Numerical algorithms of the paper:

Title: Fast, Accurate and Robust Adaptive Finite Difference Methods for Fractional Diffusion Equations

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Mathematica version...

In[1]:=
```
StringJoin[{"The running Mathematica version is ", ToString[$Version]}]
```

Out[1]=
```
The running Mathematica version is
    10.1.0  for Microsoft Windows (64-bit) (March 24, 2015)
```

Numerical algorithms for the T&E and predictive adaptive methods.

We start implementing the L1 finite difference scheme with variable timesteps for solving the fractional PDE

$$\frac{\partial^\gamma u}{\partial t^\gamma} = K\frac{\partial^2 u}{\partial x^2} + f(x,t)$$

The finite difference scheme is given by Eq. (10):

$$-S_n\, U_{j+1}^{(n)} + (1+2S_n)U_j^{(n)} - S_n\, U_{j-1}^{(n)} = G_j^{(n)}$$

where

$$G_j^{(n)} \equiv U_j^{(n-1)} - \sum_{m=0}^{n-2} \tilde{T}_{m,n}^{(\gamma)} \left[U_j^{(m+1)} - U_j^{(m)}\right] + F_j^{(n)}$$

or, in matrix-vector form, by Eq. (11):

$$A\, U^{(m)} = G\left(U^{(m-1)}, U^{(m-2)}, \ldots, U^{(0)}, F^{(m)}, t_m\right)$$

The coefficients $\tilde{T}_{m,n}^{(\gamma)}$ of the paper are given here by the function TT[m,n,γ] where t_n=tt[n]

In[2]:=
```
TT[m_, n_, gamma_] :=
 ((tt[n ] - tt[m]) ^ (1 - gamma) - (tt[n ] - tt[m + 1]) ^ (1 - gamma)) / (tt[m + 1] - tt[m])
```

The coefficient

$$G_j^{(n)} \equiv U_j^{(n-1)} - \sum_{m=0}^{n-2} \tilde{T}_{m,n}^{(\gamma)} \left[U_j^{(m+1)} - U_j^{(m)}\right] + F_j^{(n)}$$

of the paper is given here by the function GG[j,n,γ] , where $F_j^{(n)} = \Gamma(2-\gamma)\,(t_n - t_{n-1})^\gamma\, f(x_j, t_n)$ [see Eq. (9)], and $f(x_j, t_n)$ is given by funSource[$j\,\Delta$x,t_n,γ] . The solution U is stored in the matrix Umatrix. The element (1+m,1+j) of this matrix, i.e., Umatrix[[1+m,1+j]], represents $U_j^{(m)}$ with j=0,1,..., m=0,1,...

In[3]:=
```
GG[j_, 1, gamma_] := Umatrix[[1, j + 1]] +
  (tt[1] - tt[0]) ^gamma * Gamma[2 - gamma] *
   funSource[j * deltax, tt[1], gamma]  (* El elemento 1 de
    Umatrix es la solución en el instante inicial: u(x,0)   *)
```

In[4]:=
```
GG[j_, n_, gamma_] := (
  kk = (tt[n] - tt[n - 1]) ^gamma *
    Sum[ (Umatrix[[m + 2, j + 1]] - Umatrix[[m + 1, j + 1]]) *
      TT[m, n, gamma], {m, 0, n - 2}];
  Umatrix[[n, j + 1]] - kk + (tt[n] - tt[n - 1]) ^gamma *
    Gamma[2 - gamma] * funSource[j * deltax, tt[n], gamma])
```

The coefficient S_n [Eq. (7)] is given by the function Sfun[m,γ] where K=Kg and Δx=deltax

In[5]:=
```
Sfun[m_, gamma_] :=
 (Kg * Gamma[2 - gamma]) / (deltax) ^2 * (tt[m ] - tt[m - 1]) ^gamma
```

Next the matrix system

$$A\,U^{(m)} = G\left(U^{(m-1)}, U^{(m-2)}, \ldots, U^{(0)}, F^{(m)}, t_m\right)$$

is built and then solved inside the functions U1stepMatrixt and U2stepsMatrixt.
The matrix A is built by means of the instruction SparseArray[...], while the vector G is generated by means of the instruction Table[Which[j==0,.....,{j,0,Np}] . Here uL[tt[m]]=$u(x_0, t_m)$ is the boundary condition to the left (at x=x_0) and uR[tt[m]]=$u(x_L, t_m)$ is the boundary condition to the right (at x=x_R). The solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$, is stored in the vector uVectortm

U1stepMatrixt[m] provides the solution of

$$A\,U^{(m)} = G\left(U^{(m-1)}, U^{(m-2)}, \ldots, U^{(0)}, F^{(m)}, t_m\right)$$

i.e., it provides the solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$ at time t_m from the solutions

$\{\{U_0^{(m-1)}, U_1^{(m-1)}, U_2^{(m-1)}, \ldots U_{Np}^{(m-1)}\}, \{U_0^{(m-2)}, U_1^{(m-2)}, U_2^{(m-2)}, \ldots U_{Np}^{(m-2)}\}, \cdots\}$,
employing a single timestep of size t_m-t_{m-1}=ht

In[6]:=
```
U1stepMatrixt[m_] := (tt[0] = 0;
  tt[m] = tt[m - 1] + ht;
  uVectortm =
   LinearSolve[
    SparseArray[{Band[{1, 2}] → Table[Which[j == 0, 0., True, -Sfun[m, gammak]],
        {j, 0, Np - 1}], Band[{1, 1}] → Table[Which[j == 0, 1., j == Np, 1.,
         True, 1. + 2. * Sfun[m, gammak]], {j, 0, Np}], Band[{2, 1}] →
       Table[Which[j == Np - 1, 0., True, -Sfun[m, gammak]], {j, 0, Np - 1}]},
     Np + 1], Table[Which[j == 0, uL[tt[m]], j == Np, uR[tt[m]],
      True, GG[j, m, gammak]], {j, 0, Np}]])
```

