Energy Production Rates of Multicomponent Granular Gases of Rough Particles. A Unified View of Hard-Disk and Hard-Sphere Systems

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Abstract. Granular gas mixtures modeled as systems of inelastic and rough particles, either hard disks on a plane or hard spheres, are considered. Both classes of systems are embedded in a three-dimensional space ($d = 3$) but, while in the hard-sphere case the translational and angular velocities are vectors with the same dimensionality (and thus there are $d_{\text{tr}} = 3$ translational and $d_{\text{rot}} = 3$ rotational degrees of freedom), in the hard-disk case the translational velocity vectors are planar (i.e., $d_{\text{tr}} = 2$ translational degrees of freedom) and the angular velocity vectors are orthogonal to the motion plane (i.e., $d_{\text{rot}} = 1$ rotational degree of freedom). This complicates a unified presentation of both classes of systems, in contrast to what happens for smooth, spinless particles, where a treatment of d-dimensional spheres is possible. In this paper, a kinetic-theory derivation of the (collisional) energy production rates ξ_{ij}^{tr} and ξ_{ij}^{rot} (where the indices i and j label different components) in terms of the numbers of degrees of freedom d_{tr} and d_{rot} is presented. Known hard-sphere and hard-disk expressions are recovered by particularizing to $(d_{\text{tr}}, d_{\text{rot}}) = (3, 3)$ and $(d_{\text{tr}}, d_{\text{rot}}) = (2, 1)$, respectively. Moreover, in the case of spinless particles with $d = d_{\text{tr}}$, known energy production rates $\xi_{ij}^{\text{tr}} = \xi_{ij}$ of smooth d-dimensional spheres are also recovered.

INTRODUCTION

A "gas" made of identical and smooth hard disks or spheres with a constant coefficient of normal restitution is perhaps the simplest and most widely used model of a granular gas [1–8]. On the other hand, the mesoscopic or macroscopic nature of the "grains" may ask for a refinement of the model by allowing for particle-particle surface friction or "roughness" (usually accounted for by a constant coefficient of tangential restitution) [9–47], polydispersity (i.e., assuming that the particles belong to more than one component, each one characterized by different mechanical properties) [48–66], or both [67–74].

An interesting feature of multicomponent gases of rough disks or spheres is the general breakdown of energy equipartition, even in homogeneous and isotropic states (driven or undriven). This is characterized by unequal translational (T_i^{tr}) and rotational (T_i^{rot}) temperatures associated with each component i. The rate of change of the translational (rotational) mean kinetic energy of particles of component i due to collisions with particles of component j defines the energy production rate ξ_{ij}^{tr} (ξ_{ij}^{rot}). By means of kinetic-theory tools, the production rates ξ_{ij}^{tr} and ξ_{ij}^{rot} have been derived separately for disks [74] and spheres [70] as functions of T_i^{tr}, T_j^{tr}, T_i^{rot}, T_j^{rot}, and of the mechanical parameters (masses, diameters, moments of inertia, and coefficients of normal and tangential restitution) for each pair ij.

Whereas in the case of smooth, spinless particles a generic kinetic-theory treatment of d-dimensional hard spheres is possible [56, 58, 62, 63, 75], this is far less straightforward if particles have a rotational or angular motion, in addition to the translational motion of the center of mass. In fact, mathematical operations such as the cross product of *two* vectors (and hence mechanical quantities such as angular momentum and torque) are, in general, meaningful in a three-dimensional space ($d = 3$) only. Furthermore, the existence of surface friction or roughness establishes a neat separation between the cases of disks on a plane and spheres. Both classes of particles are embedded in a common three-dimensional space, but spinning spheres have $d_{\text{tr}} = 3$ translational plus $d_{\text{rot}} = 3$ rotational degrees of freedom,

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while spinning disks on a plane have $d_{\rm tr} = 2$ translational and $d_{\rm rot} = 1$ rotational degrees of freedom.

The aim of this work is to unify the derivations of $\xi_{ij}^{\rm tr}$ and $\xi_{ij}^{\rm rot}$ for disks [74] and spheres [70] so that they depend parametrically on both $d_{\rm tr}$ and $d_{\rm rot}$. On the one hand, particularization to smooth particles with $d_{\rm tr} = d$ allows us to recover known results for an arbitrary number d of spatial dimensions [56, 58, 62, 63]. On the other hand, in the case of rough particles, particularization to $(d_{\rm tr}, d_{\rm rot}) = (2, 1)$ and $(d_{\rm tr}, d_{\rm rot}) = (3, 3)$ recovers previous results for disks [74] and spheres [70], respectively.

BINARY COLLISIONS

Let us consider the binary collision of two hard spheres (or disks) of masses m_i and m_j, diameters σ_i and σ_j, and moments of inertia I_i and I_j. Before collision, the particles rotate with angular velocities $\boldsymbol{\omega}_i$ and $\boldsymbol{\omega}_j$, while their respective centers of mass move with translational velocities $\mathbf{v}_i$ and $\mathbf{v}_j$. This is sketched in Fig. 1, where $\widehat{\boldsymbol{\sigma}} \equiv (\mathbf{r}_j - \mathbf{r}_i)/|\mathbf{r}_j - \mathbf{r}_i|$ is a unit vector pointing from the center of sphere i to the center of sphere j and $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$ is the relative velocity of the centers of mass. The relative velocity ($\mathbf{w}_{ij}$) of the points of the spheres or disks which are in contact at the collision is

$$\mathbf{w}_{ij} = \mathbf{v}_{ij} - \widehat{\boldsymbol{\sigma}} \times \mathbf{S}_{ij}, \quad \mathbf{S}_{ij} \equiv \frac{\sigma_i}{2}\boldsymbol{\omega}_i + \frac{\sigma_j}{2}\boldsymbol{\omega}_j. \tag{1}$$

In the case of spheres, the vector $\widehat{\boldsymbol{\sigma}} \times \mathbf{S}_{ij}$ points in any direction of the three-dimensional space. In the case of disks, however, $\widehat{\boldsymbol{\sigma}} \times \mathbf{S}_{ij} = S_{ij}\widehat{\boldsymbol{\sigma}}_\perp$ lies on the plane of motion, where $\widehat{\boldsymbol{\sigma}}_\perp = \widehat{\boldsymbol{\sigma}} \times \widehat{\mathbf{z}} = \widehat{\sigma}_y\widehat{\mathbf{x}} - \widehat{\sigma}_x\widehat{\mathbf{y}}$.

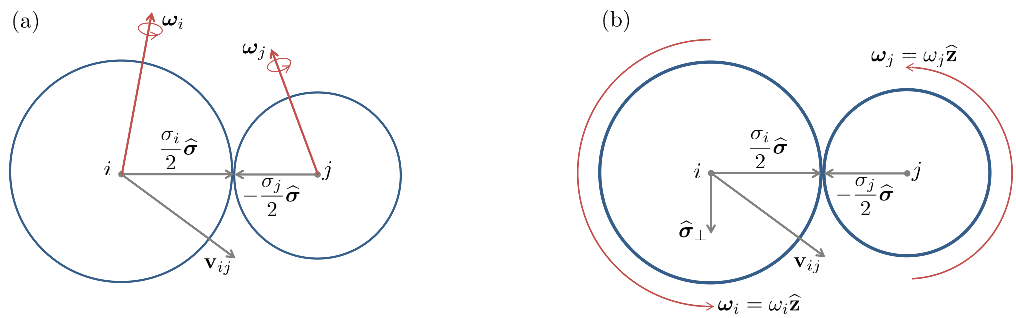

FIGURE 1. Sketch of the precollisional quantities of particles i and j in the frame of reference solidary with particle j. In panel (a) (hard spheres), the angular velocities $\boldsymbol{\omega}_i$ and $\boldsymbol{\omega}_j$ can point in any direction of the three-dimensional space. In panel (b) (hard disks), the angular velocities $\boldsymbol{\omega}_i$ and $\boldsymbol{\omega}_j$ are orthogonal to the plane (xy) of translational motion.

