F}_0 - \frac{m_0}{m}\mathbf{F}. \tag{41}$$

The vectors $\{\boldsymbol{\mathcal{A}}_0, \boldsymbol{\mathcal{B}}_0, \boldsymbol{\mathcal{C}}_0, \boldsymbol{\mathcal{E}}_0\}$, and the tensor $\mathcal{D}_0$ are functions of the true peculiar velocity $\mathbf{c}$. They are the solutions of the following linear integral equations:

$$-\left(\frac{2}{\mathrm{d}p}aP_{xy}^{(0)} + \zeta^{(0)}\right)(p\partial_p + T\partial_T)\boldsymbol{\mathcal{A}}_0 - ac_y\frac{\partial}{\partial c_x}\boldsymbol{\mathcal{A}}_0 - J_0[\boldsymbol{\mathcal{A}}_0, f^{(0)}] = \mathbf{A}_0, \tag{42}$$

$$-\left(\frac{2}{\mathrm{d}p}aP_{xy}^{(0)} + \zeta^{(0)}\right)(p\partial_p + T\partial_T)\boldsymbol{\mathcal{B}}_0 - \left(\frac{2a}{d}\partial_p P_{xy}^{(0)} + 2\zeta^{(0)} + ac_y\frac{\partial}{\partial c_x}\right)\boldsymbol{\mathcal{B}}_0$$
$$- J_0[\boldsymbol{\mathcal{B}}_0, f^{(0)}] = \mathbf{B}_0 - \left[\frac{2aT}{\mathrm{d}p^2}(1 - p\partial_p)P_{xy}^{(0)} - \frac{T\zeta^{(0)}}{p}\right]\boldsymbol{\mathcal{C}}_0 + J_0[f_0^{(0)}, \boldsymbol{\mathcal{B}}], \tag{43}$$

$$-\left(\frac{2}{\mathrm{d}p}aP_{xy}^{(0)} + \zeta^{(0)}\right)(p\partial_p + T\partial_T)\boldsymbol{\mathcal{C}}_0 - \left[\frac{2a}{\mathrm{d}p}(1 + T\partial_T)P_{xy}^{(0)} + \frac{1}{2}\zeta^{(0)}\right.$$
$$\left. + ac_y\frac{\partial}{\partial c_x}\right]\boldsymbol{\mathcal{C}}_0 - J_0[\boldsymbol{\mathcal{C}}_0, f^{(0)}] = \mathbf{C}_0 + \left(\frac{2a}{d}\partial_T P_{xy}^{(0)} - \frac{p\zeta^{(0)}}{2T}\right)\boldsymbol{\mathcal{B}}_0$$
$$+ J_0[f_0^{(0)}, \boldsymbol{\mathcal{C}}], \tag{44}$$

$$-\left(\frac{2}{\mathrm{d}p}aP_{xy}^{(0)} + \zeta^{(0)}\right)(p\partial_p + T\partial_T)\mathcal{D}_{0,\ell j} - ac_y\frac{\partial}{\partial c_x}\mathcal{D}_{0,\ell j} - a\delta_{\ell y}\mathcal{D}_{0,xj}$$
$$- J_0[\mathcal{D}_{0,\ell j}, f^{(0)}] = D_{0,\ell j} + \zeta_{u,\ell j}(p\partial_p + T\partial_T)f^{(0)} + J_0[f_0^{(0)}, \mathcal{D}_{\ell j}], \tag{45}$$

$$-\left(\frac{2}{\mathrm{d}p}aP_{xy}^{(0)} + \zeta^{(0)}\right)(p\partial_p + T\partial_T)\boldsymbol{\mathcal{E}}_0 - ac_y\frac{\partial}{\partial c_x}\boldsymbol{\mathcal{E}}_0 - J_0[\boldsymbol{\mathcal{E}}_0, f^{(0)}] = \mathbf{E}_0. \tag{46}$$

Here, $\mathbf{A}_0(\mathbf{c})$, $\mathbf{B}_0(\mathbf{c})$, $\mathbf{C}_0(\mathbf{c})$, $D_0(\mathbf{c})$, and $\mathbf{E}_0(\mathbf{c})$ are defined by equations (C.9), (C.10), (C.11), (C.12), and (C.13), respectively. In addition, upon writing equations (42)–(46), use has been made of the explicit form of $f^{(1)}$. It has been derived in [20] and reads

$$f^{(1)} = \boldsymbol{\mathcal{B}} \cdot \nabla p + \boldsymbol{\mathcal{C}} \cdot \nabla T + \boldsymbol{\mathcal{D}} : \nabla \delta\mathbf{u}, \tag{47}$$

where the coefficients $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\mathcal{C}}$, and $\boldsymbol{\mathcal{D}}$ are functions of the peculiar velocity $\mathbf{c}$ and the hydrodynamic fields. In particular, the first-order contribution to the cooling rate, $\zeta_{u,ij}$, is given by

$$\zeta_{u,ij} = -\frac{1}{\mathrm{d}p}\int \mathrm{d}\mathbf{c}\, mc^2\left(J[f^{(0)}, \mathcal{D}_{ij}] + J[\mathcal{D}_{ij}, f^{(0)}]\right). \tag{48}$$

The first-order contribution to the mass flux $\mathbf{j}_0^{(1)}$ of the impurity is defined as

$$\mathbf{j}_0^{(1)} = m_0 \int d\mathbf{c}\, \mathbf{c}\, f_0^{(1)}(\mathbf{c}). \tag{49}$$

Use of equation (40) into (49) gives the expression

$$j_{0,i}^{(1)} = -m_0 D_{ij} \frac{\partial x_0}{\partial r_j} - \frac{m}{T} D_{p,ij} \frac{\partial p}{\partial r_j} - \frac{mn}{T} D_{T,ij} \frac{\partial T}{\partial r_j} + \chi_{ij} \mathcal{F}_j, \tag{50}$$

where

$$D_{ij} = -\int d\mathbf{c}\, c_i\, \mathcal{A}_{0,j}(\mathbf{c}), \tag{51}$$

$$D_{p,ij} = -\frac{T m_0}{m} \int d\mathbf{c}\, c_i\, \mathcal{B}_{0,j}(\mathbf{c}), \tag{52}$$

$$D_{T,ij} = -\frac{T m_0}{mn} \int d\mathbf{c}\, c_i\, \mathcal{C}_{0,j}(\mathbf{c}), \tag{53}$$

$$\chi_{ij} = m_0 \int d\mathbf{c}\, c_i\, \mathcal{E}_{0,j}(\mathbf{c}). \tag{54}$$

Upon writing equations (51)–(54) use has been made of the symmetry properties of $\boldsymbol{\mathcal{A}}_0$, $\boldsymbol{\mathcal{B}}_0$, $\boldsymbol{\mathcal{C}}_0$, and $\boldsymbol{\mathcal{E}}_0$. In general, the set of *generalized* transport coefficients D_{ij}, $D_{p,ij}$, $D_{T,ij}$, and χ_{ij} are non-linear functions of the shear rate and the coefficients of restitution α and α_0. It is apparent that the anisotropy induced by the presence of shear flow gives rise to new transport coefficients for the mass flux, reflecting broken symmetry. According to equation (50), the mass flux of the impurity is expressed in terms of a diffusion tensor D_{ij}, a pressure diffusion tensor $D_{p,ij}$, a thermal diffusion tensor $D_{T,ij}$, and a mobility tensor χ_{ij}. Note that in the particular case of the gravitational force $\mathbf{F} = m\mathbf{g}$ and $\mathbf{F}_0 = m_0\mathbf{g}$, where $\mathbf{g}$ is the gravity acceleration. In this case, the combined force $\boldsymbol{\mathcal{F}}$ defined in equation (41) vanishes. Consequently, the external force $\boldsymbol{\mathcal{F}}$ does not occur in equation (40) when the system is only subjected to a gravity field.