U2stepsMatrixt[n] provides the solution $\{U_0^{(n)}, U_1^{(n)}, U_2^{(n)}, \ldots U_{Np}^{(n)}\}$ at time t_n from the solutions
$\{\{U_0^{(n-1)}, U_1^{(n-1)}, U_2^{(n-1)}, \ldots U_{Np}^{(n-1)}\}, \{U_0^{(n-2)}, U_1^{(n-2)}, U_2^{(n-2)}, \ldots U_{Np}^{(n-2)}\}, \cdots\}$,
employing two timesteps of size (t_n-t_{n-1})/2=ht/2=hthalf

In[7]:=
```
U2stepsMatrixt[n_] := (
   hthalf = ht / 2.;
  Do[tt[m] = tt[m - 1] + hthalf;
   uVectortm =
    LinearSolve[
     SparseArray[{Band[{1, 2}] → Table[Which[j == 0, 0., True, -Sfun[m, gammak]],
         {j, 0, Np - 1}], Band[{1, 1}] → Table[Which[j == 0, 1., j == Np, 1.,
          True, 1. + 2. * Sfun[m, gammak]], {j, 0, Np}], Band[{2, 1}] →
        Table[Which[j == Np - 1, 0., True, -Sfun[m, gammak]], {j, 0, Np - 1}]},
      Np + 1], Table[Which[j == 0, uL[tt[m]], j == Np, uR[tt[m]],
       True, GG[j, m, gammak]], {j, 0, Np}]];
   Umatrix = Append[Umatrix, uVectortm],
   {m, n, n + 1}
  ];
  Umatrix = Drop[Umatrix, -2];
  uVectortm)
```

uMsolutionFixedTimesteps[t,Δ] provides the full solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$ with m=0,1,2,....n for fixed timesteps of size Δ from t=0 to t=t_n where n=t_n/Δ . The full solution is stored in the matrix Umatrix so that its element (1+m,1+j), i.e., Umatrix[[1+m,1+j]], gives $U_j^{(m)}$.

In[8]:=
```
uMsolutionFixedTimesteps[t_, htFixed_] := Module[{numpasos},
  ht = htFixed;
  numpasos = Round[t / htFixed];
  Umatrix = {}; Umatrix = AppendTo[Umatrix, UmatrixIni];
  Do[
       Umatrix = Append[Umatrix, U1stepMatrixt[m]],
   {m, 1, numpasos}];
  Umatrix
 ]
```

uMsolutionTE[n,τ] provides the full solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{\mathrm{Np}}^{(m)}\}$ with m=0,1,2,....n for variable timesteps whose size is determined by the *trial and error* (T&E) method for a tolerance τ. The full solution is stored in the matrix Umatrix so that its element (1+m,1+j), i.e., Umatrix[[1+m,1+j]], gives $U_j^{(m)}$.

Some arbitrary parameters one has to chose:
• The initial timestep Δ_0=Delta0TE. Here is fixed to the arbitrary (but hardly relevant) value of 0.01
• The maximum value of the timestep, DeltatcapTE, one is ready to accept. Here we choose Deltatmax=10^6 (the actual value could be somewhat larger). [When the solution goes towards zero, the absolute value of this solution will be smaller than the tolerance for times long enough. In this case, the condition $\mathcal{E}^{(n)}$=udif<tolerance always holds and the search of ever-increasing timesteps would never end. Besides, to cap the size of the timesteps guarantees a minimum sampling of the solution.]

In[9]:=
```
Delta0TE = 0.01; DeltatcapTE = 10.^6;
```

In[10]:=
```
uMsolutionTE[numpasos_, tolerance_] := (
  Umatrix = {}; Umatrix = AppendTo[Umatrix, UmatrixIni];
  ht = Delta0TE;
  Do[
   usingle = U1stepMatrixt[m];
   uhalf = U2stepsMatrixt[m];
   udif = Max[Abs[usingle - uhalf]];
   If[udif < tolerance,
        While[
         ht < DeltatcapTE && udif < tolerance,
         ht = 2. * ht;
         usingle = U1stepMatrixt[m];
         uhalf = U2stepsMatrixt[m];
         udif = Max[Abs[usingle - uhalf]];
                ];
        ht = ht / 2.;
      ,
        While[
         udif > tolerance,
         ht = ht / 2;
         usingle = U1stepMatrixt[m];
         uhalf = U2stepsMatrixt[m];
         udif = Max[Abs[usingle - uhalf]];

            ]
     ];
    usingle = U1stepMatrixt[m];
   uhalf = U2stepsMatrixt[m];
   udif = Max[Abs[usingle - uhalf]];

   Umatrix = Append[Umatrix, U1stepMatrixt[m]],
   {m, 1, numpasos}];
  Umatrix
 )
```

uMsolutionPredictive[n,τ] provides the full solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$ with m=0,1,2,....n for variable timesteps whose size is determined by the *predictive* method for a tolerance τ. The full solution is stored in the matrix Umatrix so that its element (1+m,1+j), i.e., Umatrix[[1+m,1+j]], gives $U_j^{(m)}$.

Some arbitrary parameters one has to chose:

• The initial timestep Δ_0=Delta0Predictive. Here is fixed to the arbitrary (but hardly relevant)

value of 0.01
• The maximum value of the timestep, DeltatcapTE, one is ready to accept. Here we choose Deltatmax=10^6 (the actual value could be somewhat larger). [When the solution goes towards zero, the absolute value of this solution will be smaller than the tolerance for times long enough. In this case, the condition $\mathcal{E}^{(n)}$=udif<fac2*tolerance always holds and the search of ever-increasing timesteps would never end. Besides, to cap the size of the timesteps guarantees a minimum sampling of the solution.]
• The under-relaxation parameter ω=omegaParameter. Here we use ω=1/2.
• The scaling exponent θ=thetaParameter. Here we use θ=3/2.
• The fac1 and fac2 parameteres. The timestep is accepted if the difference $\mathcal{E}^{(n)}$=udif is larger than fac1*tolerance and smaller than fac2*tolerance. Here we use fac1=1/2 and fac2=2 [see Eq. (18)]