The relative velocity $\mathbf{w}_{ij}$ can be decomposed into a normal component (parallel to $\widehat{\boldsymbol{\sigma}}$) and a tangential component (orthogonal to $\widehat{\boldsymbol{\sigma}}$):

$$\mathbf{w}_{ij} = (\mathbf{w}_{ij} \cdot \widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}} - \widehat{\boldsymbol{\sigma}} \times (\widehat{\boldsymbol{\sigma}} \times \mathbf{w}_{ij}). \tag{2}$$

In the case of disks, $-\widehat{\boldsymbol{\sigma}} \times (\widehat{\boldsymbol{\sigma}} \times \mathbf{w}_{ij}) = (\mathbf{w}_{ij} \cdot \widehat{\boldsymbol{\sigma}}_\perp)\widehat{\boldsymbol{\sigma}}_\perp$. After collision, the normal and tangential components of $\mathbf{w}_{ij}$ are modified by constant factors α_{ij} (coefficient of normal restitution) and β_{ij} (coefficient of tangential restitution), respectively, i.e.,

$$\mathfrak{B}_{ij,\widehat{\boldsymbol{\sigma}}}\mathbf{w}_{ij} \cdot \widehat{\boldsymbol{\sigma}} = -\alpha_{ij}\mathbf{w}_{ij} \cdot \widehat{\boldsymbol{\sigma}}, \quad \mathfrak{B}_{ij,\widehat{\boldsymbol{\sigma}}}\widehat{\boldsymbol{\sigma}} \times \mathbf{w}_{ij} = -\beta_{ij}\widehat{\boldsymbol{\sigma}} \times \mathbf{w}_{ij}, \tag{3}$$

where the operator $\mathfrak{B}_{ij,\widehat{\boldsymbol{\sigma}}}$ acting on a precollisional quantity gives the associated postcollisional quantity as the result of a collision with unit vector $\widehat{\boldsymbol{\sigma}}$ between particles of components i and j. The coefficient of normal restitution ranges from $\alpha_{ij} = 0$ (perfectly inelastic particles) to $\alpha_{ij} = 1$ (perfectly elastic particles), while the coefficient of tangential restitution ranges from $\beta_{ij} = -1$ (perfectly smooth particles) to $\beta_{ij} = 1$ (perfectly rough particles).

Equation (3), together with the laws of conservation of linear and angular momenta, yield the following collision rules [30, 70, 74]

$$\mathfrak{B}_{ij,\widehat{\boldsymbol{\sigma}}}\mathbf{v}_i = \mathbf{v}_i - \frac{1}{m_i}\mathbf{Q}_{ij}, \quad \mathfrak{B}_{ij,\widehat{\boldsymbol{\sigma}}}\mathbf{v}_j = \mathbf{v}_j + \frac{1}{m_j}\mathbf{Q}_{ij}, \quad \mathfrak{B}_{ij,\widehat{\boldsymbol{\sigma}}}\boldsymbol{\omega}_i = \boldsymbol{\omega}_i - \frac{\sigma_i}{2I_i}\widehat{\boldsymbol{\sigma}} \times \mathbf{Q}_{ij}, \quad \mathfrak{B}_{ij,\widehat{\boldsymbol{\sigma}}}\boldsymbol{\omega}_j = \boldsymbol{\omega}_j - \frac{\sigma_j}{2I_j}\widehat{\boldsymbol{\sigma}} \times \mathbf{Q}_{ij}, \tag{4}$$

where

$$\mathbf{Q}_{ij} = m_{ij}\overline{\alpha}_{ij}(\mathbf{v}_{ij} \cdot \widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}} + m_{ij}\overline{\beta}_{ij}\left[\mathbf{v}_{ij} - (\mathbf{v}_{ij} \cdot \widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}} - \widehat{\boldsymbol{\sigma}} \times \mathbf{S}_{ij}\right]. \tag{5}$$

TABLE 1. Relevant collisional changes.

$\psi_{ij}(\mathbf{c}_i,\mathbf{c}_j)$	$\left(\mathfrak{B}_{ij,\widehat{\sigma}}-1\right)\psi_{ij}(\mathbf{c}_i,\mathbf{c}_j)$
$m_i\mathbf{v}_i$	$-m_{ij}\overline{\alpha}_{ij}(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}-m_{ij}\overline{\beta}_{ij}\left[\mathbf{v}_{ij}-(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}-\widehat{\boldsymbol{\sigma}}\times\mathbf{S}_{ij}\right]$
$I_i\omega_i$	$-\frac{m_{ij}\sigma_i}{2}\overline{\beta}_{ij}\left[\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij}+\mathbf{S}_{ij}-(\mathbf{S}_{ij}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}\right]$
$m_iv_i^2$	$\frac{m_{ij}^2\overline{\alpha}_{ij}^2}{m_i}(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}})^2-2m_{ij}\overline{\alpha}_{ij}(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}})(\mathbf{v}_i\cdot\widehat{\boldsymbol{\sigma}})+\frac{m_{ij}^2\overline{\beta}_{ij}^2}{m_i}\left[(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})^2+(\widehat{\boldsymbol{\sigma}}\times\mathbf{S}_{ij})^2+2(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})\cdot\mathbf{S}_{ij}\right]$ $-2m_{ij}\overline{\beta}_{ij}\left[(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})\cdot(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_i)+(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_i)\cdot\mathbf{S}_{ij}\right]$
$I_i\omega_i^2$	$\frac{m_{ij}^2\overline{\beta}_{ij}^2}{m_i\kappa_i}\left[(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})^2+(\widehat{\boldsymbol{\sigma}}\times\mathbf{S}_{ij})^2+2(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})\cdot\mathbf{S}_{ij}\right]-m_{ij}\overline{\beta}_{ij}\sigma_i\left[(\widehat{\boldsymbol{\sigma}}\times\mathbf{S}_{ij})\cdot(\widehat{\boldsymbol{\sigma}}\times\boldsymbol{\omega}_i)+(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})\cdot\boldsymbol{\omega}_i\right]$
$m_iv_i^2+m_jv_j^2$	$-m_{ij}\left(1-\alpha_{ij}^2\right)(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}})^2+m_{ij}\overline{\beta}_{ij}^2\left[(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})^2+(\widehat{\boldsymbol{\sigma}}\times\mathbf{S}_{ij})^2+2(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})\cdot\mathbf{S}_{ij}\right]$ $-2m_{ij}\overline{\beta}_{ij}\left[(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})^2+(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})\cdot\mathbf{S}_{ij}\right]$
$I_i\omega_i^2+I_j\omega_j^2$	$\frac{m_{ij}\overline{\beta}_{ij}^2}{\kappa_{ij}}\left[(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})^2+(\widehat{\boldsymbol{\sigma}}\times\mathbf{S}_{ij})^2+2(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})\cdot\mathbf{S}_{ij}\right]-2m_{ij}\overline{\beta}_{ij}\left[(\widehat{\boldsymbol{\sigma}}\times\mathbf{S}_{ij})^2+(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})\cdot\mathbf{S}_{ij}\right]$
E_{ij}	$-m_{ij}\frac{1-\alpha_{ij}^2}{2}(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}})^2-m_{ij}\frac{\kappa_{ij}}{1+\kappa_{ij}}\frac{1-\beta_{ij}^2}{2}\left[(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})^2+(\widehat{\boldsymbol{\sigma}}\times\mathbf{S}_{ij})^2+2(\widehat{\boldsymbol{\sigma}}\times\mathbf{v}_{ij})\cdot\mathbf{S}_{ij}\right]$

Here,

$$m_{ij}\equiv\frac{m_im_j}{m_i+m_j},\quad \overline{\alpha}_{ij}\equiv 1+\alpha_{ij},\quad \overline{\beta}_{ij}\equiv\frac{\kappa_{ij}}{1+\kappa_{ij}}\left(1+\beta_{ij}\right),\quad \kappa_{ij}\equiv\kappa_i\kappa_j\frac{m_i+m_j}{\kappa_im_i+\kappa_jm_j},\quad \kappa_i\equiv\frac{4I_i}{m_i\sigma_i^2},\quad \kappa_j\equiv\frac{4I_j}{m_j\sigma_j^2}. \tag{6}$$

The values of the reduced moments of inertia κ_i run from $\kappa_i=0$, if the mass is completely concentrated in the center of the body, to $\kappa_i=1$ (disks) or $\kappa_i=\frac{2}{3}$ (spheres), if the mass is concentrated on the perimeter of the particle. In the case of a uniform mass distribution, $\kappa_i=\frac{1}{2}$ (disks) or $\kappa_i=\frac{2}{5}$ (spheres). Note that for perfectly smooth particles ($\beta_{ij}=-1$) one has $\overline{\beta}_{ij}=0$, and thus the angular velocities are not affected by collisions.