3.3. Steady state conditions

The evaluation of the above transport coefficients requires to know the complete dependence of $P_{ij}^{(0)}$ and $P_{0,ij}^{(0)}$ on the pressure p and the temperature T. This involves the corresponding numerical integrations of the differential equations obeying the pressure tensors $P_{ij}^{(0)}$ and $P_{0,ij}^{(0)}$ (see equation (38) for $P_{0,ij}^{(0)}$). Needless to say, this is quite an intricate problem. However, some simplifications occur if attention is restricted to linear deviations from the USF steady state described in section 2. In particular, since the contributions to the mass flux (50) are already of first order in the deviations from the steady state, one only needs to know the transport coefficients to zero order in the deviations. This means that $\partial_t^{(0)} T = \partial_t^{(0)} p = 0$ and so the term $(2/dp)aP_{xy}^{(0)} + \zeta^{(0)} = 0$ in the left-hand side of equations (42)–(46). The differential equations for the transport coefficients thus become simple coupled algebraic equations.

The dependence of $P_{ij}^{(0)}$ on the pressure p and temperature T occurs explicitly and through its dependence on the reduced shear rate $a^* \propto \sqrt{T}/p$. Consequently,

$$p\partial_p P_{ij}^{(0)} = p\partial_p p P_{ij}^*(a^*) = p\left(1 - a^*\frac{\partial}{\partial a^*}\right)P_{ij}^*(a^*), \tag{55}$$

$$T\partial_T P_{ij}^{(0)} = T\partial_T p P_{ij}^*(a^*) = \frac{1}{2}pa^*\frac{\partial}{\partial a^*}P_{ij}^*(a^*), \tag{56}$$

where $P_{ij}^* = P_{ij}^{(0)}/p$ and $a^* = a/\nu$. The dependence of P_{ij}^* on a^* near the steady state was determined in the one-component problem [20] so that all the terms appearing in the integral equations are explicitly known in the steady state. Under the above conditions, equations (42)–(46) become

$$-ac_y\frac{\partial}{\partial c_x}\boldsymbol{\mathcal{A}}_0 - J_0[\boldsymbol{\mathcal{A}}_0, f^{(0)}] = \mathbf{A}_0, \tag{57}$$

$$-\left[\frac{2a}{d}(1 - a^*\partial_{a^*})P_{xy}^* + 2\zeta^{(0)} + ac_y\frac{\partial}{\partial c_x}\right]\boldsymbol{\mathcal{B}}_0 - J_0[\boldsymbol{\mathcal{B}}_0, f^{(0)}] = \mathbf{B}_0$$
$$- \left(\frac{2aT}{\mathrm{d}p}a^*\partial_{a^*}P_{xy}^* - \frac{T\zeta^{(0)}}{p}\right)\boldsymbol{\mathcal{C}}_0 + J_0[f_0^{(0)}, \boldsymbol{\mathcal{B}}], \tag{58}$$

$$-\left[\frac{2a}{d}\left(1 + \frac{1}{2}a^*\partial_{a^*}\right)P_{xy}^* + \frac{1}{2}\zeta^{(0)} + ac_y\frac{\partial}{\partial c_x}\right]\boldsymbol{\mathcal{C}}_0 - J_0[\boldsymbol{\mathcal{C}}_0, f^{(0)}] = \mathbf{C}_0$$
$$+ \frac{p}{T}\left(\frac{a}{d}a^*\partial_{a^*}P_{xy}^* - \frac{\zeta^{(0)}}{2}\right)\boldsymbol{\mathcal{B}}_0 + J_0[f_0^{(0)}, \boldsymbol{\mathcal{C}}], \tag{59}$$

$$-ac_y\frac{\partial}{\partial c_x}\mathcal{D}_{0,\ell j} - a\delta_{\ell y}\mathcal{D}_{0,xj} - J_0[\mathcal{D}_{0,\ell j}, f^{(0)}] = D_{0,\ell j} + \zeta_{u,\ell j}\left(p\partial_p + T\partial_T\right)f^{(0)} + J_0[f_0^{(0)}, \mathcal{D}_{\ell j}], \tag{60}$$

$$-ac_y\frac{\partial}{\partial c_x}\boldsymbol{\mathcal{E}}_0 - J_0[\boldsymbol{\mathcal{E}}_0, f^{(0)}] = \mathbf{E}_0. \tag{61}$$

It must be recalled that in equations (57)–(61) all the quantities are evaluated in the steady state, namely, P_{ij}^* and a^* are given by equations (A.3)–(A.5) and (A.6), respectively, while $\partial P_{ij}^*/\partial a^*$ is given by equations (B.1)–(B.3). Henceforth, the calculations will be restricted to the particular condition (20).

4. Mass transport of the impurity

This section is devoted to the determination of the generalized transport coefficients associated with the mass transport of the impurity. In order to get explicit expressions for these coefficients, one has to know the quantities $\boldsymbol{\mathcal{A}}_0$, $\boldsymbol{\mathcal{B}}_0$, $\boldsymbol{\mathcal{C}}_0$, and $\boldsymbol{\mathcal{E}}_0$ which verify the coupled integral equations (57), (58), (59), and (61), respectively. To determine the explicit dependence of these quantities on the coefficients of restitution α and α_0, one needs to make use of certain approximations. The standard approach is to consider the leading term in a Sonine polynomial expansion. In a previous work on diffusion in shear flow [15],

the isotropic part of $f_0^{(1)}$ in this expansion was assumed for simplicity to be a Maxwellian $f_{0,M}(\mathbf{c})$. However, given that the system is strongly sheared, it is reasonable to expect that the isotropic part of $f_0^{(1)}$ is mainly governed by the shear flow distribution $f_0^{(0)}$ rather than by the Maxwellian distribution. For this reason, here we keep the usual structure of the standard Sonine approximation, except that the Maxwellian weight function $f_{0,M}$ is replaced by $f_0^{(0)}$. A similar type of modified Sonine approximation has been recently considered [30] to estimate the Navier–Stokes transport coefficients of a single granular gas.

According to the above arguments, in the case of the mass flux $\mathbf{j}_0^{(1)}$, a good estimate of $\{\mathcal{A}_0, \mathcal{B}_0, \mathcal{C}_0, \mathcal{E}_0\}$ is given by the first Sonine approximation:

$$\mathcal{A}_{0,i} \to -c_j \overline{D}_{ji}\, f_0^{(0)}(\mathbf{c}), \qquad \mathcal{B}_{0,i} \to -c_j \overline{D}_{p,ji}\, f_0^{(0)}(\mathbf{c}), \tag{62a}$$

$$\mathcal{C}_{0,i} \to -c_j \overline{D}_{T,ji}\, f_0^{(0)}(\mathbf{c}), \qquad \mathcal{E}_{0,i} \to c_j \overline{\chi}_{ji}\, f_0^{(0)}(\mathbf{c}), \tag{62b}$$

where $f_0^{(0)}(\mathbf{c})$ is the solution of equation (36). The relationship between the tensors $\{\overline{D}_{ij}, \overline{D}_{p,ij}, \overline{D}_{T,ij}, \overline{\chi}_{ij}\}$ and their corresponding counterparts $\{D_{ij}, D_{p,ij}, D_{T,ij}, \chi_{ij}\}$ is consistently obtained from equations (51)–(54). A simple calculation yields

$$\overline{D}_{ij} = m_0 Q_{ik} D_{kj}, \qquad \overline{D}_{p,ij} = \frac{m}{T} Q_{ik} D_{p,kj}, \tag{63}$$

$$\overline{D}_{T,ij} = \frac{mn}{T} Q_{ik} D_{T,kj}, \qquad \overline{\chi}_{ij} = Q_{ik} \chi_{kj}, \tag{64}$$

where $\mathsf{Q} = \mathsf{P}_0^{-1}$. When one takes $f_{0,M}$ instead of $f_0^{(0)}$ in the Sonine expansion, then $Q_{ij} = (n_0 T_0)^{-1} \delta_{ij}$ and one recovers previous results [15]. Consistently, the quantities $\mathcal{B}$ and $\mathcal{C}$ corresponding to the distribution $f^{(1)}$ of the granular gas (see equation (47)) must be similarly approximated. However, as shown in [20], these quantities vanish in the first Sonine approximation and so there is no contribution to the mass flux coming from the terms of the form $J_0[f_0^{(0)}, \mathcal{B}]$ and $J_0[f_0^{(0)}, \mathcal{C}]$. Substitution of the expressions (62a) and (62b) into equations (57), (58), (59), and (61) gives a closed set of integral equations for D_{ij}, $D_{p,ij}$, $D_{T,ij}$, and χ_{ij}. Multiplication of these equations by $m_0 c_i$ and integration over $\mathbf{c}$ yields