In[11]:=
```
Delta0Predictive = 0.01; DeltatcapPredictive = 10.^6;
omegaParameter = 0.5; thetaParameter = 3./2.; fac1 = 0.5; fac2 = 2.;
```

In[12]:=
```
uMsolutionPredictive[numpasos_, tolerance_] := (
  Umatrix = {}; Umatrix = AppendTo[Umatrix, UmatrixIni];
  ht = Delta0Predictive;
  Do[
   usingle = U1stepMatrixt[m];
   uhalf = U2stepsMatrixt[m];
   udif = Max[Abs[usingle - uhalf]];
       While[     ht < DeltatcapPredictive &&
      (udif < fac1 * tolerance || udif > fac2 * tolerance),
    ht = (1. - omegaParameter) * ht + omegaParameter * ht *
       (tolerance / udif) ^ (1. / thetaParameter);
        uhalf = U2stepsMatrixt[m];
        usingle = U1stepMatrixt[m];
        udif = Max[Abs[usingle - uhalf]];
              ];

   Umatrix = Append[Umatrix, U1stepMatrixt[m]],
   {m, 1, numpasos}];
  htVariableUltimo = ht;
  Umatrix
 )
```

■ Example I

$$\frac{\partial^\gamma u}{\partial t^\gamma} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le \pi,$$
$$u(x = 0, t) = u(x = \pi, t) = 0,$$
$$u(x, 0) = \sin x.$$

Here we provide the parameters of the problem and its exact solution

Parameter: K_γ=Kg=1

```
In[13]:= Kg = 1.;
```

Exact solution: $u(x, t) = E_\gamma(-t^\gamma)\sin(x)$

```
In[14]:= uSol[x_, t_, gamma_] := Sin[x] * MittagLefflerE[gamma, 1, -t^gamma]
In[15]:= uSol[x_, t_, 1] := Sin[x] * e^-t
```

Source term. In this example is zero.

```
In[16]:= funSource[x_, t_, ga_] := 0.
```

Order of the fractional derivative: γ=gammak. Here we choose γ=1/4

```
In[17]:= gammak = 0.25;
```

Spatial discretization: Δx≡deltax=Pi/Np where, in this case, we choose Np=2Nx=40

```
In[18]:= Nx = 20; Np = 2 * Nx; deltax = Pi / (2. * Nx);
```

We set the initial condition: $U^{(0)}$=UmatrixIni

```
In[19]:= UmatrixIni = Table[ Sin[j * deltax] , {j, 0, Np}] // N
Out[19]= {0., 0.0784591, 0.156434, 0.233445, 0.309017, 0.382683, 0.45399,
  0.522499, 0.587785, 0.649448, 0.707107, 0.760406, 0.809017, 0.85264,
  0.891007, 0.92388, 0.951057, 0.97237, 0.987688, 0.996917, 1.,
  0.996917, 0.987688, 0.97237, 0.951057, 0.92388, 0.891007, 0.85264,
  0.809017, 0.760406, 0.707107, 0.649448, 0.587785, 0.522499, 0.45399,
  0.382683, 0.309017, 0.233445, 0.156434, 0.0784591, 1.22465×10^-16}
```

Boundary conditions: u(x=0,t)=uL[t]=0; u(x=π,t)=uR[t]=0.

```
In[20]:= uL[t_] = 0.;
uR[t_] = 0.;
```

■ Numerical results with the T&E method for tolerance τ=0.001

uMs is a matrix that contains the full solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$ for m=0,1,2,....,numSteps.

In[22]:=
```
tolerance = 0.001;
numSteps = 20;
CPUtime = Timing[uMs = uMsolutionTE[numSteps, tolerance]];
Print["Example 1. T&E method. Case with τ=", tolerance, " for γ=",
  gammak, ", Np=", Np, ", Δ0=", Delta0TE, ", Δmax=" , DeltatcapTE ];
Print["CPU time=", CPUtime[[1]], " seconds"];
Print["The solution is saved in the matrix uMs"];
```

```
Example 1. T&E method. Case with τ=0.001 for γ=0.25, Np=40, Δ0=0.01, Δmax=1. × 10^6
CPU time=3.58802 seconds
The solution is saved in the matrix uMs
```

The solution for the m-th timestep is just the m+1 element of this matrix:
uMs[[m+1]]= $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$

In[28]:=
```
mStep = 5;
Print["The numerical solution at time ", tt[mStep], " is:"];
uMs[[mStep]]
```

```
The numerical solution at time 0.0000257969 is:
```

Out[29]=
```
{-1.26814 × 10^-16, 0.0744668, 0.148474, 0.221567, 0.293293, 0.363211,
 0.43089, 0.495912, 0.557876, 0.616402, 0.671126, 0.721714, 0.767851,
 0.809254, 0.845669, 0.876869, 0.902663, 0.922892, 0.937431, 0.94619,
 0.949116, 0.94619, 0.937431, 0.922892, 0.902663, 0.876869, 0.845669,
 0.809254, 0.767851, 0.721714, 0.671126, 0.616402, 0.557876, 0.495912,
 0.43089, 0.363211, 0.293293, 0.221567, 0.148474, 0.0744668, 0.}
```

The [[m+1,j+1] element of this matrix is the numerical solution for time t_m and position x_j:
uMs[[m+1,j+1]]= $U_j^{(m)}$

In[30]:=
```
mStep = 5; jPos = Nx + 1;
Print["The numerical solution at time t=",
 tt[mStep], " and position x= " , jPos * deltax , " is:"];
uMs[[mStep, jPos]]
```

```
The numerical solution at time t=0.0000257969 and position x= 1.64934 is:
```