The rules for restituting collisions are

$$\mathfrak{B}_{ij,\widehat{\sigma}}^{-1}\mathbf{v}_i=\mathbf{v}_i-\frac{1}{m_i}\mathbf{Q}_{ij}^{-},\quad \mathfrak{B}_{ij,\widehat{\sigma}}^{-1}\mathbf{v}_j=\mathbf{v}_j+\frac{1}{m_j}\mathbf{Q}_{ij}^{-},\quad \mathfrak{B}_{ij,\widehat{\sigma}}^{-1}\boldsymbol{\omega}_i=\boldsymbol{\omega}_i-\frac{\sigma_i}{2I_i}\widehat{\boldsymbol{\sigma}}\times\mathbf{Q}_{ij}^{-},\quad \mathfrak{B}_{ij,\widehat{\sigma}}^{-1}\boldsymbol{\omega}_j=\boldsymbol{\omega}_j-\frac{\sigma_j}{2I_j}\widehat{\boldsymbol{\sigma}}\times\mathbf{Q}_{ij}^{-}, \tag{7}$$

where

$$\mathbf{Q}_{ij}^{-}=m_{ij}\frac{\overline{\alpha}_{ij}}{\alpha_{ij}}(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}+m_{ij}\frac{\overline{\beta}_{ij}}{\beta_{ij}}\left[\mathbf{v}_{ij}-(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}-\widehat{\boldsymbol{\sigma}}\times\mathbf{S}_{ij}\right]. \tag{8}$$

Given a certain dynamic variable $\psi_{ij}(\mathbf{c}_i,\mathbf{c}_j)$, where the short-hand notation $\mathbf{c}_i\equiv\{\mathbf{v}_i,\boldsymbol{\omega}_i\}$, $\mathbf{c}_j\equiv\{\mathbf{v}_j,\boldsymbol{\omega}_j\}$ has been introduced, Eqs. (4) and (5) provide its collisional change $\left(\mathfrak{B}_{ij,\widehat{\sigma}}-1\right)\psi_{ij}(\mathbf{c}_i,\mathbf{c}_j)$. The most relevant cases are presented in Table 1 [70]. In the last row,

$$E_{ij}=\frac{1}{2}m_iv_i^2+\frac{1}{2}m_jv_j^2+\frac{1}{2}I_i\omega_i^2+\frac{1}{2}I_j\omega_j^2 \tag{9}$$

is the total kinetic energy (translational plus rotational) of both colliding particles. Energy is conserved only if the particles are elastic ($\alpha_{ij}=1$) and either perfectly smooth ($\beta_{ij}=-1$) or perfectly rough ($\beta_{ij}=1$). Otherwise, $\left(\mathfrak{B}_{ij,\widehat{\sigma}}-1\right)E_{ij}<0$ and kinetic energy is dissipated upon collisions.

The collision rules (4) and (5), as well as those in Table 1, hold both for spheres and disks. In the latter case, however, some terms may simplify [74]. On the other hand, the Jacobian of the transformation $\{\mathbf{c}_i, \mathbf{c}_j\} \to \{\mathfrak{B}_{ij,\widehat{\sigma}}\mathbf{c}_i, \mathfrak{B}_{ij,\widehat{\sigma}}\mathbf{c}_j\}$ is different for spheres ($d_{\text{tr}} = d_{\text{rot}} = 3$) and disks ($d_{\text{tr}} = 2$, $d_{\text{rot}} = 1$), namely [70, 74]

$$\mathfrak{J}_{ij} \equiv \left| \frac{\partial\left(\mathfrak{B}_{ij,\widehat{\sigma}}\mathbf{c}_i, \mathfrak{B}_{ij,\widehat{\sigma}}\mathbf{c}_j\right)}{\partial\left(\mathbf{c}_i, \mathbf{c}_j\right)} \right| = \left| \frac{\partial\left(\mathbf{c}_i, \mathbf{c}_j\right)}{\partial\left(\mathfrak{B}^{-1}_{ij,\widehat{\sigma}}\mathbf{c}_i, \mathfrak{B}^{-1}_{ij,\widehat{\sigma}}\mathbf{c}_j\right)} \right| = \begin{cases} \alpha_{ij}|\beta_{ij}| & \text{(disks)}, \\ \alpha_{ij}\beta_{ij}^2 & \text{(spheres)}. \end{cases} \tag{10}$$

BOLTZMANN EQUATION

Let $f_i(\mathbf{r}, \mathbf{c}_i; t)$ be the one-body velocity distribution function of particles of component i. In the low-density limit, by application of the molecular chaos assumption on the first equation of the Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy, $f_i(\mathbf{r}, \mathbf{c}_i; t)$ obeys the Boltzmann equation [70, 74]

$$\partial_t f_i(\mathbf{r}, \mathbf{c}_i; t) + \mathbf{v}_i \cdot \nabla f_i(\mathbf{r}, \mathbf{c}_i; t) = \sum_j J_{ij}[\mathbf{r}, \mathbf{c}_i; t|f_i, f_j], \tag{11}$$

where

$$J_{ij}[\mathbf{r}, \mathbf{c}_i; t|f_i, f_j] = \chi_{ij}\sigma_{ij}^{d_{\text{tr}}-1} \int d\mathbf{c}_j \int_+ d\widehat{\boldsymbol{\sigma}}\,(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}}) \left(\frac{1}{\alpha_{ij}\mathfrak{J}_{ij}} \mathfrak{B}^{-1}_{ij,\widehat{\sigma}} - 1 \right) f_i(\mathbf{r}, \mathbf{c}_i; t) f_j(\mathbf{r}, \mathbf{c}_j; t) \tag{12}$$

is the bilinear Boltzmann collision operator. In Eq. (12), χ_{ij} is the contact value of the pair correlation function, $\sigma_{ij} \equiv \frac{1}{2}(\sigma_i + \sigma_j)$, $\int d\mathbf{c}_j \equiv \int d\mathbf{v}_j \int d\boldsymbol{\omega}_j$, and $\int_+ d\widehat{\boldsymbol{\sigma}} \equiv \int d\widehat{\boldsymbol{\sigma}}\,\Theta(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}})$, $\Theta(x)$ being the Heaviside step function. The following mathematical integrals over $\widehat{\boldsymbol{\sigma}}$ will be needed [76]:

$$\int_+ d\widehat{\boldsymbol{\sigma}}\,(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}})^\ell = B_\ell v_{ij}^\ell, \quad \int_+ d\widehat{\boldsymbol{\sigma}}\,(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}})^\ell \widehat{\boldsymbol{\sigma}} = B_{\ell+1} v_{ij}^{\ell-1}\mathbf{v}_{ij}, \tag{13a}$$

$$\int_+ d\widehat{\boldsymbol{\sigma}}\,(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}})^\ell \widehat{\boldsymbol{\sigma}}\widehat{\boldsymbol{\sigma}} = B_{\ell+2} v_{ij}^{\ell-2}\mathbf{v}_{ij}\mathbf{v}_{ij} + \frac{B_\ell - B_{\ell+2}}{d_{\text{tr}} - 1} v_{ij}^{\ell-2}\left(v_{ij}^2 \mathsf{I}_{\text{tr}} - \mathbf{v}_{ij}\mathbf{v}_{ij} \right), \tag{13b}$$

where I_{tr} is the $d_{\text{tr}} \times d_{\text{tr}}$ unit tensor and $B_\ell = \pi^{(d_{\text{tr}}-1)/2}\Gamma(\ell/2 + 1/2)/\Gamma(\ell/2 + d_{\text{tr}}/2)$.