$$(a_{ik} + \Omega_{ik})\, D_{kj} = \frac{p}{m_0} P_{0,ij}^*, \tag{65}$$

$$\left[\frac{2a}{d}(1 - a^*\partial_{a^*})P_{xy}^* + 2\zeta^{(0)}\right] D_{p,ij} - (a_{ik} + \Omega_{ik})\, D_{p,kj} = \frac{T x_0}{m}(1 - a^*\partial_{a^*})\left(\frac{m_0}{m}P_{ij}^* - P_{0,ij}^*\right)$$
$$+ \left(\frac{2a}{d}a^*\partial_{a^*} P_{xy}^* - \zeta^{(0)}\right) D_{T,ij}, \tag{66}$$

$$\left[\frac{2a}{d}\left(1 + \frac{1}{2}a^*\partial_{a^*}\right)P_{xy}^* + \frac{1}{2}\zeta^{(0)}\right] D_{T,ij} - (a_{ik} + \Omega_{ik})\, D_{T,kj} = \frac{1}{2}\frac{T x_0}{m}a^*\partial_{a^*}\left(\frac{m_0}{m}P_{ij}^* - P_{0,ij}^*\right)$$
$$- \left(\frac{a}{d}a^*\partial_{a^*} P_{xy}^* - \frac{1}{2}\zeta^{(0)}\right) D_{p,ij}, \tag{67}$$

$$(a_{ik} + \Omega_{ik})\, \chi_{kj} = n_0 \delta_{ij}. \tag{68}$$

In the above equations, $P_{0,ij}^* = P_{0,ij}^{(0)}/x_0 p$ and $\Omega_{ij} = n_0 T_0 \Lambda_{ik} Q_{kj}$, where

$$\Lambda_{ij} = -\frac{m_0}{n_0 T_0} \int d\mathbf{c}\, c_i\, J_0[c_j f_0^{(0)}, f^{(0)}]. \tag{69}$$

This quantity can be evaluated by using standard integration techniques with the result

$$\Lambda_{ij} = \frac{\sqrt{2}}{4d} \left(\frac{\overline{\sigma}}{\sigma}\right)^{d-1} \nu\mu(1+\alpha_0)\left[(1+\theta)\theta\right]^{-1/2} \left\{ (d+2)(1+\theta)\delta_{ij} \right.$$
$$\left. + \theta\left(P_{ij}^* - \delta_{ij}\right) + [d+3+(d+2)\theta]\left(\gamma^{-1}P_{0,ij}^* - \delta_{ij}\right) \right\}, \tag{70}$$

where ν is defined by equation (21), $\mu = m/(m+m_0)$, $\gamma = T_0/T$ is the temperature ratio and $\theta = m_0 T/m T_0$ is the mean square velocity of the gas particles relative to that of the impurity. Upon deriving equation (70), use has been made of the Sonine approximations (A.1) and (A.8) for the distributions $f^{(0)}$ and $f_0^{(0)}$, respectively, and the non-linear term proportional to $\left(P_{ij}^* - \delta_{ij}\right)\left(\gamma^{-1}P_{0,ij}^* - \delta_{ij}\right)$ has been neglected.

The coefficients D_{ij} and χ_{ij} decouple from the other ones and hence can be obtained more easily. By using matrix notation, they are given by

$$\mathsf{D} = \frac{p}{m_0}\left(\mathsf{a} + \mathbf{\Omega}\right)^{-1} \cdot \mathsf{P}_0^*, \tag{71}$$

$$\boldsymbol{\chi} = n_0\left(\mathsf{a} + \mathbf{\Omega}\right)^{-1} \cdot \mathsf{I}. \tag{72}$$

The remaining coefficients $D_{p,ij}$ and $D_{T,ij}$ are coupled and they obey the set of simple algebraic equations (66) and (67).

In the elastic limit ($\alpha = \alpha_0 = 1$, which implies $a^* = 0$ in the steady state conditions), $T = T_0$, $P_{ij}^* = P_{0,ij}^* = \delta_{ij}$, and $\Omega_{ij} = \Omega_0 \delta_{ij}$, so that equations (65)–(68) have the solutions $D_{ij} = D_0 \delta_{ij}$, $D_{p,ij} = D_{p0}\delta_{ij}$, $D_{T,ij} = 0$, and $\chi_{ij} = \chi_0 \delta_{ij}$, where D_0, D_{p0} and χ_0 are the conventional Navier–Stokes transport coefficients for ordinary gases [1]. Their expressions are

$$D_0 = \frac{p}{m_0 \Omega_0}, \qquad D_{p0} = \frac{T x_0}{m \Omega_0}\left(1 - \frac{m_0}{m}\right), \qquad \chi_0 = \frac{n_0}{\Omega_0}, \tag{73}$$

where

$$\Omega_0 = \frac{4}{d}\frac{\pi^{(d-1)/2}}{\Gamma(d/2)}n\overline{\sigma}^{d-1}\sqrt{\frac{2Tm}{m_0(m+m_0)}}. \tag{74}$$

To illustrate the dependence of the tensors $T_{ij} \equiv \{D_{ij}, D_{p,ij}, D_{T,ij}, \chi_{ij}\}$ on dissipation, let us consider a three-dimensional system. In this case, according to equations (65)–(68), $T_{xz} = T_{zx} = T_{yz} = T_{zy} = 0$, in agreement with the symmetry of the problem. As a consequence, there are five relevant elements: the three diagonal (T_{xx}, T_{yy}, and T_{zz}) and two off-diagonal elements (T_{xy}, and T_{yx}). The integral equations (65)–(68) also show that $T_{xx} \neq T_{yy} \neq T_{zz}$ and $T_{xy} \neq T_{yx}$. In figures 1–4, we plot the relevant reduced elements of tensors D_{ij}^*, $D_{p,ij}^*$, $D_{T,ij}^*$ and χ_{ij}^* as functions of the (common) coefficient of restitution $\alpha = \alpha_0$ when $\sigma = \overline{\sigma}$ and $m_0/m = 0.5$. Here, the tensors have been reduced with respect to their values in the elastic case, namely, $D_{ij}^* = D_{ij}/D_0$, $D_{p,ij}^* = D_{p,ij}/D_{p0}$, and $\chi_{ij}^* = \chi_{ij}/\chi_0$, while $D_{T,ij}^* = D_{T,ij}/(x_0 T/m\nu)$. We observe that in general the influence of dissipation on the transport coefficients is quite significant. This means that the deviation of the
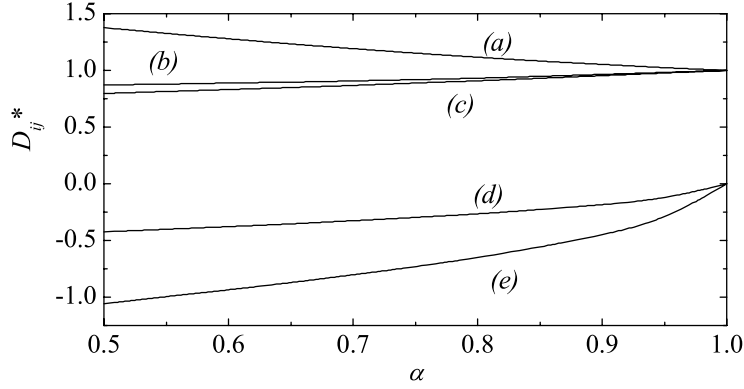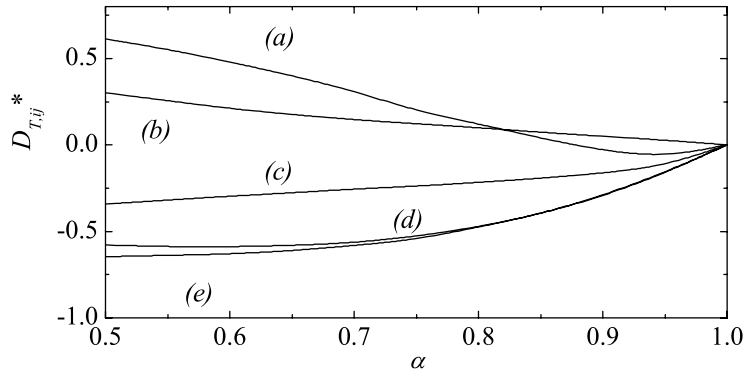
Figure 1. Plot of the reduced elements (a) D_{xx}^*, (b) D_{zz}^*, (c) D_{yy}^*, (d) D_{yx}^*, and (e) D_{xy}^* as functions of the (common) coefficient of restitution $\alpha = \alpha_0$ for a three-dimensional system in the case $\sigma = \overline{\sigma}$ and $m_0/m = 0.5$.
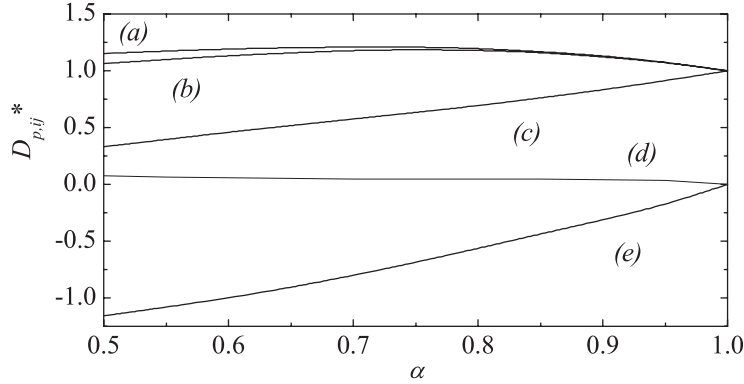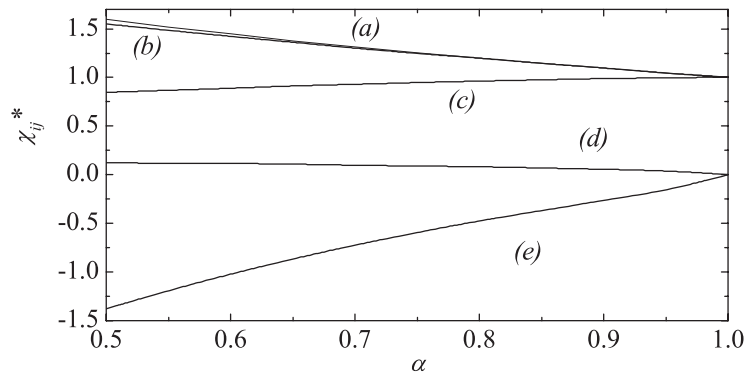