Out[31]=
```
0.949116
```

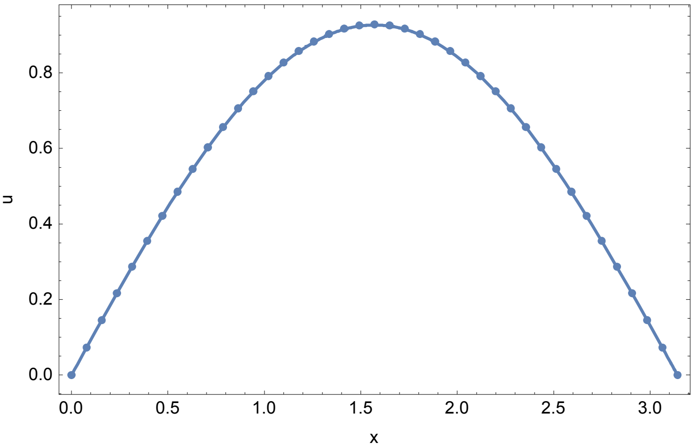
A plot comparing the numerical and the exact solution at time t_{mStep}

In[32]:=
```
mStep = 5;
Print["T&E method"];
Print["Numerical solution (points) and exact solution (line)  at time t=",
  tt[mStep]];
 FigNum = ListPlot[Table[{xj = j * deltax, uMs[[mStep + 1, j + 1]]}, {j, 0, Np}],
  PlotRange → All];
FigExact = Plot[uSol[x, tt[mStep], gammak], {x, 0, Pi}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"x", "u"}]
```

T&E method

Numerical solution (points) and exact solution (line) at time t=0.0000257969

Out[35]=



Here we give a table that compares the numerical and exact solution at the middle point, x=π/2, for the first numSteps timesteps.

In[36]:=
```
headTE = {"step", "t", "UT&E(π/2,t)", "uexact(π/2,t)", "error"};
```

In[37]:=
```
tabTE = Table[{m, tt[m], uMs[[m + 1, Nx + 1]], uSol[Pi / 2, tt[m], gammak],
    uMs[[m + 1, Nx + 1]] - uSol[Pi / 2, tt[m], gammak]}, {m, 0, numSteps}] // N;
Insert[tabTE, headTE, 1] // TableForm
```

Out[37]//TableForm=

step	t	$U_{T\&E}(\pi/2,t)$	$u_{exact}(\pi/2,t)$	error
0.	0.	1.	1.	0.
1.	9.53674×10^{-9}	0.991004	0.989207	0.00179746
2.	1.62125×10^{-7}	0.979909	0.978308	0.00160158
3.	1.38283×10^{-6}	0.965182	0.963451	0.00173057
4.	6.26564×10^{-6}	0.949116	0.947497	0.00161946
5.	0.0000257969	0.928564	0.926735	0.00182893
6.	0.0000648594	0.910841	0.909353	0.00148807
7.	0.000142984	0.89288	0.891558	0.00132247
8.	0.000299234	0.873482	0.872207	0.00127468
9.	0.000611734	0.852004	0.850705	0.00129939
10.	0.00123673	0.828042	0.826673	0.0013694
11.	0.00248673	0.801325	0.799859	0.00146637
12.	0.00498673	0.771696	0.770119	0.00157624
13.	0.00998673	0.739105	0.737418	0.00168699
14.	0.0199867	0.703622	0.701835	0.00178771
15.	0.0399867	0.665446	0.663577	0.00186831
16.	0.0799867	0.624906	0.622986	0.00191989
17.	0.159987	0.582463	0.580528	0.00193535
18.	0.319987	0.538694	0.536784	0.00191025
19.	0.639987	0.494262	0.492418	0.00184352
20.	1.27999	0.449876	0.448138	0.00173784

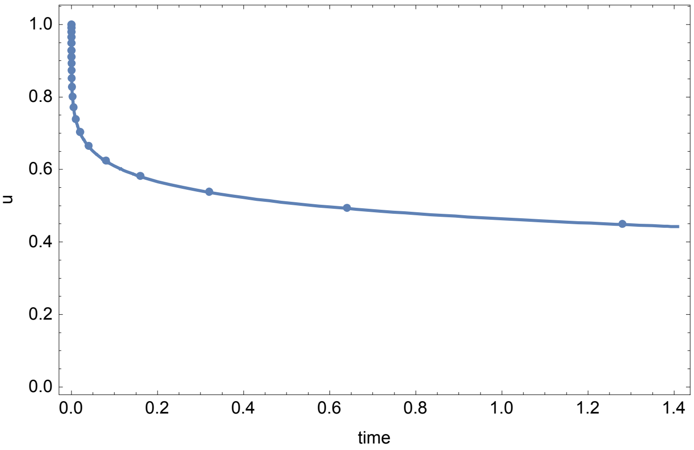
A plot comparing the numerical and the exact solution at position x=π/2 for times t with 0≤t≤ $t_{numSteps}$

In[38]:=
```
Print["T&E method"];
Print["Numerical solution (points) and exact solution
   (line) at position x=π/2 vs. time"]; FigNum = ListPlot[
  Table[{tabTE[[n, 2]], tabTE[[n, 3]]}, {n, 1, numSteps + 1}], PlotRange → All];
FigExact = Plot[uSol[Pi / 2, time, gammak],
   {time, 0, 1.1 * tabTE[[numSteps + 1, 2]]}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"time", "u"}, PlotRange → All]
```

```
T&E method

Numerical solution (points) and exact solution (line) at position x=π/2 vs. time
```

Out[40]=



■ Numerical results with the *predictive* method for tolerance τ=0.001

uMs is a matrix that contains the full solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$ for m=0,1,2,....,numSteps.