Given a one-body dynamic variable $\psi_i(\mathbf{c}_i)$, its average value is

$$\langle\psi_i(\mathbf{c}_i)\rangle \equiv \frac{1}{n_i}\int d\mathbf{c}_i\,\psi_i(\mathbf{c}_i) f_i(\mathbf{c}_i), \quad n_i = \int d\mathbf{c}_i\, f_i(\mathbf{c}_i), \quad n = \sum_i n_i, \tag{14}$$

n_i and n being the number density of component i and the total number density, respectively. For the sake of brevity, in Eq. (14) and henceforth the spatial and temporal arguments are omitted. Analogously, the average of a two-body dynamic variable $\psi_{ij}(\mathbf{c}_i, \mathbf{c}_j)$ is

$$\langle\!\langle\psi_{i,j}(\mathbf{c}_i, \mathbf{c}_j)\rangle\!\rangle \equiv \frac{1}{n_i n_j}\int d\mathbf{c}_i \int d\mathbf{c}_j\,\psi_{ij}(\mathbf{c}_i, \mathbf{c}_j) f_i(\mathbf{c}_i) f_j(\mathbf{c}_j). \tag{15}$$

Multiplying both sides of the Boltzmann equation (11) by $\psi_i(\mathbf{c}_i)$ and integrating over $\mathbf{c}_i$, we obtain the balance equation

$$\frac{\partial}{\partial t}\left[n_i\langle\psi_i(\mathbf{c}_i)\rangle\right] + \nabla\cdot\left[n_i\langle\mathbf{v}_i\psi_i(\mathbf{c}_i)\rangle\right] = \sum_j \mathcal{J}_{ij}[\psi_i|f_i, f_j], \tag{16}$$

where

$$\mathcal{J}_{ij}[\psi_i|f_i, f_j] \equiv \int d\mathbf{c}_i\,\psi_i(\mathbf{c}_i) J_{ij}[\mathbf{c}_i|f_i, f_j] = \chi_{ij}\sigma_{ij}^{d_{\text{tr}}-1}\int d\mathbf{c}_i \int d\mathbf{c}_j \int_+ d\widehat{\boldsymbol{\sigma}}\,(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}}) f_i(\mathbf{c}_i) f_j(\mathbf{c}_j)\left(\mathfrak{B}_{ij,\widehat{\sigma}} - 1\right)\psi_i(\mathbf{c}_i). \tag{17}$$

Therefore, $n_i^{-1}\mathcal{J}_{ij}[\psi_i|f_i, f_j] = \partial_t\langle\psi_i\rangle|_{\text{coll},j}$ represents the rate of change of the quantity $\psi_i(\mathbf{c}_i)$ due to collisions with particles of component j. Analogously, in the case of a two-body dynamic variable $\psi_{ij}(\mathbf{c}_i, \mathbf{c}_j)$, the collisional production rate $\mathcal{J}_{ij}[\psi_{ij}|f_i, f_j]$ is defined by the second equality of Eq. (17) with the replacement $\psi_i(\mathbf{c}_i) \to \psi_{ij}(\mathbf{c}_i, \mathbf{c}_j)$.

TABLE 2. Relevant collisional integrals in terms of two-body averages.

$\psi_{ij}(\mathbf{c}_i,\mathbf{c}_j)$	$\mathcal{K}_{ij}[\psi_{ij}\vert f_i,f_j]$
$m_i\mathbf{v}_i$	$\left(\overline{\alpha}_{ij}+\frac{d_{\mathrm{tr}}-1}{2}\overline{\beta}_{ij}\right)\langle\langle v_{ij}\mathbf{v}_{ij}\rangle\rangle-\frac{\sqrt{\pi}\Gamma(3/2+d_{\mathrm{tr}}/2)}{2\Gamma(1+d_{\mathrm{tr}}/2)}\overline{\beta}_{ij}\langle\langle\mathbf{v}_{ij}\times\mathbf{S}_{ij}\rangle\rangle$
$I_i\boldsymbol{\omega}_i$	$\frac{\sigma_i}{4}\overline{\beta}_{ij}\left[3\langle\langle v_{ij}\mathbf{S}_{ij}\rangle\rangle-\langle\langle v_{ij}^{-1}(\mathbf{v}_{ij}\cdot\mathbf{S}_{ij})\mathbf{v}_{ij}\rangle\rangle\right]$
$m_iv_i^2$	$2\left(\overline{\alpha}_{ij}+\frac{d_{\mathrm{tr}}-1}{2}\overline{\beta}_{ij}\right)\langle\langle v_{ij}(\mathbf{v}_i\cdot\mathbf{v}_{ij})\rangle\rangle-\frac{m_{ij}}{m_i}\left(\overline{\alpha}_{ij}^2+\frac{d_{\mathrm{tr}}-1}{2}\overline{\beta}_{ij}^2\right)\langle\langle v_{ij}^3\rangle\rangle$ $-\frac{\sqrt{\pi}\Gamma(3/2+d_{\mathrm{tr}}/2)}{\Gamma(1+d_{\mathrm{tr}}/2)}\overline{\beta}_{ij}\langle\langle\mathbf{S}_{ij}\cdot(\mathbf{v}_i\times\mathbf{v}_{ij})\rangle\rangle-\frac{m_{ij}\overline{\beta}_{ij}^2}{2m_i}\left[3\langle\langle v_{ij}S_{ij}^2\rangle\rangle-\langle\langle v_{ij}^{-1}(\mathbf{v}_{ij}\cdot\mathbf{S}_{ij})^2\rangle\rangle\right]$
$I_i\omega_i^2$	$\frac{\sigma_i}{2}\overline{\beta}_{ij}\left[3\langle\langle v_{ij}\boldsymbol{\omega}_i\cdot\mathbf{S}_{ij}\rangle\rangle-\langle\langle v_{ij}^{-1}(\mathbf{v}_{ij}\cdot\mathbf{S}_{ij})(\mathbf{v}_{ij}\cdot\boldsymbol{\omega}_i)\rangle\rangle\right]-\frac{\overline{\beta}_{ij}^2m_{ij}}{2m_i\kappa_i}\left[(d_{\mathrm{tr}}-1)\langle\langle v_{ij}^3\rangle\rangle\right.$ $\left.+3\langle\langle v_{ij}S_{ij}^2\rangle\rangle-\langle\langle v_{ij}^{-1}(\mathbf{v}_{ij}\cdot\mathbf{S}_{ij})^2\rangle\rangle\right]$
$m_iv_i^2+m_jv_j^2$	$\left[1-\alpha_{ij}^2+\frac{d_{\mathrm{tr}}-1}{2}\overline{\beta}_{ij}\left(2-\overline{\beta}_{ij}\right)\right]\langle\langle v_{ij}^3\rangle\rangle-\frac{\overline{\beta}_{ij}^2}{2}\left[3\langle\langle v_{ij}S_{ij}^2\rangle\rangle-\langle\langle v_{ij}^{-1}(\mathbf{v}_{ij}\cdot\mathbf{S}_{ij})^2\rangle\rangle\right]$
$I_i\omega_i^2+I_j\omega_j^2$	$\frac{\overline{\beta}_{ij}}{2\kappa_{ij}}\left(2\kappa_{ij}-\overline{\beta}_{ij}\right)\left[3\langle\langle v_{ij}S_{ij}^2\rangle\rangle-\langle\langle v_{ij}^{-1}(\mathbf{v}_{ij}\cdot\mathbf{S}_{ij})^2\rangle\rangle\right]-\frac{d_{\mathrm{tr}}-1}{2}\frac{\overline{\beta}_{ij}^2}{\kappa_{ij}}\langle\langle v_{ij}^3\rangle\rangle$
E_{ij}	$\frac{1-\alpha_{ij}^2}{2}\langle\langle v_{ij}^3\rangle\rangle+\frac{\kappa_{ij}(1-\beta_{ij}^2)}{4(1+\kappa_{ij})}\left[(d_{\mathrm{tr}}-1)\langle\langle v_{ij}^3\rangle\rangle+3\langle\langle v_{ij}S_{ij}^2\rangle\rangle-\langle\langle v_{ij}^{-1}(\mathbf{v}_{ij}\cdot\mathbf{S}_{ij})^2\rangle\rangle\right]$