Figure 2. Plot of the reduced elements (a) $D_{p,zz}^*$, (b) $D_{p,yy}^*$, (c) $D_{p,xx}^*$, (d) $D_{p,yx}^*$, and (e) $D_{p,xy}^*$ as functions of the (common) coefficient of restitution $\alpha = \alpha_0$ for a three-dimensional system in the case $\sigma = \overline{\sigma}$ and $m_0/m = 0.5$.

elements T_{ij} from their functional forms for elastic collisions is important for moderate dissipation. It is also apparent that the anisotropy of the system, as measured by the differences $|T_{xx} - T_{yy}|$ and $|T_{yy} - T_{zz}|$, grows with the inelasticity. This anisotropy is much more important in the plane of shear flow ($|T_{xx} - T_{yy}|$) than in the plane perpendicular to the flow velocity ($|T_{yy} - T_{zz}|$).

As expected, the usual Einstein relation between the diffusion and mobility coefficients for ordinary fluids [31] is no longer valid in this non-equilibrium situation for granular gases. A similar conclusion has been found when the gas is under HCS [32]–[34]. There are basically two independent reasons for this violation: the occurrence of different kinetic temperatures between the impurity and gas particles and the inherent non-Newtonian properties of the reference state. The deviation of the ratio D_{ij}^*/χ_{ij}^* from unity is a measure of the violation of Einstein's relation in the USF state. This ratio is plotted in figure 5 for the diagonal elements in the case $\sigma = \overline{\sigma}$ and $m_0/m = 0.5$. As figure 5 clearly shows, the ratio $D_{ij}^*/\chi_{ij}^* \neq 1$ and quickly decays in this situation (when the impurity is lighter than the gas particles) for the yy and zz elements, while the opposite happens

Figure 3. Plot of the reduced elements (a) $D_{T,xy}^*$, (b) $D_{T,xx}^*$, (c) $D_{T,yx}^*$, (d) $D_{T,yy}^*$, and (e) $D_{T,zz}^*$ as functions of the (common) coefficient of restitution $\alpha = \alpha_0$ for a three-dimensional system in the case $\sigma = \overline{\sigma}$ and $m_0/m = 0.5$.
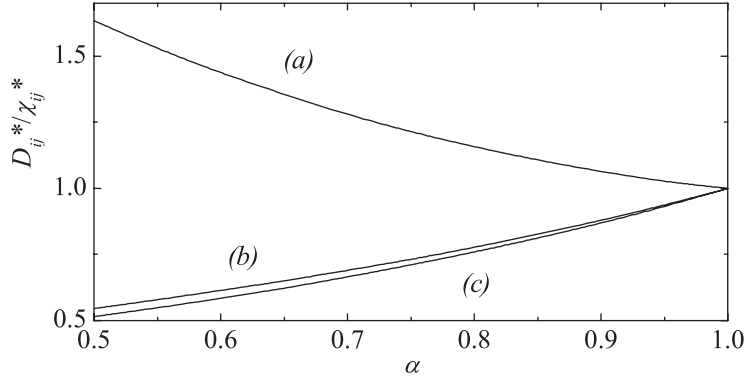

Figure 4. Plot of the reduced elements (a) χ_{zz}^*, (b) χ_{yy}^*, (c) χ_{xx}^*, (d) χ_{yx}^*, and (e) χ_{xy}^* as functions of the (common) coefficient of restitution $\alpha = \alpha_0$ for a three-dimensional system in the case $\sigma = \overline{\sigma}$ and $m_0/m = 0.5$.

for the xx elements. The violation of the Einstein relation obtained here contrasts with some numerical experiments performed by Makse and Kurchan [35] who applied uniform shear to measure diffusivity and mobility in bidisperse mixtures. Their results show that Einstein's relation is verified so that a temperature can be defined in the system by analogy with ordinary fluids. However, this conclusion also disagrees with the lack of equipartition in a granular sheared gas when different particles are present [10].

5. Concluding remarks

In this paper, mass transport of an impurity or intruder immersed in a strongly sheared granular gas at low density has been analysed. We have been interested in a situation where *weak* spatial gradients of concentration, temperature and pressure coexist with a *strong* shear rate. In addition, we have also assumed that the system (gas plus impurity) feels the action of external forces which are considered to be at least of first order in the spatial gradients. Under these conditions, the resulting mass transport process is anisotropic and thus it cannot be described by scalar transport coefficients.
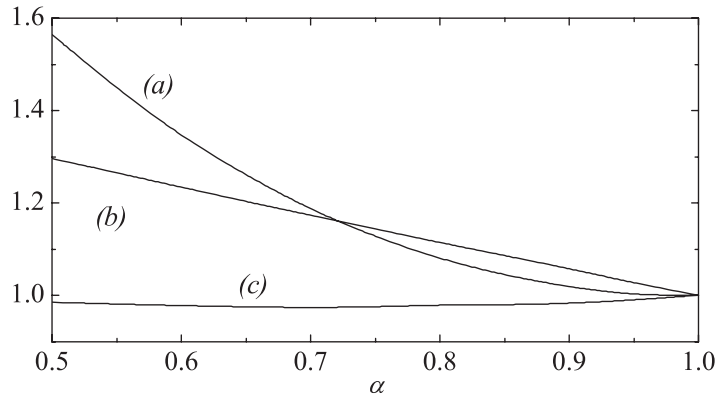
Figure 5. Plot of the ratios (a) D^*_{xx}/χ^*_{xx}, (b) D^*_{zz}/χ^*_{zz}, and (c) D^*_{yy}/χ^*_{yy} as functions of the (common) coefficient of restitution $\alpha = \alpha_0$ for a three-dimensional system in the case $\sigma = \overline{\sigma}$ and $m_0/m = 0.5$.