In[41]:=
```
tolerance = 0.001;
numSteps = 20;
CPUtime = Timing[uMs = uMsolutionPredictive[numSteps, tolerance]];
Print["Example 1. Predictive method. Case with τ=", tolerance,
  " for γ=", gammak, ", Np=", Np, ", Δ0=", Delta0Predictive, ", Δmax=" ,
  DeltatcapPredictive , ",ω=", omegaParameter, ", θ=", thetaParameter ];
Print["CPU time=", CPUtime[[1]], " seconds"];
Print["The solution is saved in the matrix uMs"];
```

Example 1. Predictive method. Case with τ=0.001
  for γ=0.25, Np=40, Δ_0=0.01, Δ_{max}=1. × 10^6,ω=0.5, θ=1.5

CPU time=2.04361 seconds

The solution is saved in the matrix uMs

The solution for the m-th timestep is just the m+1 element of this matrix:
uMs[[m+1]]=$\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$

In[47]:=
```
mStep = 5;
Print["The numerical solution at time ", tt[mStep], " is:"];
uMs[[mStep]]
```

```
The numerical solution at time 0.0000221186 is:
```

Out[48]= $\{6.88793\times10^{-16}$, 0.0742876, 0.148117, 0.221034, 0.292587, 0.362337, 0.429853, 0.494719, 0.556534, 0.614918, 0.669512, 0.719977, 0.766004, 0.807307, 0.843634, 0.874759, 0.900491, 0.920671, 0.935175, 0.943914, 0.946832, 0.943914, 0.935175, 0.920671, 0.900491, 0.874759, 0.843634, 0.807307, 0.766004, 0.719977, 0.669512, 0.614918, 0.556534, 0.494719, 0.429853, 0.362337, 0.292587, 0.221034, 0.148117, 0.0742876, 0.$\}$

The [[m+1,j+1]] element of this matrix is the numerical solution for time t_m and position x_j: uMs[[m+1,j+1]]=$U_j^{(m)}$

In[49]:=
```
mStep = 5; jPos = Nx + 1;
Print["The numerical solution at time t=",
 tt[mStep], " and position x= " , jPos * deltax , " is:"];
uMs[[mStep, jPos]]
```

```
The numerical solution at time t=0.0000221186 and position x= 1.64934 is:
```

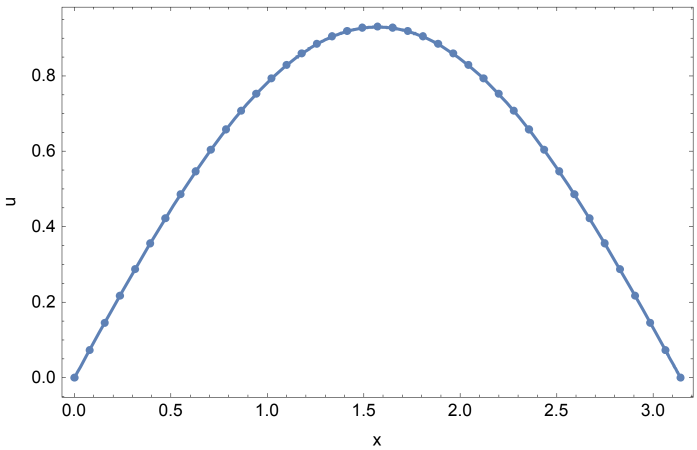
Out[50]= 0.946832

A plot comparing the numerical and the exact solution at time t_{mStep}

In[51]:=
```
mStep = 5;
Print["Predictive method"];
Print["Numerical solution (points) and exact solution (line)  at time t=",
  tt[mStep]];
 FigNum = ListPlot[Table[{xj = j * deltax, uMs[[mStep + 1, j + 1]]}, {j, 0, Np}],
  PlotRange → All];
FigExact = Plot[uSol[x, tt[mStep], gammak], {x, 0, Pi}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"x", "u"}]
```

```
Predictive method
Numerical solution (points) and exact solution (line)  at time t=0.0000221186
```

Out[54]=



Here we give a table that compares the numerical and exact solution at the middle point, x=π/2, for the first numSteps timesteps.

In[55]:=
```
headPredictive = {"step", "t", "UPredic(π/2,t)", "uexact(π/2,t)", "error"};
```

In[56]:=
```
tabPred = Table[{m, tt[m], uMs[[m + 1, Nx + 1]], uSol[Pi / 2, tt[m], gammak],
    uMs[[m + 1, Nx + 1]] - uSol[Pi / 2, tt[m], gammak]}, {m, 0, numSteps}] // N;
Insert[tabPred, headPredictive, 1] // TableForm
```

Out[56]//TableForm=

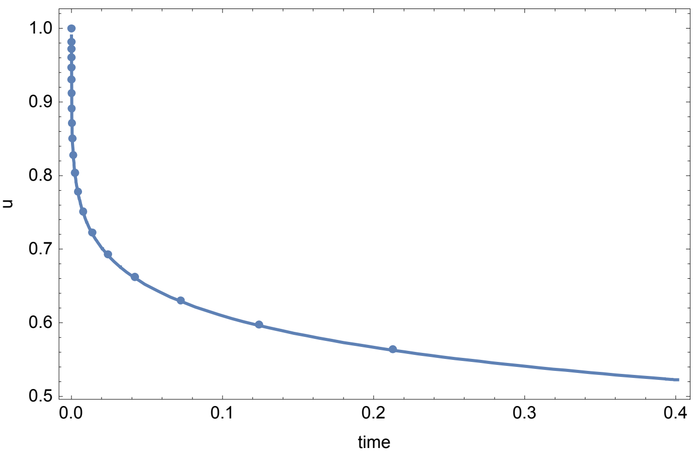
step	t	$U_{Predic}(\pi/2,t)$	$u_{exact}(\pi/2,t)$	error
0.	0.	1.	1.	0.
1.	1.67505×10^{-7}	0.981755	0.978133	0.0036221
2.	5.91669×10^{-7}	0.972067	0.970247	0.00182016
3.	2.1607×10^{-6}	0.960639	0.959301	0.00133796
4.	7.29445×10^{-6}	0.946832	0.945566	0.00126659
5.	0.0000221186	0.930626	0.929316	0.00130953
6.	0.0000609236	0.912025	0.910643	0.00138275
7.	0.000154849	0.891061	0.889601	0.00145986
8.	0.000322321	0.87142	0.870103	0.00131683
9.	0.000641044	0.850478	0.849198	0.00128078
10.	0.00123797	0.827925	0.826636	0.00128854
11.	0.00233021	0.803792	0.802482	0.00130943
12.	0.00428921	0.77815	0.776818	0.00133234
13.	0.00774617	0.75108	0.749727	0.00135273
14.	0.0137684	0.722669	0.721301	0.00136843
15.	0.0241574	0.693012	0.691634	0.00137825
16.	0.0419587	0.662212	0.66083	0.00138153
17.	0.0723422	0.630373	0.628995	0.00137788
18.	0.12415	0.597609	0.596242	0.00136709
19.	0.212664	0.564036	0.562686	0.00134907
20.	0.364655	0.529771	0.528447	0.00132378