Let us focus on the quantities $\psi_{ij}(\mathbf{c}_i,\mathbf{c}_j)$ listed on the first column of Table 1. When obtaining the corresponding production rates $\mathcal{J}_{ij}[\psi_{ij}|f_i,f_j]$, the angular integrals $\int_+ d\widehat{\boldsymbol{\sigma}}\,(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}})\left(\mathfrak{B}_{ij,\widehat{\boldsymbol{\sigma}}}-1\right)\psi_{ij}(\mathbf{c}_i,\mathbf{c}_j)$ can be evaluated with the help of Eqs. (13), so that $\mathcal{J}_{ij}[\psi_{ij}|f_i,f_j]$ are expressed in terms of two-body averages. Some care is needed when applying Eq. (13b) and contracting the tensor I_{tr} with an angular velocity vector, for instance $\mathsf{I}_{\mathrm{tr}}\cdot\mathbf{S}_{ij}$. In the case of spheres ($d_{\mathrm{tr}}=3$), one obviously have $\mathsf{I}_{\mathrm{tr}}\cdot\mathbf{S}_{ij}=\mathbf{S}_{ij}$. However, in the case of disks ($d_{\mathrm{tr}}=2$) the vector $\mathbf{S}_{ij}$ is orthogonal to the subspace where the identity tensor I_{tr} acts, and thus $\mathsf{I}_{\mathrm{tr}}\cdot\mathbf{S}_{ij}=0$. To unify both possibilities, and taking into account that the rotational degrees of freedom are meaningless except in the cases of disks and spheres, it is convenient to write $\mathsf{I}_{\mathrm{tr}}\cdot\mathbf{S}_{ij}=(d_{\mathrm{tr}}-2)\mathbf{S}_{ij}$. Analogously, $\mathsf{I}_{\mathrm{tr}}:\mathbf{S}_{ij}\mathbf{S}_{ij}=(d_{\mathrm{tr}}-2)S_{ij}^2$ and $\mathsf{I}_{\mathrm{tr}}:\mathbf{S}_{ij}\boldsymbol{\omega}_i=(d_{\mathrm{tr}}-2)\boldsymbol{\omega}_i\cdot\mathbf{S}_{ij}$. The final results are displayed in Table 2, where we have introduced the scaled collisional integrals

$$\mathcal{K}_{ij}[\psi_{ij}|f_i,f_j]\equiv-\frac{\Gamma(3/2+d_{\mathrm{tr}}/2)}{\pi^{(d_{\mathrm{tr}}-1)/2}}\frac{\mathcal{J}_{ij}[\psi_{ij}|f_if_j]}{\chi_{ij}m_{ij}n_in_j\sigma_{ij}^{d_{\mathrm{tr}}-1}}. \tag{18}$$

The most important one-body averages are

$$\mathbf{u}_i=\langle\mathbf{v}_i\rangle,\quad \mathbf{u}=\frac{\sum_i m_in_i\mathbf{u}_i}{\sum_i m_in_i},\quad \boldsymbol{\Omega}_i=\langle\boldsymbol{\omega}_i\rangle,\quad \frac{d_{\mathrm{tr}}}{2}T_i^{\mathrm{tr}}=\frac{m_i}{2}\langle(\mathbf{v}_i-\mathbf{u})^2\rangle,\quad \frac{d_{\mathrm{rot}}}{2}T_i^{\mathrm{rot}}=\frac{I_i}{2}\langle\omega_i^2\rangle,\quad T=\sum_i\frac{n_i}{n}\frac{d_{\mathrm{tr}}T_i^{\mathrm{tr}}+d_{\mathrm{rot}}T_i^{\mathrm{rot}}}{d_{\mathrm{tr}}+d_{\mathrm{rot}}}, \tag{19}$$

where $\mathbf{u}_i$ and $\boldsymbol{\Omega}_i$ are partial flow and angular velocities, respectively, $\mathbf{u}$ is the global flow velocity, T_i^{tr} and T_i^{rot} are partial granular temperatures associated with the translational and rotational degrees of freedom, respectively, and T is the global granular temperature. The partial energy production rates associated with T_i^{tr} and T_i^{rot} are defined as [70, 74]

$$\xi_{ij}^{\mathrm{tr}}\equiv-\frac{1}{d_{\mathrm{tr}}n_iT_i^{\mathrm{tr}}}\mathcal{J}_{ij}[m_i(\mathbf{v}_i-\mathbf{u})^2|f_i,f_j],\quad \xi_{ij}^{\mathrm{rot}}\equiv-\frac{1}{d_{\mathrm{rot}}n_iT_i^{\mathrm{rot}}}\mathcal{J}_{ij}[I_i\omega_i^2|f_i,f_j]. \tag{20}$$

These quantities can be expressed in terms of two-body averages with the help of Table 2. From ξ_{ij}^{tr} and ξ_{ij}^{rot} one can obtain the total production rates ξ_i^{tr} and ξ_i^{rot}, as well as the global cooling rate ζ, by the relations

$$\xi_i^{\text{tr}} \equiv -\frac{\partial_t T_i^{\text{tr}}\big|_{\text{coll}}}{T_i^{\text{tr}}} = \sum_j \xi_{ij}^{\text{tr}}, \quad \xi_i^{\text{rot}} \equiv -\frac{\partial_t T_i^{\text{rot}}\big|_{\text{coll}}}{T_i^{\text{rot}}} = \sum_j \xi_{ij}^{\text{rot}}, \quad \zeta \equiv -\frac{\partial_t T|_{\text{coll}}}{T} = \sum_i \frac{n_i\left(d_{\text{tr}} T_i^{\text{tr}} \xi_i^{\text{tr}} + d_{\text{rot}} T_i^{\text{rot}} \xi_i^{\text{rot}}\right)}{(d_{\text{tr}} + d_{\text{rot}}) nT}. \tag{21}$$

Before closing this section, let us consider the mean collision frequency of a particle of component i with particles of component j as given by [77]

$$\nu_{ij} = \frac{1}{n_i}\chi_{ij}\sigma_{ij}^{d_{\text{tr}}-1}\int \mathrm{d}\mathbf{c}_i \int \mathrm{d}\mathbf{c}_j \int_+ \mathrm{d}\widehat{\boldsymbol{\sigma}}\,(\mathbf{v}_{ij}\cdot\widehat{\boldsymbol{\sigma}}) f_i(\mathbf{c}_i) f_j(\mathbf{c}_j) = \frac{\langle\!\langle v_{ij}\rangle\!\rangle}{\sqrt{2}\lambda_{ij}}, \quad \lambda_{ij} = \frac{\Gamma(1/2 + d_{\text{tr}}/2)}{\sqrt{2}\pi^{(d_{\text{tr}}-1)/2}\chi_{ij} n_j \sigma_{ij}^{d_{\text{tr}}-1}}, \tag{22}$$

where in the second equality, namely $\nu_{ij} = \langle\!\langle v_{ij}\rangle\!\rangle/\sqrt{2}\lambda_{ij}$, use has been made of Eq. (13a), λ_{ij} being the mean free path of a particle of component i with respect to collisions with particles of component j. Note that $n_i\nu_{ij} = n_j\nu_{ji}$. The total collision frequency of a particle of component i is $\nu_i = \sum_j \nu_{ij}$, while the global mean collision frequency is $\nu = \sum_i n_i\nu_i/n$.

All the results in this section are exact within the framework of the Boltzmann equation. The expressions of the main collisional integrals in terms of two-body averages are useful to evaluate those integrals by computer simulations. On the other hand, exact analytic expressions are not possible unless $f_i(\mathbf{c}_i)$ and $f_j(\mathbf{c}_j)$ are known. The next section provides analytic approximations for the energy production rates based on a Maxwellian approximation.

ESTIMATES OF TWO-BODY AVERAGES AND APPROXIMATE ENERGY PRODUCTION RATES

Henceforth, we particularize to mixtures without mutual diffusion (i.e., $\mathbf{u}_i = \mathbf{u}$ for all i) and with isotropic distributions of translational velocities, $\mathbf{v}_i - \mathbf{u}$, relative to the flow velocity.