Instead, it must be described by shear rate dependent tensorial quantities whose explicit determination has been the main objective of this paper.

In the tracer limit, the inelastic Boltzmann and Boltzmann–Lorentz kinetic equations describe the state of the gas and the impurity, respectively. Since the state of the gas slightly deviates from the USF by small spatial gradients, a generalized Chapman–Enskog method has been recently proposed [19, 20] to analyse transport around non-equilibrium states. Here, a similar perturbative scheme has been used to solve the Boltzmann–Lorentz equation to first order in the deviations of the hydrodynamic field gradients from their values in the reference shear flow state $f_0^{(0)}$. As noted in previous works [19, 20], the zeroth-order distribution $f_0^{(0)}$ is not in general stationary and only in very special conditions has a simple relation with the (steady) USF distribution. Since we are mainly interested in determining mass transport of the impurity $\mathbf{j}_0^{(1)}$ in the USF state, for practical purposes the results have been specialized to the steady state conditions, namely, when the hydrodynamic variables satisfy the relation (20). This implies that the reduced shear rate a^* is coupled with the coefficient of restitution α, so that the latter is the relevant parameter of the problem. Under these conditions, $\mathbf{j}_0^{(1)}$ is given by equation (50) where the corresponding set of generalized transport coefficients $\{D_{ij}, D_{p,ij}, D_{T,ij}, \chi_{ij}\}$ are the solutions of the linear integral equations (65)–(68). To get explicit results, a first Sonine polynomial approximation has been considered to estimate some collisional integrals. The reliability of this approach has been assessed in the USF problem, where it has been shown to agree very well with Monte Carlo simulations [12, 28].

The results show that the coefficients $\{D_{ij}, D_{p,ij}, D_{T,ij}, \chi_{ij}\}$ present a complex dependence on the coefficients of restitution α and α_0 and on the masses and sizes of the system. This is clearly illustrated in figures 1–4. The deviations of $\{D_{ij}, D_{p,ij}, D_{T,ij}, \chi_{ij}\}$ from their elastic counterparts are basically due to three different reasons. First, the presence of shear flow modifies the collision frequency of the elastic diffusion problem Ω_0 (defined by equation (74)) by the tensorial term $a_{ij} + \Omega_{ij}$. Second, given that in general the impurity and gas particles are mechanically different, the reference shear flow state of the impurity $f_0^{(0)}$ is completely different from that of the gas particles $f^{(0)}$. This effect gives rise to terms proportional to $(m_0/m)P^*_{ij} - P^*_{0,ij}$. Third, there is a coupling between

Figure 6. Plot of the ratios (a) $D_p/((1/d)D_{p,kk})$, (b) $D/((1/d)D_{kk})$, and (c) $\chi/((1/d)\chi_{kk})$ as functions of the (common) coefficient of restitution $\alpha = \alpha_0$ for a three-dimensional system in the case $\sigma = \overline{\sigma}$ and $m_0/m = 0.5$.

the coefficients $D_{p,ij}$ and $D_{T,ij}$ due to the inherent non-Newtonian features of the USF state of the gas. Each one of the three effects is a different reflection of the dissipation present in the system.

Most of the works [36] on granular mixtures have been based on the CE expansion around an elastic (local) equilibrium state up to the Navier–Stokes order, and therefore they are limited to nearly elastic systems in the USF. A more recent CE expansion [2, 37] around the (local) HCS takes into account energy non-equipartition and provides expressions for the Navier–Stokes transport coefficients of the mixture without any restriction on the level of inelasticity. In particular, the mass transport of impurity is characterized by the single scalar coefficients D, D_p, D_T, and χ. Although the base state considered in [2] and [37] is different from the one chosen here, it is still worthwhile to carry out some comparison between the two descriptions. To that end, let us define the scalars $(1/d)D_{kk}$, $(1/d)D_{p,kk}$, and $(1/d)\chi_{kk}$. These coefficients can be understood as the generalized diffusion coefficient, pressure diffusion coefficient, and mobility coefficient in a strongly sheared mixture. In figure 6, the ratios $D/((1/d)D_{kk})$, $D_p/((1/d)D_{p,kk})$, and $\chi/((1/d)\chi_{kk})$ are plotted versus the (common) coefficient of restitution $\alpha = \alpha_0$ for a three-dimensional system in the case $\sigma = \overline{\sigma}$ and $m_0/m = 0.5$. Here, D, D_p and χ refer to the coefficients obtained in [2] and [32]. As expected, we observe that in general the transport coefficients of the *perturbed* USF state differ from the usual Navier–Stokes coefficients as the collisions become more inelastic. These discrepancies are specially significant in the cases of the diffusion and the pressure diffusion coefficients.

It is apparent that the results presented here are relevant to make a comparison with numerical simulations. In the self-diffusion problem (when the impurity and gas particles are mechanically equivalent), previous results obtained for the diffusion tensor [15] have shown good qualitative agreement with molecular dynamics simulations [16]. Beyond this particular case, to my knowledge no previous studies on the tensors $\{D_{ij}, D_{p,ij}, D_{T,ij}, \chi_{ij}\}$ have been performed. I hope that this paper stimulates the performance of such computer studies to check the relevance of kinetic theory to describe mass transport under shear flow.

A possible application of the results reported in this paper is to study segregation induced by a thermal gradient. Thermal diffusion is caused by the relative motion of the components of a mixture due to the presence of a temperature gradient. Due to this motion, concentration gradients subsequently appear in the mixture producing diffusion that tends to oppose those gradients. In the steady state, $\mathbf{j}_0^{(1)} = \mathbf{0}$ and the thermal diffusion factor provides a segregation criterion. Recent kinetic theory results [38] based on the Navier–Stokes transport coefficients have been able to explain some experimental and/or molecular dynamics segregation results obtained in agitated granular mixtures at large shaking amplitudes [39]. Another possible direction of study is the extension of the present approach to mixtures with finite composition. Given the mathematical difficulties associated with the description of multicomponent systems in far from equilibrium situations, one could perhaps use a kinetic model of the Boltzmann equation. Once the transport coefficients of the mixture are known, a linear stability analysis of the hydrodynamic equations could be carried out to identify the conditions for instabilities at long wavelengths [20]. Finally, it must be noted that the results reported here have been made in the context of a very simple collision model where the coefficients of restitution α and α_0 are constant. However, experiments and simulations show that the coefficients of restitution depend in general on the relative velocity of colliding particles [40]. Recent results [41, 42] derived for these viscoelastic models in the case of the Navier–Stokes transport coefficients show qualitative differences from the ones obtained with the simplifying assumption of a constant coefficient of restitution. For this reason, it would be interesting to extend the present description to this kind of models (where collisions are described by an impact velocity dependent coefficient of restitution) in order to check whether the behaviour predicted here for the generalized transport coefficients also occurs (at least from a qualitative level) in the case of granular gases of viscoelastic particles. As in the Navier–Stokes description [41], explicit results for this more realistic model could be obtained only for small enough dissipation. This contrasts with the results reported here since they are not restricted to any level of inelasticity. Work along the above lines will be done in the near future.