A plot comparing the numerical and the exact solution at position x=π/2 for times t with 0≤t≤ $t_{numSteps}$

In[57]:=
```
Print["Predictive method"];
Print["Numerical solution (points) and exact solution
   (line) at position x=π/2 vs. time"]; FigNum = ListPlot[
  Table[{tabPred[[n, 2]], tabPred[[n, 3]]}, {n, 1, numSteps}], PlotRange → All];
FigExact = Plot[uSol[Pi/2, time, gammak],
   {time, 0, 1.1 * tabPred[[numSteps + 1, 2]]}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"time", "u"}, PlotRange → All]
```

```
Predictive method
Numerical solution (points) and exact solution (line) at position x=π/2 vs. time
```

Out[59]=



■ Numerical results with the method with *fixed timesteps* for Δ=0.01=htFT

uMs is a matrix that contains the full solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$ for m=0,1,2,....,numSteps.

In[60]:=
```
htFT = 0.01;
numSteps = 50;
CPUtime = Timing[uMs = uMsolutionFixedTimesteps[numSteps * htFT, htFT]];
Print["Example 1. Method with fixed timesteps of size ",
  htFT, " for γ=", gammak, ", Np=", Np ];
Print["CPU time=", CPUtime[[1]], " seconds"];
Print["The solution is saved in the matrix uMs"];
```

```
Example 1. Method with fixed timesteps of size 0.01 for γ=0.25, Np=40
CPU time=1.40401 seconds
The solution is saved in the matrix uMs
```

The solution for the m-th timestep is just the m+1 element of this matrix:
uMs[[m+1]]=$\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$

In[66]:=
```
mStep = 5;
Print["The numerical solution at time ", tt[mStep], " is:"];
uMs[[mStep]]
```

The numerical solution at time 0.05 is:

Out[67]= $\{-6.37165\times10^{-15}$, 0.0527189, 0.105113, 0.156859, 0.207637, 0.257136, 0.305049, 0.351082, 0.39495, 0.436383, 0.475125, 0.510939, 0.543602, 0.572913, 0.598693, 0.620781, 0.639042, 0.653363, 0.663656, 0.669857, 0.671929, 0.669857, 0.663656, 0.653363, 0.639042, 0.620781, 0.598693, 0.572913, 0.543602, 0.510939, 0.475125, 0.436383, 0.39495, 0.351082, 0.305049, 0.257136, 0.207637, 0.156859, 0.105113, 0.0527189, 0.$\}$

The [[m+1,j+1] element of this matrix is the numerical solution for time t_m and position x_j: uMs[[m+1,j+1]]=$U_j^{(m)}$

In[68]:=
```
mStep = 5; jPos = Nx + 1;
Print["The numerical solution at time t=",
 tt[mStep], " and position x= " , jPos * deltax , " is:"];
uMs[[mStep, jPos]]
```

The numerical solution at time t=0.05 and position x= 1.64934 is:

Out[69]= 0.671929

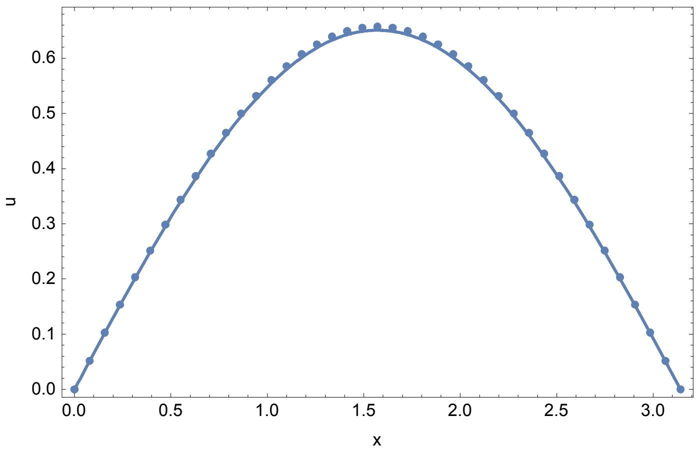
A plot comparing the numerical and the exact solution at time t_{mStep}

In[70]:=
```
mStep = 5;
Print["Method with fixed timesteps"];
Print["Numerical solution (points) and exact solution (line)  at time t=",
  tt[mStep]];
 FigNum = ListPlot[Table[{xj = j * deltax, uMs[[mStep + 1, j + 1]]}, {j, 0, Np}],
  PlotRange → All];
FigExact = Plot[uSol[x, tt[mStep], gammak], {x, 0, Pi}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"x", "u"}]
```

Method with fixed timesteps

Numerical solution (points) and exact solution (line) at time t=0.05

Out[73]=



Here we give a table that compares the numerical and exact solution at the middle point, x=π/2, for the first numSteps timesteps.

```
In[74]:= headFixedTimesteps = {"step", "t", "UFixed△(π/2,t)", "uexact(π/2,t)", "error"};

In[75]:= tabFT = Table[{m, tt[m], uMs[[m + 1, Nx + 1]], uSol[Pi / 2, tt[m], gammak],
            uMs[[m + 1, Nx + 1]] - uSol[Pi / 2, tt[m], gammak]}, {m, 0, numSteps}] // N;
         Insert[tabFT, headFixedTimesteps, 1] // TableForm
```