In order to get practical estimates of the two-body averages appearing in Table 2, let us approximate the unknown velocity distribution functions $f_i(\mathbf{c}_i)$ by means of two assumptions: (i) statistical independence between translational and rotational velocities, i.e., $f_i(\mathbf{c}_i) = n_i^{-1} f_i^{\text{tr}}(\mathbf{v}_i) f_i^{\text{rot}}(\boldsymbol{\omega}_i)$, where $f_i^{\text{tr}}(\mathbf{v}_i)$ and $f_i^{\text{rot}}(\boldsymbol{\omega}_i)$ are the marginal distribution functions associated with the translational and rotational degrees of freedom, respectively; (ii) Maxwellian form for $f_i^{\text{tr}}(\mathbf{v}_i)$. Therefore,

$$f_i(\mathbf{c}_i) \to \left(\frac{m_i}{2\pi T_i^{\text{tr}}}\right)^{d_{\text{tr}}/2} \exp\left[-\frac{m_i(\mathbf{v}_i - \mathbf{u})^2}{2T_i^{\text{tr}}}\right] f_i^{\text{rot}}(\boldsymbol{\omega}_i). \tag{23}$$

When Eq. (23), together with the equivalent approximation for $f_j(\mathbf{c}_j)$ is inserted into Eq. (15), the two-body averages appearing in Table 2 can be explicitly evaluated in terms of the material parameters (m_i, m_j, σ_i, σ_j, κ_i, κ_j, α_{ij}, and β_{ij}) and of the physical quantities $\boldsymbol{\Omega}_i$, $\boldsymbol{\Omega}_j$, T_i^{tr}, T_j^{tr}, T_i^{rot}, and T_j^{rot}. Analogously to what happened with Table 2, the evaluation of averages involving angular velocities must be done with care to treat the cases of spheres and disks in a common setting. For instance, $\langle\!\langle v_{ij}^{-1}(\mathbf{v}_{ij}\cdot\mathbf{S}_{ij})\mathbf{v}_{ij}\rangle\!\rangle = \langle\!\langle v_{ij}^{-1}\mathbf{v}_{ij}\mathbf{v}_{ij}\rangle\!\rangle\cdot\langle\!\langle \mathbf{S}_{ij}\rangle\!\rangle = d_{\text{tr}}^{-1}\langle\!\langle v_{ij}\rangle\!\rangle \mathsf{I}_{\text{tr}}\cdot\langle\!\langle \mathbf{S}_{ij}\rangle\!\rangle = d_{\text{tr}}^{-1}(d_{\text{tr}}-2)\langle\!\langle v_{ij}\rangle\!\rangle\langle\!\langle \mathbf{S}_{ij}\rangle\!\rangle$. Similarly, $\langle\!\langle v_{ij}^{-1}(\mathbf{v}_{ij}\cdot\mathbf{S}_{ij})^2\rangle\!\rangle = d_{\text{tr}}^{-1}(d_{\text{tr}}-2)\langle\!\langle v_{ij}\rangle\!\rangle\langle\!\langle S_{ij}^2\rangle\!\rangle$ and $\langle\!\langle v_{ij}^{-1}(\mathbf{v}_{ij}\cdot\mathbf{S}_{ij})(\mathbf{v}_{ij}\cdot\boldsymbol{\omega}_i)\rangle\!\rangle = d_{\text{tr}}^{-1}(d_{\text{tr}}-2)\langle\!\langle v_{ij}\rangle\!\rangle\langle\!\langle \boldsymbol{\omega}_i\cdot\mathbf{S}_{ij}\rangle\!\rangle$. The results for the two-body averages are summarized in Table 3. Note that $\langle\!\langle S_{ij}^2\rangle\!\rangle = d_{\text{rot}}\left(T_i^{\text{rot}}/m_i\kappa_i + T_j^{\text{rot}}/m_j\kappa_j + \sigma_i\sigma_j\boldsymbol{\Omega}_i\cdot\boldsymbol{\Omega}_j/2d_{\text{rot}}\right)$ is positive definite.

Substitution of the expression for $\langle\!\langle v_{ij}\rangle\!\rangle$ into Eq. (22) provides the approximate expression

$$\nu_{ij} = \frac{\sqrt{2}\pi^{(d_{\text{tr}}-1)/2}}{\Gamma(d_{\text{tr}}/2)}\chi_{ij} n_j \sigma_{ij}^{d_{\text{tr}}-1}\left(\frac{T_i^{\text{tr}}}{m_i} + \frac{T_j^{\text{tr}}}{m_j}\right)^{1/2}. \tag{24}$$

Analogously, substitution into Table 2 of the expressions for other two-body averages shown in Table 3 allows us to obtain the energy production rates ξ_{ij}^{tr} and ξ_{ij}^{rot} defined by Eq. (20). The results are given in Table 4, where the equality $d_{\text{rot}} = \frac{1}{2}d_{\text{tr}}(d_{\text{tr}}-1)$ (valid only for disks and spheres) has been used in order to present the expressions in a compact form. In fact, we can observe that the number of degrees of freedom d_{tr} and d_{rot} intervene in ξ_{ij}^{tr} and ξ_{ij}^{rot} by following

TABLE 3. Expressions, as obtained from the approximation (23), for the two-body averages appearing in Table 2.

$\langle\langle v_{ij}\mathbf{v}_{ij}\rangle\rangle$, $\langle\langle \mathbf{v}_{ij}\times\mathbf{S}_{ij}\rangle\rangle$, $\langle\langle \mathbf{S}_{ij}\cdot(\mathbf{v}_i\times\mathbf{v}_{ij})\rangle\rangle$	0
$\langle\langle v_{ij}\rangle\rangle$	$\dfrac{\sqrt{2}\Gamma(1/2+d_\text{tr}/2)}{\Gamma(d_\text{tr}/2)}\left(\dfrac{T_i^\text{tr}}{m_i}+\dfrac{T_j^\text{tr}}{m_j}\right)^{1/2}$
$\langle\langle v_{ij}^3\rangle\rangle$	$\dfrac{2\sqrt{2}\Gamma(3/2+d_\text{tr}/2)}{\Gamma(d_\text{tr}/2)}\left(\dfrac{T_i^\text{tr}}{m_i}+\dfrac{T_j^\text{tr}}{m_j}\right)^{3/2}$
$\langle\langle v_{ij}\mathbf{v}_i\cdot\mathbf{v}_{ij}\rangle\rangle$	$\dfrac{T_i^\text{tr}}{m_i}\left(\dfrac{T_i^\text{tr}}{m_i}+\dfrac{T_j^\text{tr}}{m_j}\right)^{-1}\langle\langle v_{ij}^3\rangle\rangle$
$3\langle\langle v_{ij}\mathbf{S}_{ij}\rangle\rangle-\langle\langle v_{ij}^{-1}(\mathbf{v}_{ij}\cdot\mathbf{S}_{ij})\mathbf{v}_{ij}\rangle\rangle$	$\dfrac{d_\text{tr}+1}{d_\text{tr}}\left(\sigma_i\mathbf{\Omega}_i+\sigma_j\mathbf{\Omega}_j\right)\langle\langle v_{ij}\rangle\rangle$
$3\langle\langle v_{ij}S_{ij}^2\rangle\rangle-\langle\langle v_{ij}^{-1}(\mathbf{v}_{ij}\cdot\mathbf{S}_{ij})^2\rangle\rangle$	$2\dfrac{d_\text{tr}+1}{d_\text{tr}}d_\text{rot}\left(\dfrac{T_i^\text{rot}}{m_i\kappa_i}+\dfrac{T_j^\text{rot}}{m_j\kappa_j}+\dfrac{\sigma_i\sigma_j\mathbf{\Omega}_i\cdot\mathbf{\Omega}_j}{2d_\text{rot}}\right)\langle\langle v_{ij}\rangle\rangle$
$3\langle\langle v_{ij}\boldsymbol{\omega}_i\cdot\mathbf{S}_{ij}\rangle\rangle-\langle\langle v_{ij}^{-1}(\mathbf{v}_{ij}\cdot\mathbf{S}_{ij})(\mathbf{v}_{ij}\cdot\boldsymbol{\omega}_i)\rangle\rangle$	$2\dfrac{d_\text{tr}+1}{d_\text{tr}}\dfrac{d_\text{rot}}{\sigma_i}\left(\dfrac{2T_i^\text{rot}}{m_i\kappa_i}+\dfrac{\sigma_i\sigma_j\mathbf{\Omega}_i\cdot\mathbf{\Omega}_j}{2d_\text{rot}}\right)\langle\langle v_{ij}\rangle\rangle$

three simple rules: (i) ξ_{ij}^tr and ξ_{ij}^rot are divided by d_tr and d_rot, respectively, as a consequence of their definitions in Eq. (20); (ii) a factor d_rot/d_tr is attached to $\overline{\beta}_{ij}$ and $\overline{\beta}_{ij}^2$; (iii) a factor d_rot^{-1} is attached to $\mathbf{\Omega}_i\cdot\mathbf{\Omega}_j$.