Acknowledgment

Appendix A. Rheological properties in the steady USF

The explicit expressions for the pressure tensors $P_{ij}^* \equiv P_{ij}/p$ and $P_{0,ij}^* \equiv P_{0,ij}/x_0 p$ of the gas and the impurity in the steady USF are provided in this appendix. To get the explicit expressions of the elements of P_{ij}^*, one takes the following Sonine approximation for f:

$$f(\mathbf{V}) \rightarrow f_M(\mathbf{V}) \left[1 + \frac{m}{2T} \left(\frac{P_{k\ell}}{nT} - \delta_{k\ell} \right) \left(V_k V_\ell - \frac{1}{d} V^2 \delta_{k\ell} \right) \right], \qquad (A.1)$$

where

$$f_M(\mathbf{V}) = n \left(\frac{m}{2\pi T} \right)^{d/2} \exp \left(-\frac{mV^2}{2T} \right). \qquad (A.2)$$

By using this approximation, the non-zero elements of P_{ij}^* are given by [21]

$$P_{yy}^* = P_{zz}^* = \cdots = P_{dd}^* = \frac{d+1+(d-1)\alpha}{2d+3-3\alpha}, \tag{A.3}$$

$$P_{xy}^* = -4d\frac{d+1+(d-1)\alpha}{(1+\alpha)(2d+3-3\alpha)^2}a^*, \tag{A.4}$$

$$P_{xx}^* = d - (d-1)P_{yy}^*. \tag{A.5}$$

The relationship between the reduced shear rate $a^* = a/\nu$ (where ν is defined by equation (21)) and the coefficient of restitution α is

$$a^{*2} = \frac{d+2}{32d}\frac{(1+\alpha)(2d+3-3\alpha)^2(1-\alpha^2)}{d+1+(d-1)\alpha}. \tag{A.6}$$

Moreover, the (reduced) cooling rate $\zeta^* = \zeta^{(0)}/\nu$ is

$$\zeta^* = \frac{d+2}{4d}(1-\alpha^2). \tag{A.7}$$

In the case of $P_{0,ij}^*$, one considers the leading Sonine approximation for $f_0^{(0)}$ given by

$$f_0(\mathbf{V}) \rightarrow f_{0,M}(\mathbf{V})\left[1+\frac{m_0}{2T_0}\left(\frac{P_{0,k\ell}}{n_0 T_0}-\delta_{k\ell}\right)\left(V_k V_\ell - \frac{1}{d}V^2\delta_{k\ell}\right)\right], \tag{A.8}$$

where

$$f_{0,M}(\mathbf{V}) = n_0\left(\frac{m_0}{2\pi T_0}\right)^{d/2}\exp\left(-\frac{m_0 V^2}{2T_0}\right). \tag{A.9}$$

Another different approach based on an anisotropic Gaussian distribution has been considered by Lutsko [13]. This latter approximation has the additional advantage of being positive definite. However, the algebraic equations defining the pressure tensor $P_{0,ij}^*$ cannot be solved explicitly since it requires to numerically solve some collision integrals. Here, for the sake of simplicity, we have preferred to estimate the rheological properties of the system by means of Grad's solution (A.8). This allows us to get explicit expressions for $P_{0,ij}^*$. It has been shown, by comparison to DSMC simulations [12, 13], that the results derived from the two approaches compare very well with computer simulations, even for strong dissipation. Using (A.8), the non-zero elements of $P_{0,ij}^*$ can be written as [15]

$$P_{0,yy}^* = P_{0,zz}^* = \cdots = P_{0,dd}^* = -\frac{F+HP_{yy}^*}{G}, \tag{A.10}$$

$$P_{0,xy}^* = \frac{a^* P_{0,yy}^* - HP_{xy}^*}{G}, \tag{A.11}$$

$$P_{0,xx}^* = d\gamma - (d-1)P_{0,yy}^*, \tag{A.12}$$

where $\gamma = T_0/T$ is the temperature ratio and

$$F = \frac{\sqrt{2}}{2d} \left(\frac{\overline{\sigma}}{\sigma}\right)^{d-1} \mu_0 \left(\frac{1+\theta}{\theta^3}\right)^{1/2} (1+\alpha_0) \left[1 + \frac{\mu}{2}(d-1)(1+\theta)(1+\alpha_0)\right], \tag{A.13}$$

$$G = -\frac{\sqrt{2}}{4d} \left(\frac{\overline{\sigma}}{\sigma}\right)^{d-1} \mu \left(\frac{1}{\theta(1+\theta)}\right)^{1/2} (1+\alpha_0)$$
$$\times \ \{2[(d+2)\theta + d + 3] - 3\mu(1+\theta)(1+\alpha_0)\}, \tag{A.14}$$

$$H = \frac{\sqrt{2}}{4d} \left(\frac{\overline{\sigma}}{\sigma}\right)^{d-1} \mu_0 \left(\frac{1}{\theta(1+\theta)}\right)^{1/2} (1+\alpha_0) \left[3\mu(1+\theta)(1+\alpha_0) - 2\right]. \tag{A.15}$$

Here, $\theta = m_0/m\gamma$, $\mu = m/(m+m_0)$, and $\mu_0 = 1 - \mu = m_0/(m+m_0)$. The temperature ratio γ is determined from the condition

$$\gamma = \frac{\zeta^* P^*_{0,xy}}{\zeta^*_0 P^*_{xy}}, \tag{A.16}$$

where the 'cooling rate' $\zeta^*_0 = \zeta_0/\nu$ for the impurity is given by

$$\zeta^*_0 = \frac{(d+2)\sqrt{2}}{2d} \left(\frac{\overline{\sigma}}{\sigma}\right)^{d-1} \mu \left(\frac{1+\theta}{\theta}\right)^{1/2} (1+\alpha_0) \left[1 - \frac{\mu}{2}(1+\theta)(1+\alpha_0)\right]. \tag{A.17}$$

For elastic collisions ($\alpha = \alpha_0 = 1$), equations (A.3)–(A.5) and (A.10)–(A.15) lead to $P^*_{ij} = P^*_{0,ij} = \delta_{ij}$. In this case, as expected, the solution to equation (A.16) is $\gamma = 1$. In addition, if we assume that particles of the gas and the impurity are mechanically equivalent (i.e., $m = m_0$, $\sigma = \sigma_0$, and $\alpha = \alpha_0$), then $P^*_{ij} = P^*_{0,ij}$, $\zeta^* = \zeta^*_0$ and so $\gamma = 1$. Beyond these two limit cases, the temperatures T and T_0 are different so that there is a violation of energy equipartition.

Appendix B. Behaviour of the zeroth-order pressure tensors near the steady state

This appendix addresses the behaviour of the pressure tensors P^*_{ij} and $P^*_{0,ij}$ of the gas particles and the impurity, respectively, near the steady state. The behaviour of the second-degree moment P^*_{ij} was studied in [20] in the three-dimensional case ($d = 3$). The extension to an arbitrary number of dimensions is straightforward and here only the final expressions are displayed. The behaviour of the yy element is given by

$$\left(\frac{\partial P^*_{yy}}{\partial a^*}\right)_s = 4P^*_{yy} \frac{a^*_s \Delta + P^*_{xy}}{2a^{*2}_s \Delta + d(2\beta + \zeta^*)}, \tag{B.1}$$

where $a^*_s(\alpha)$ is the steady state value of $a^*(p,T)$ given by equation (A.6),

$$\beta = \frac{1+\alpha}{2} \left[1 - \frac{d-1}{2d}(1-\alpha)\right], \tag{B.2}$$

and $\Delta \equiv \left(\partial P^*_{xy}/\partial a^*\right)_s$ is the real root of the cubic equation

$$2a^{*4}_s \Delta^3 + 4da^{*2}_s(\zeta^* + \beta)\Delta^2 + \frac{d^2}{2}(7\zeta^{*2} + 14\zeta^*\beta + 4\beta^2)\Delta$$
$$+ d^2\beta(\zeta^* + \beta)^{-2}(2\beta^2 - 2\zeta^{*2} - \beta\zeta^*). \tag{B.3}$$

In the above equations, it is understood that all the quantities are computed in the steady state.

Let us consider now the elements of $P_{0,ij}^*$. In dimensionless form, they verify the equation

$$-\left(\frac{2}{d}a^*P_{xy}^* + \zeta^*\right)\left(1 - \frac{1}{2}a^*\frac{\partial}{\partial a^*}\right)P_{0,ij}^* + a_{i\ell}^*P_{0,j\ell}^* + a_{j\ell}^*P_{0,i\ell}^* = B_{ij}^*, \qquad (B.4)$$

where

$$B_{ij}^* = \frac{m_0}{x_0 \nu p}\int \mathrm{d}\mathbf{c}\, c_i c_j\, J_0[f_0^{(0)}, f^{(0)}]. \qquad (B.5)$$