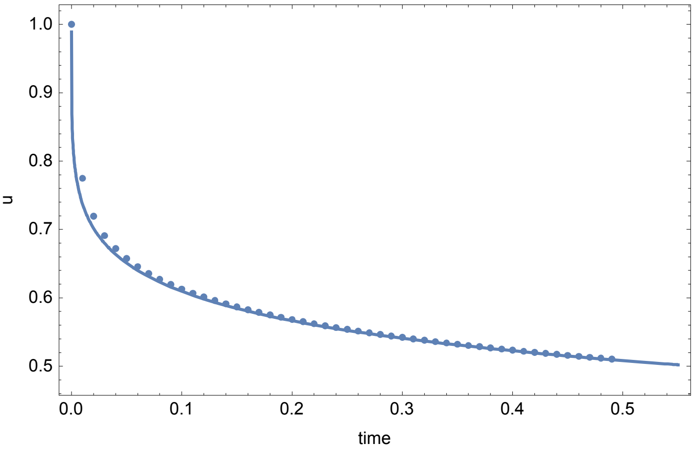
Out[75]//TableForm=

step	t	$U_{Fixed\,\Delta}(\pi/2,t)$	$u_{exact}(\pi/2,t)$	error
0.	0.	1.	1.	0.
1.	0.01	0.774903	0.737353	0.0375506
2.	0.02	0.719399	0.701799	0.0175995
3.	0.03	0.691047	0.679737	0.0113101
4.	0.04	0.671929	0.663559	0.00837013
5.	0.05	0.6574	0.650729	0.00667066
6.	0.06	0.645637	0.640076	0.00556061
7.	0.07	0.635733	0.630956	0.00477704
8.	0.08	0.627169	0.622976	0.00419354
9.	0.09	0.619621	0.615879	0.00374169
10.	0.1	0.612868	0.609487	0.00338121
11.	0.11	0.606758	0.603671	0.00308675
12.	0.12	0.601176	0.598334	0.00284161
13.	0.13	0.596038	0.593403	0.00263428
14.	0.14	0.591277	0.588821	0.00245659
15.	0.15	0.586842	0.58454	0.00230257
16.	0.16	0.582691	0.580523	0.00216778
17.	0.17	0.578788	0.576739	0.0020488
18.	0.18	0.575106	0.573163	0.00194299
19.	0.19	0.571622	0.569774	0.00184826
20.	0.2	0.568314	0.566551	0.00176297
21.	0.21	0.565166	0.563481	0.00168575
22.	0.22	0.562163	0.560548	0.00161551
23.	0.23	0.559293	0.557742	0.00155134
24.	0.24	0.556543	0.555051	0.00149248
25.	0.25	0.553905	0.552467	0.0014383
26.	0.26	0.55137	0.549982	0.00138825
27.	0.27	0.548929	0.547587	0.00134189
28.	0.28	0.546577	0.545278	0.00129881
29.	0.29	0.544306	0.543048	0.00125868
30.	0.3	0.542113	0.540891	0.0012212
31.	0.31	0.53999	0.538804	0.00118612
32.	0.32	0.537935	0.536782	0.00115322
33.	0.33	0.535942	0.53482	0.00112229
34.	0.34	0.534009	0.532916	0.00109317
35.	0.35	0.532132	0.531066	0.0010657
36.	0.36	0.530307	0.529268	0.00103974
37.	0.37	0.528533	0.527518	0.00101517
38.	0.38	0.526805	0.525813	0.000991889
39.	0.39	0.525123	0.524153	0.000969788
40.	0.4	0.523483	0.522534	0.000948782
41.	0.41	0.521883	0.520954	0.000928792
42.	0.42	0.520322	0.519412	0.000909746
43.	0.43	0.518798	0.517906	0.000891578
44.	0.44	0.517309	0.516434	0.000874229
45.	0.45	0.515853	0.514995	0.000857644
46.	0.46	0.514429	0.513587	0.000841774
47.	0.47	0.513036	0.51221	0.000826574
48.	0.48	0.511673	0.510861	0.000812001
49.	0.49	0.510337	0.509539	0.000798018
50.	0.5	0.509029	0.508245	0.000784589

A plot comparing the numerical and the exact solution at position x=π/2 for times t with $0 \le t \le t_{numSteps}$

In[76]:=
```
Print["Method with fixed timesteps"];
Print["Numerical solution (points) and exact solution
   (line) at position x=π/2 vs. time"]; FigNum = ListPlot[
  Table[{tabFT[[n, 2]], tabFT[[n, 3]]}, {n, 1, numSteps}], PlotRange → All];
FigExact = Plot[uSol[Pi / 2, time, gammak],
   {time, 0, 1.1 * tabFT[[numSteps + 1, 2]]}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"time", "u"}, PlotRange → All]
```

```
Method with fixed timesteps
Numerical solution (points) and exact solution (line) at position x=π/2 vs. time
```

Out[78]=


Example 2

$$\frac{\partial^\gamma u}{\partial t^\gamma} = \frac{\partial^2 u}{\partial x^2} + a\left[\sin(\nu t) + \nu^\gamma \sin\left(\nu t + \gamma\pi/2\right)\right], \quad 0 \le x \le \pi,$$
$$u(x=0,t) = u(x=\pi,t) = 0,$$
$$u(x,0) = \sin x.$$

Here we provide the parameters of the problem and its exact solution

Parameters: K_γ=Kg=1, ν=nu=2*Pi; a=aParameter=1/10

In[79]:=
```
Kg = 1.; nu = 2. * Pi; aParameter = 0.1;
```

Exact solution: $u(x,t) = \left[E_\gamma(-t^\gamma) + a\sin(\nu t)\right]\sin x$

In[80]:=
```
uSol[x_, t_, ga_] := Sin[x] * (MittagLefflerE[ga, -t^ga] + aParameter * Sin[nu * t])
```

Source term: $f(x,t) = a\left[\sin(\nu t) + \nu^\gamma \sin\left(\nu t + \gamma\pi/2\right)\right]$

In[81]:=
```
funSource[x_, t_, ga_] :=
 aParameter * (nu^ga * Sin[nu * t + ga * Pi / 2] + Sin[nu * t]) * Sin[x]
```