In the special case of frictionless, smooth particles ($\beta_{ij}=-1\Rightarrow\overline{\beta}_{ij}=0$), one has $\xi_{ij}^\text{rot}=0$ and

$$\xi_{ij}^\text{tr}=\frac{2\nu_{ij}m_{ij}^2}{d_\text{tr}m_iT_i^\text{tr}}\left[2\overline{\alpha}_{ij}\frac{T_i^\text{tr}}{m_{ij}}-\overline{\alpha}_{ij}^2\left(\frac{T_i^\text{tr}}{m_i}+\frac{T_j^\text{tr}}{m_j}\right)\right]. \tag{25}$$

This coincides with previous results for an arbitrary number of dimensions $d=d_\text{tr}$ [56, 58, 62, 63]. On the other hand, in the case of rough disks ($d_\text{tr}=2$, $d_\text{rot}=1$) or spheres ($d_\text{tr}=3$, $d_\text{rot}=3$), the expressions for ξ_{ij}^tr and ξ_{ij}^rot in Table 4 reduce to results derived in Refs. [74] and [70], respectively.

The global cooling rate ζ defined in Eq. (21) is also shown in Table 4. Now, a factor d_rot/d_tr is attached to $(1-\beta_{ij}^2)$. While ζ is a positive definite quantity, the partial production rates ξ_{ij}^tr and ξ_{ij}^rot can in general be positive or negative since the energy dissipation and equipartition effects are mixed together. To disentangle them, it is convenient to carry out the decompositions [71, 74]

$$\xi_{ij}^\text{tr}=\frac{d_\text{rot}\kappa_iT_i^\text{rot}}{d_\text{tr}T_i^\text{tr}}\xi_{ij}^\text{rot}+\zeta_{ij}^\text{tr}+\Xi_{ij}^{(1)}+\Xi_{ij}^{(2)},\quad \xi_{ij}^\text{rot}=\zeta_{ij}^\text{rot}+\Xi_{ij}^{(3)}, \tag{26}$$

where ζ_{ij}^tr and ζ_{ij}^rot are true cooling rates (positive definite), whereas $\Xi_{ij}^{(1-3)}$ represent *equipartition* rates and do not have a definite sign. The expressions for ζ_{ij}^tr, ζ_{ij}^rot, and $\Xi_{ij}^{(1-3)}$ are also included in Table 4. It can be easily checked that $n_iT_i^\text{tr}\Xi_{ij}^{(1)}+n_jT_j^\text{tr}\Xi_{ji}^{(1)}=0$ and $n_i\left[d_\text{tr}T_i^\text{tr}\Xi_{ij}^{(2)}+d_\text{rot}(1+\kappa_i)T_i^\text{rot}\Xi_{ij}^{(3)}\right]+n_j\left[d_\text{tr}T_j^\text{tr}\Xi_{ji}^{(2)}+d_\text{rot}(1+\kappa_j)T_j^\text{rot}\Xi_{ji}^{(3)}\right]=0$. Therefore, as expected on physical grounds, the equipartition rates $\Xi_{ij}^{(1-3)}$ do not contribute to the net cooling rate ζ, so that

$$\zeta=\frac{1}{2(d_\text{tr}+d_\text{rot})nT}\sum_{i,j}\left\{n_i\left[d_\text{tr}T_i^\text{tr}\zeta_{ij}^\text{tr}+d_\text{rot}(1+\kappa_i)T_i^\text{rot}\zeta_{ij}^\text{rot}\right]+n_j\left[d_\text{tr}T_j^\text{tr}\zeta_{ji}^\text{tr}+d_\text{rot}(1+\kappa_j)T_j^\text{rot}\zeta_{ji}^\text{rot}\right]\right\}. \tag{27}$$

TABLE 4. Collisional energy production rates for polydisperse systems

ξ_{ij}^{tr}	$\frac{\nu_{ij}}{d_{\text{tr}}}\frac{2m_{ij}^2}{m_i T_i^{\text{tr}}}\left\{2\left(\overline{\alpha}_{ij}+\frac{d_{\text{rot}}}{d_{\text{tr}}}\overline{\beta}_{ij}\right)\frac{T_i^{\text{tr}}}{m_{ij}}-\left(\overline{\alpha}_{ij}^2+\frac{d_{\text{rot}}}{d_{\text{tr}}}\overline{\beta}_{ij}^2\right)\left(\frac{T_i^{\text{tr}}}{m_i}+\frac{T_j^{\text{tr}}}{m_j}\right)-\frac{d_{\text{rot}}}{d_{\text{tr}}}\overline{\beta}_{ij}^2\left(\frac{T_i^{\text{rot}}}{m_i\kappa_i}+\frac{T_j^{\text{rot}}}{m_j\kappa_j}+\frac{\sigma_i\sigma_j\mathbf{\Omega}_i\cdot\mathbf{\Omega}_j}{2d_{\text{rot}}}\right)\right\}$
ξ_{ij}^{rot}	$\frac{\nu_{ij}}{d_{\text{tr}}}\frac{4m_{ij}^2\overline{\beta}_{ij}}{m_i\kappa_i T_i^{\text{rot}}}\left[\frac{T_i^{\text{rot}}}{m_{ij}}+\frac{m_i\kappa_i}{m_{ij}}\frac{\sigma_i\sigma_j\mathbf{\Omega}_i\cdot\mathbf{\Omega}_j}{4d_{\text{rot}}}-\frac{\overline{\beta}_{ij}}{2}\left(\frac{T_i^{\text{tr}}}{m_i}+\frac{T_j^{\text{tr}}}{m_j}+\frac{T_i^{\text{rot}}}{m_i\kappa_i}+\frac{T_j^{\text{rot}}}{m_j\kappa_j}+\frac{\sigma_i\sigma_j\mathbf{\Omega}_i\cdot\mathbf{\Omega}_j}{2d_{\text{rot}}}\right)\right]$
ζ	$\sum_{i,j}\frac{n_i\nu_{ij}m_{ij}}{(d_{\text{tr}}+d_{\text{rot}})nT}\left[\left(1-\alpha_{ij}^2\right)\left(\frac{T_i^{\text{tr}}}{m_i}+\frac{T_j^{\text{tr}}}{m_j}\right)+\frac{d_{\text{rot}}\kappa_{ij}(1-\beta_{ij}^2)}{d_{\text{tr}}(1+\kappa_{ij})}\left(\frac{T_i^{\text{tr}}}{m_i}+\frac{T_j^{\text{tr}}}{m_j}+\frac{T_i^{\text{rot}}}{m_i\kappa_i}+\frac{T_j^{\text{rot}}}{m_j\kappa_j}+\frac{\sigma_i\sigma_j\mathbf{\Omega}_i\cdot\mathbf{\Omega}_j}{2d_{\text{rot}}}\right)\right]$
ζ_{ij}^{tr}	$\frac{\nu_{ij}}{d_{\text{tr}}}\frac{2m_{ij}^2(1-\alpha_{ij}^2)}{m_i T_i^{\text{tr}}}\left(\frac{T_i^{\text{tr}}}{m_i}+\frac{T_j^{\text{tr}}}{m_j}\right)$
ζ_{ij}^{rot}	$\frac{\nu_{ij}}{d_{\text{tr}}}\frac{2m_{ij}^2\kappa_{ij}^2(1-\beta_{ij}^2)}{m_i\kappa_i(1+\kappa_{ij})^2T_i^{\text{rot}}}\left(\frac{T_i^{\text{tr}}}{m_i}+\frac{T_j^{\text{tr}}}{m_j}+\frac{T_i^{\text{rot}}}{m_i\kappa_i}+\frac{T_j^{\text{rot}}}{m_j\kappa_j}+\frac{\sigma_i\sigma_j\mathbf{\Omega}_i\cdot\mathbf{\Omega}_j}{2d_{\text{rot}}}\right)$
$\Xi_{ij}^{(1)}$	$\frac{\nu_{ij}}{d_{\text{tr}}}\frac{4m_{ij}^2(1+\alpha_{ij})}{m_i m_j T_i^{\text{tr}}}\left(T_i^{\text{tr}}-T_j^{\text{tr}}\right)$
$\Xi_{ij}^{(2)}$	$\frac{\nu_{ij}}{d_{\text{tr}}^2}\frac{4d_{\text{rot}}m_{ij}\kappa_{ij}(1+\beta_{ij})}{m_i(1+\kappa_{ij})T_i^{\text{tr}}}\left(T_i^{\text{tr}}-T_i^{\text{rot}}-\frac{m_i\kappa_i\sigma_i\sigma_j\mathbf{\Omega}_i\cdot\mathbf{\Omega}_j}{4d_{\text{rot}}}\right)$
$\Xi_{ij}^{(3)}$	$\frac{\nu_{ij}}{d_{\text{tr}}}\frac{4m_{ij}^2\kappa_{ij}^2(1+\beta_{ij})}{(1+\kappa_{ij})^2T_i^{\text{rot}}}\left[\frac{T_i^{\text{rot}}-T_j^{\text{rot}}}{m_i m_j\kappa_i\kappa_j}+\frac{T_i^{\text{tr}}-T_j^{\text{tr}}}{m_i m_j\kappa_i}+\frac{T_i^{\text{rot}}-T_i^{\text{tr}}}{m_i m_{ij}\kappa_i}+\left(\frac{m_i\kappa_i-m_j\kappa_j}{m_i m_j\kappa_i\kappa_j}+\frac{1}{m_{ij}}\right)\frac{\sigma_i\sigma_j\mathbf{\Omega}_i\cdot\mathbf{\Omega}_j}{4d_{\text{rot}}}\right]$