Upon deriving equation (B.4), use has been made of the fact that in the hydrodynamic regime the dimensionless pressure tensor depends on p and T only through its dependence on the reduced shear rate $a^* = a/\nu(p,T)$ (see equations (55) and (56)). The collisional moment B_{ij}^* can be estimated by using Grad's approximation (A.8) with the result [15]

$$B_{ij}^* = Y\delta_{ij} + X_0 P_{0,ij}^* + X P_{ij}^*, \qquad (B.6)$$

where

$$Y = \frac{d+2}{2\sqrt{2}d}\left(\frac{\overline{\sigma}}{\sigma}\right)^{d-1}\mu_0(1+\alpha_0)\left(\frac{1+\theta}{\theta}\right)^{3/2}\left[\frac{\lambda_0}{d+2} + \frac{d}{d+3}\mu(1+\alpha_0)\right], \qquad (B.7)$$

$$X_0 = -\frac{d+2}{\sqrt{2}d}\left(\frac{\overline{\sigma}}{\sigma}\right)^{d-1}\mu_0(1+\alpha_0)\left[\theta(1+\theta)\right]^{-1/2}\left[1 + \frac{(d+3)}{2(d+2)}\frac{1+\theta}{\theta}\lambda_0\right]\gamma^{-1}, \qquad (B.8)$$

$$X = \frac{d+2}{\sqrt{2}d}\left(\frac{\overline{\sigma}}{\sigma}\right)^{d-1}\mu_0(1+\alpha_0)\left[\theta(1+\theta)\right]^{-1/2}\left[1 - \frac{(d+3)}{2(d+2)}(1+\theta)\lambda_0\right], \qquad (B.9)$$

with

$$\lambda_0 = \frac{2}{1+\theta} - \frac{3}{d+3}\mu_0(1+\alpha_0). \qquad (B.10)$$

Let us consider the elements $P_{0,xy}^*$ and $P_{0,yy}^*$. From equation (B.4), one gets

$$-\left(\frac{2}{d}a^*P_{xy}^* + \zeta^*\right)\left(1 - \frac{1}{2}a^*\frac{\partial}{\partial a^*}\right)P_{0,xy}^* + a^*P_{0,yy}^* - X_0 P_{0,xy}^* = X P_{xy}^*, \quad (B.11)$$

$$-\left(\frac{2}{d}a^*P_{xy}^* + \zeta^*\right)\left(1 - \frac{1}{2}a^*\frac{\partial}{\partial a^*}\right)P_{0,yy}^* - X_0 P_{0,yy}^* = Y + X P_{yy}^*. \qquad (B.12)$$

This set of equations can be written as

$$\frac{\partial P_{0,xy}^*}{\partial a^*} = 2\frac{(\zeta^* + (2/d)a^*P_{xy}^* + X_0)P_{0,xy}^* - a^*P_{0,yy}^* + X P_{xy}^*}{a^*\left(\zeta^* + (2/d)a^*P_{xy}^*\right)}, \qquad (B.13)$$

$$\frac{\partial P_{0,yy}^*}{\partial a^*} = 2\frac{(\zeta^* + (2/d)a^*P_{xy}^* + X_0)P_{0,yy}^* + Y + X P_{yy}^*}{a^*\left(\zeta^* + (2/d)a^*P_{xy}^*\right)}. \qquad (B.14)$$

It must be remarked that the temperature ratio $\gamma = T_0/T$ is also a function of a^* in the hydrodynamic solution. Since $P_{0,xx}^* + (d-1)P_{0,yy}^* = d\gamma$, the corresponding equation for the derivative $(\partial\gamma/\partial a^*)$ can be obtained from equation (B.4) as

$$\gamma - \frac{a^*}{2}\frac{\partial\gamma}{\partial a^*} = \frac{\gamma\zeta_0^* + (2/d)a^*P_{0,xy}^*}{\zeta^* + (2/d)a^*P_{xy}^*}, \tag{B.15}$$

where ζ_0^* is given by equation (A.17). The set of three coupled equations (B.13)–(B.15) has a singular point corresponding to the steady state solution, i.e., when $a^*(p,T) = a_s^*(\alpha)$. In this limit $(a^* \to a_s^*)$, the numerators and denominators of equations (B.13)–(B.15) vanish. The limit can be evaluated by means of l'Hopital's rule. Thus, when one differentiates with respect to a^* the numerators and denominators of equations (B.13)–(B.15) and then takes the limit $a \to a_s^*$, one gets the relations

$$\left(\frac{\partial P_{0,xy}^*}{\partial a^*}\right)_s = \left(\frac{1}{2}a_s^*\chi_s - X_0\right)^{-1}\left\{\chi_s P_{0,xy}^* + \left(X_0'P_{0,xy}^* + X'P_{xy}^*\right)(\partial\gamma/\partial a^*)_s\right.$$
$$\left. - \left[P_{0,yy}^* + a_s^*(\partial P_{0,yy}^*/\partial a^*)_s\right] + X(\partial P_{xy}^*/\partial a^*)_s\right\}, \tag{B.16}$$

$$\left(\frac{\partial P_{0,yy}^*}{\partial a^*}\right)_s = \frac{\chi_s P_{0,yy}^* + \left(Y' + X_0'P_{0,yy}^* + X'P_{yy}^*\right)(\partial\gamma/\partial a^*)_s + X(\partial P_{yy}^*/\partial a^*)_s}{(1/2)a_s^*\chi_s - X_0}, \tag{B.17}$$

$$\left(\frac{\partial\gamma}{\partial a^*}\right)_s = \frac{\chi_s\gamma - (2/d)\left[P_{0,xy}^* + a_s^*(\partial P_{0,xy}^*/\partial a^*)_s\right]}{(1/2)a_s^*\chi_s + \zeta_0^* + \gamma\zeta_0'}, \tag{B.18}$$

where the subscript s means that the derivatives are computed in the steady state. In addition, we have introduced the quantities

$$\chi_s = \frac{2}{d}\left[P_{xy}^* + a_s^*(\partial P_{xy}^*/\partial a^*)_s\right], \tag{B.19}$$

and $Y' \equiv (\partial Y/\partial\gamma)$, $X' \equiv (\partial X/\partial\gamma)$, $X_0' \equiv (\partial Y/\partial\gamma)$, and $\zeta_0' \equiv (\partial\zeta_0^*/\partial\gamma)$. Note that all the quantities appearing on the right-hand side of equations (B.16)–(B.18) are evaluated in the steady state. This set of equations can be easily solved to give the corresponding derivatives. The result is

$$\left(\frac{\partial\gamma}{\partial a^*}\right)_s = \frac{\Lambda_1}{\Lambda_2}, \tag{B.20}$$

where

$$\Lambda_1 = d\left(\frac{1}{2}a_s^*\chi_s - X_0\right)\left\{\left(\frac{1}{2}a_s^*\chi_s - X_0\right)\left(\chi_s\gamma - \frac{2}{d}P_{0,xy}^*\right)\right.$$
$$\left. - \frac{2}{d}a_s^*\left[\chi_s P_{0,xy}^* - P_{0,yy}^* + X(\partial P_{xy}^*/\partial a^*)_s\right]\right\}$$
$$+ 2a_s^{*2}\left[\chi_s P_{0,yy}^* + X(\partial P_{yy}^*/\partial a^*)_s\right], \tag{B.21}$$

$$\Lambda_2 = d\left(\frac{1}{2}a_s^*\chi_s - X_0\right)\left[\left(\frac{1}{2}a_s^*\chi_s - X_0\right)\left(\zeta_0^* + \frac{1}{2}\chi_s + \gamma\zeta_0'\right)\right.$$
$$\left. + \frac{2}{d}a_s^*\left(X_0'P_{0,xy}^* + X'P_{yy}^*\right)\right] - 2a_s^{*2}\left(Y' + X_0'P_{0,yy}^* + X'P_{yy}^*\right). \tag{B.22}$$

Substitution of equation (B.20) into equations (B.16) and (B.17) gives the derivatives $(\partial_{a^*} P_{0,xy}^*)_s$ and $(\partial_{a^*} P_{0,yy}^*)_s$, respectively.