Apart from the energy production rates shown in Table 4, one can introduce a spin production rate ζ_{ij}^{Ω} by [41]

$$\frac{\sigma_i}{n_i}\mathcal{J}_{ij}[\omega_i|f_i,f_j]=-\frac{\zeta_{ij}^{\Omega}}{2}\left(\sigma_i\mathbf{\Omega}_i+\sigma_j\mathbf{\Omega}_j\right),\quad \zeta_{ij}^{\Omega}=\frac{\nu_{ij}}{d_{\text{tr}}}\frac{4m_{ij}\overline{\beta}_{ij}}{m_i\kappa_i}, \tag{28}$$

where in the last equality use has been made of Tables 2 and 3.

The expressions of Table 4 simplify considerably in the case of monodisperse systems, i.e., if $m_i=2m_{ij}=m$, $\kappa_i=\kappa_{ij}=\kappa$, $\sigma_i=\sigma_{ij}=\sigma$, $\alpha_{ij}=\alpha$, $\beta_{ij}=\beta$, $\chi_{ij}=\chi$, $T_i^{\text{tr}}=T^{\text{tr}}$, $T_i^{\text{rot}}=T^{\text{rot}}$, $\mathbf{\Omega}_i=\mathbf{\Omega}$, and $\nu_{ij}=\nu$, with

$$\nu=\frac{2\pi^{(d_{\text{tr}}-1)/2}}{\Gamma(d_{\text{tr}}/2)}\chi n\sigma^{d_{\text{tr}}-1}\sqrt{\frac{T^{\text{tr}}}{m}}, \tag{29}$$

for all i and j. From those conditions, Table 4 reduces to Table 5. Moreover, the spin production rate becomes $\zeta_{ij}^{\Omega}=\zeta^{\Omega}=2\nu(1+\beta)/d_{\text{tr}}(1+\kappa)$.

CONCLUSION

Arguably, the most distinctive feature of granular gases is collisional energy dissipation due to inelasticity and surface roughness of the particles. Moreover, there are in general two classes of contributions to the kinetic energy, one associated with d_{tr} translational degrees of freedom and the other one associated with d_{rot} rotational degrees of freedom. In a multicomponent gas, additionally, the (translational or rotational) kinetic energy is split into different components. To characterize all these separate contributions, the (partial) granular temperatures T_i^{tr} and T_i^{rot} are defined as twice the mean (translational or rotational) kinetic energy per particle and per degree of freedom associated with component

TABLE 5. Collisional energy production rates for monodisperse systems

ξ^{tr}	$\frac{\nu}{d_{\text{tr}}}\left\{1-\alpha^2+\frac{2d_{\text{rot}}\kappa(1+\beta)}{d_{\text{tr}}(1+\kappa)^2T^{\text{tr}}}\left[\frac{\kappa(1-\beta)}{2}\left(T^{\text{tr}}+\frac{T^{\text{rot}}}{\kappa}+\frac{m\sigma^2\Omega^2}{4d_{\text{rot}}}\right)+T^{\text{tr}}-T^{\text{rot}}-\frac{\kappa m\sigma^2\Omega^2}{4d_{\text{rot}}}\right]\right\}$
ξ^{rot}	$\frac{\nu}{d_{\text{tr}}}\frac{2\kappa(1+\beta)}{(1+\kappa)^2T^{\text{rot}}}\left[\frac{1-\beta}{2}\left(T^{\text{tr}}+\frac{T^{\text{rot}}}{\kappa}+\frac{m\sigma^2\Omega^2}{4d_{\text{rot}}}\right)+T^{\text{rot}}-T^{\text{tr}}+\frac{\kappa m\sigma^2\Omega^2}{4d_{\text{rot}}}\right]$
ζ	$\frac{\nu}{(d_{\text{tr}}+d_{\text{rot}})T}\left[\left(1-\alpha^2\right)T^{\text{tr}}+\frac{d_{\text{rot}}\kappa(1-\beta^2)}{d_{\text{tr}}(1+\kappa)}\left(T^{\text{tr}}+\frac{T^{\text{rot}}}{\kappa}+\frac{m\sigma^2\Omega^2}{4d_{\text{rot}}}\right)\right]$
ζ^{tr}	$\frac{\nu}{d_{\text{tr}}}\left(1-\alpha^2\right)$
ζ^{rot}	$\frac{\nu}{d_{\text{tr}}}\frac{\kappa(1-\beta^2)}{(1+\kappa)^2T^{\text{rot}}}\left(T^{\text{tr}}+\frac{T^{\text{rot}}}{\kappa}+\frac{m\sigma^2\Omega^2}{4d_{\text{rot}}}\right)$
$\Xi^{(1)}$	0
$\Xi^{(2)}$	$\frac{\nu}{d_{\text{tr}}^2}\frac{2d_{\text{rot}}\kappa(1+\beta)}{(1+\kappa)T^{\text{tr}}}\left(T^{\text{tr}}-T^{\text{rot}}-\frac{\kappa m\sigma^2\Omega^2}{4d_{\text{rot}}}\right)$
$\Xi^{(3)}$	$-\frac{d_{\text{tr}}}{d_{\text{rot}}(1+\kappa)}\frac{T^{\text{tr}}}{T^{\text{rot}}}\Xi^{(2)}$

i. Collisions of particles of component i with those of component j produce two main competing effects: on the one hand, T_i^{tr}, T_j^{tr}, T_i^{rot}, and T_j^{rot} tend to decay due to a dissipative cooling effect but, on the other hand, those partial temperatures also tend to equal each other due to an equipartition effect. These basic effects are entangled in the energy production rates ξ_{ij}^{tr} and ξ_{ij}^{rot} defined by the rate equations $\partial_t T_i^{\text{tr}}\big|_{\text{coll},j}=-\xi_{ij}^{\text{tr}}T_i^{\text{tr}}$ and $\partial_t T_i^{\text{rot}}\big|_{\text{coll},j}=-\xi_{ij}^{\text{rot}}T_i^{\text{rot}}$.

The aim of this work has been the unified derivation of ξ_{ij}^{tr} and ξ_{ij}^{rot} for disks ($d_{\text{tr}}=2$, $d_{\text{rot}}=1$) and spheres ($d_{\text{tr}}=d_{\text{rot}}=3$) from the Boltzmann equation (i.e., under the molecular chaos ansatz). In order to obtain analytic results, statistical independence of the distributions of translational and angular velocities and a Maxwellian form for the translational distribution have been assumed. The expressions for ξ_{ij}^{tr} and ξ_{ij}^{rot}, together with those for the global cooling rate ζ, the partial cooling rates ζ_{ij}^{tr} and ζ_{ij}^{rot}, and the equipartition rates $\Xi_{ij}^{(1)\text{–}(3)}$, are presented in Table 4. They encapsulate isolated previous results for smooth d-dimensional spheres [56, 58, 62, 63], rough disks [74], and rough spheres [70] in a common framework. The parametric dependence of ξ_{ij}^{tr} and ξ_{ij}^{rot} on the numbers of degrees of freedom d_{tr} and d_{rot} turns out to be quite simple: $d_{\text{tr}}\xi_{ij}^{\text{tr}}/\nu_{ij}$ and $d_{\text{rot}}\xi_{ij}^{\text{rot}}/\nu_{ij}$, where the collision frequency ν_{ij} depends on d_{tr} [see Eq. (24)], have a factor $d_{\text{rot}}/d_{\text{tr}}$ attached to $\overline{\beta}_{ij}$ and $\overline{\beta}_{ij}^2$, and a factor d_{rot}^{-1} attached to $\mathbf{\Omega}_i\cdot\mathbf{\Omega}_j$.

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