Appendix C. Chapman–Enskog-like expansion

In this appendix, we provide some technical details on the determination of the first-order approximation $f_0^{(1)}$ by means of the Chapman–Enskog-like expansion. Inserting the expansions (29) and (31) into equation (23), one gets the kinetic equation for $f_0^{(1)}$

$$\left(\partial_t^{(0)} - aV_y \frac{\partial}{\partial V_x}\right) f_0^{(1)} - J_0[f_0^{(1)}, f^{(0)}] = -\left[\partial_t^{(1)} + (\mathbf{V} + \mathbf{u}_s) \cdot \nabla + \frac{\mathbf{F}_0}{m_0} \cdot \frac{\partial}{\partial \mathbf{V}}\right] f^{(0)}. \quad \text{(C.1)}$$

The velocity dependence on the right side of equation (C.1) can be obtained from the macroscopic balance equations (24)–(26) to first order in the gradients. They are given by

$$\partial_t^{(1)} x_0 = -(\mathbf{u}_s + \delta \mathbf{u}) \cdot \nabla x_0, \quad \text{(C.2)}$$

$$\partial_t^{(1)} \delta \mathbf{u} = -(\mathbf{u}_s + \delta \mathbf{u}) \cdot \nabla \delta \mathbf{u} - \frac{1}{\rho} \left(\nabla \cdot \mathsf{P}^{(0)} - n\mathbf{F}\right), \quad \text{(C.3)}$$

$$\partial_t^{(1)} p = -(\mathbf{u}_s + \delta \mathbf{u}) \cdot \nabla p - p(\nabla \cdot \delta \mathbf{u} + \zeta^{(1)}) - \frac{2}{d} \left(aP_{xy}^{(1)} + \mathsf{P}^{(0)} : \nabla \delta \mathbf{u}\right), \quad \text{(C.4)}$$

$$\partial_t^{(1)} T = -(\mathbf{u}_s + \delta \mathbf{u}) \cdot \nabla T - \frac{2}{dn} \left(aP_{xy}^{(1)} + \mathsf{P}^{(0)} : \nabla \delta \mathbf{u}\right) - p\zeta^{(1)}, \quad \text{(C.5)}$$

where $\rho = mn$ is the mass density,

$$P_{ij}^{(1)} = \int d\mathbf{c}\, mc_ic_j f^{(1)}(\mathbf{c}), \quad \text{(C.6)}$$

and

$$\zeta^{(1)} = -\frac{1}{dp} \int d\mathbf{c}\, mc^2 \left(J[f^{(0)}, f^{(1)}] + J[f^{(1)}, f^{(0)}]\right). \quad \text{(C.7)}$$

Use of equations (C.2)–(C.5) in equation (C.1) yields

$$\left(\partial_t^{(0)} - aV_y \frac{\partial}{\partial V_x}\right) f_0^{(1)} - J_0[f_0^{(1)}, f^{(0)}] = \mathbf{A}_0 \cdot \nabla x_0 + \mathbf{B}_0 \cdot \nabla p$$

$$+ \mathbf{C}_0 \cdot \nabla T + \mathsf{D}_0 : \nabla \delta \mathbf{u} + \mathbf{E}_0 \cdot \boldsymbol{\mathcal{F}}$$

$$+ \zeta^{(1)}(p\partial_p + T\partial_T) f_0^{(0)} + J_0[f_0^{(0)}, f^{(1)}], \quad \text{(C.8)}$$

where we have introduced the force $\boldsymbol{\mathcal{F}} = \mathbf{F}_0 - (m_0/m)\mathbf{F}$. The coefficients of the field gradients on the right-hand side of (C.8) are functions of $\mathbf{c}$ and the hydrodynamic fields.

They are given by

$$A_{0,i}(\mathbf{c}) = -\frac{\partial f_0^{(0)}}{\partial x_0} c_i, \tag{C.9}$$

$$B_{0,i}(\mathbf{c}) = -\frac{\partial f_0^{(0)}}{\partial p} c_i + \frac{1}{\rho} \frac{\partial f_0^{(0)}}{\partial \delta u_j} \frac{\partial P_{ij}^{(0)}}{\partial p}, \tag{C.10}$$

$$C_{0,i}(\mathbf{c}) = -\frac{\partial f_0^{(0)}}{\partial T} c_i + \frac{1}{\rho} \frac{\partial f_0^{(0)}}{\partial \delta u_j} \frac{\partial P_{ij}^{(0)}}{\partial T}, \tag{C.11}$$

$$D_{0,ij}(\mathbf{c}) = p \frac{\partial f_0^{(0)}}{\partial p} \delta_{ij} - \frac{\partial f_0^{(0)}}{\partial \delta u_i} c_j + \frac{2}{dp} \left(P_{ij}^{(0)} - a\eta_{xyij} \right) \left(p \frac{\partial}{\partial p} + T \frac{\partial}{\partial T} \right) f_0^{(0)}, \tag{C.12}$$

$$E_{0,i}(\mathbf{c}) = -\frac{1}{m_0} \frac{\partial f_0^{(0)}}{\partial c_i}. \tag{C.13}$$

Upon writing equation (C.12) use has been made of the expression of the pressure tensor $P_{ij}^{(1)}$ of the gas [20]

$$P_{ij}^{(1)} = -\eta_{ijk\ell} \frac{\partial \delta u_k}{\partial r_\ell}, \tag{C.14}$$

where $\eta_{ijk\ell}$ is the viscosity tensor. Moreover, the expression for the distribution $f^{(1)}$ derived in [20] is given by equation (47).

The solution to equation (C.8) has the form

$$f_0^{(1)} = \boldsymbol{\mathcal{A}}_0 \cdot \nabla x_0 + \boldsymbol{\mathcal{B}}_0 \cdot \nabla p + \boldsymbol{\mathcal{C}}_0 \cdot \nabla T + \mathcal{D}_0 : \nabla \delta \mathbf{u} + \boldsymbol{\mathcal{E}}_0 \cdot \boldsymbol{\mathcal{F}}. \tag{C.15}$$

The coefficients $\boldsymbol{\mathcal{A}}_0$, $\boldsymbol{\mathcal{B}}_0$, $\boldsymbol{\mathcal{C}}_0$, $\mathcal{D}_0$, and $\boldsymbol{\mathcal{E}}_0$ are functions of the peculiar velocity and the hydrodynamic fields x_0, p, T, and $\delta \mathbf{u}$. The cooling rate $\zeta^{(0)}$ depends on space through its dependence on p and T. Moreover, there are contributions from $\partial_t^{(0)}$ acting on the pressure, temperature, and velocity gradients given by

$$\partial_t^{(0)} \nabla p = -\nabla \left(\frac{2}{d} a P_{xy}^{(0)} + p\zeta^{(0)} \right)$$
$$= -\left(\frac{2a}{d} \frac{\partial P_{xy}^{(0)}}{\partial p} + 2\zeta^{(0)} \right) \nabla p - \left(\frac{2a}{d} \frac{\partial P_{xy}^{(0)}}{\partial T} - \frac{1}{2} \frac{p\zeta^{(0)}}{T} \right) \nabla T, \tag{C.16}$$

$$\partial_t^{(0)} \nabla T = -\nabla \left(\frac{2T}{dp} a P_{xy}^{(0)} + \zeta^{(0)} \right)$$
$$= \left[\frac{2aT}{dp^2} \left(1 - p\frac{\partial}{\partial p} \right) P_{xy}^{(0)} - \frac{T\zeta^{(0)}}{p} \right] \nabla p$$
$$- \left[\frac{2a}{dp} \left(1 + T\frac{\partial}{\partial T} \right) P_{xy}^{(0)} + \frac{1}{2} \zeta^{(0)} \right] \nabla T, \tag{C.17}$$

$$\partial_t^{(0)} \nabla_i \delta u_j = \nabla_i \partial_t^{(0)} \delta u_j = -a_{jk} \nabla_i \delta u_k. \tag{C.18}$$

The corresponding integral equations (42)–(46) can be obtained when one identifies coefficients of independent gradients in (C.8) and takes into account equations (C.16)–(C.18) and the mathematical property

$$
\begin{aligned}
\partial_t^{(0)} X &= \frac{\partial X}{\partial p} \partial_t^{(0)} p + \frac{\partial X}{\partial T} \partial_t^{(0)} T + \frac{\partial X}{\partial \delta u_i} \partial_t^{(0)} \delta u_i \\
&= -\left(\frac{2}{\mathrm{d}p} a P_{xy}^{(0)} + \zeta^{(0)} \right) \left(p \frac{\partial}{\partial p} + T \frac{\partial}{\partial T} \right) X + a_{ij} \delta u_j \frac{\partial X}{\partial c_i},
\end{aligned}
\tag{C.19}
$$

where in the last step we have taken into account that X depends on $\delta\mathbf{u}$ through $\mathbf{c} = \mathbf{V} - \delta\mathbf{u}$.

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