Order of the fractional derivative: γ=gammak. Here we choose γ=1/4

In[82]:=
```
gammak = 0.25;
```

Spatial discretization: Δx=deltax=Pi/Np where, in this case, we choose Np=2Nx=40

In[83]:=
```
Nx = 20; Np = 2 * Nx; deltax = Pi / (2. * Nx);
```

We set the initial condition: $U^{(0)}$=UmatrixIni

In[84]:=
```
UmatrixIni = Table[ Sin[j * deltax] , {j, 0, Np}] // N
```

Out[84]=
```
{0., 0.0784591, 0.156434, 0.233445, 0.309017, 0.382683, 0.45399,
 0.522499, 0.587785, 0.649448, 0.707107, 0.760406, 0.809017, 0.85264,
 0.891007, 0.92388, 0.951057, 0.97237, 0.987688, 0.996917, 1.,
 0.996917, 0.987688, 0.97237, 0.951057, 0.92388, 0.891007, 0.85264,
 0.809017, 0.760406, 0.707107, 0.649448, 0.587785, 0.522499, 0.45399,
 0.382683, 0.309017, 0.233445, 0.156434, 0.0784591, 1.22465×10^-16}
```

Boundary conditions: u(x=0,t)=uL[t]=0; u(x=π,t)=uR[t]=0.

In[85]:=
```
uL[t_] = 0.;
uR[t_] = 0.;
```

■ Numerical results with the T&E method for tolerance τ=0.001

uMs is a matrix that contains the full solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$ for m=0,1,2,....,numSteps.

In[87]:=
```
tolerance = 0.001;
numSteps = 40;
CPUtime = Timing[uMs = uMsolutionTE[numSteps, tolerance]];
Print["Example 2. T&E method. Case with τ=", tolerance, " for γ=",
  gammak, ", Np=", Np, ", Δ0=", Delta0TE, ", Δmax=" , DeltatcapTE ];
Print["CPU time=", CPUtime[[1]], " seconds"];
Print["The solution is saved in the matrix uMs"];
```

```
Example 2. T&E method. Case with τ=0.001 for γ=0.25, Np=40, Δ0=0.01, Δmax=1.×10^6
CPU time=10.0153 seconds
The solution is saved in the matrix uMs
```

The solution for the m-th timestep is just the m+1 element of this matrix:
uMs[[m+1]]=$\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$

In[93]:=
```
mStep = 5;
Print["The numerical solution at time ", tt[mStep], " is:"];
uMs[[mStep]]
```

The numerical solution at time 0.0000257969 is:

Out[94]= $\{2.21925 \times 10^{-16}$, 0.0747088, 0.148957, 0.222287, 0.294246, 0.364392, 0.43229, 0.497524, 0.55969, 0.618405, 0.673308, 0.724059, 0.770347, 0.811885, 0.848417, 0.879719, 0.905597, 0.925892, 0.940478, 0.949266, 0.952201, 0.949266, 0.940478, 0.925892, 0.905597, 0.879719, 0.848417, 0.811885, 0.770347, 0.724059, 0.673308, 0.618405, 0.55969, 0.497524, 0.43229, 0.364392, 0.294246, 0.222287, 0.148957, 0.0747088, 0.$\}$

The [[m+1,j+1] element of this matrix is the numerical solution for time t_m and position x_j: uMs[[m+1,j+1]]=$U_j^{(m)}$

In[95]:=
```
mStep = 5; jPos = Nx + 1;
Print["The numerical solution at time t=",
 tt[mStep], " and position x= " , jPos * deltax , " is:"];
uMs[[mStep, jPos]]
```

The numerical solution at time t=0.0000257969 and position x= 1.64934 is:

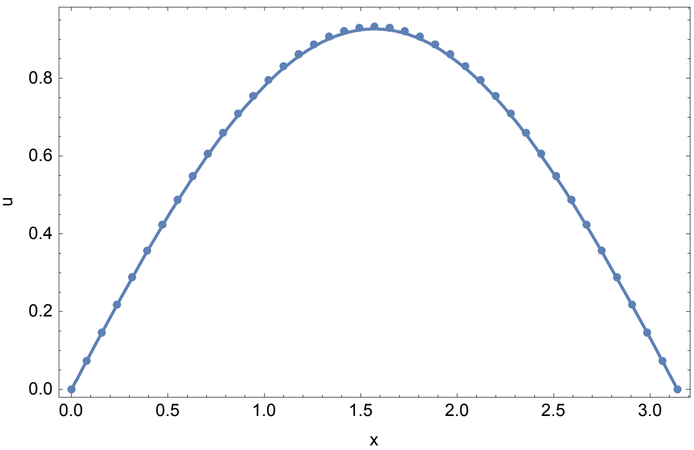
Out[96]= 0.952201

A plot comparing the numerical and the exact solution at time t_{mStep}

In[97]:=
```
mStep = 5;
Print["T&E method"];
Print["Numerical solution (points) and exact solution (line)  at time t=",
  tt[mStep]];
 FigNum = ListPlot[Table[{xj = j * deltax, uMs[[mStep + 1, j + 1]]}, {j, 0, Np}],
  PlotRange → All];
FigExact = Plot[uSol[x, tt[mStep], gammak], {x, 0, Pi}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"x", "u"}]
```

T&E method

Numerical solution (points) and exact solution (line) at time t=0.0000257969

Out[100]=


Here we give a table that compares the numerical and exact solution at the middle point, x=π/2, for the first numSteps timesteps.

In[101]:=
```
headTE = {"step", "t", "UT& E (π/2,t)", "uexact (π/2,t)", "error"};
```

In[102]:=
```
tabTE = Table[{m, tt[m], uMs[[m + 1, Nx + 1]], uSol[Pi / 2, tt[m], gammak],
    uMs[[m + 1, Nx + 1]] - uSol[Pi / 2, tt[m], gammak]}, {m, 0, numSteps}] // N;
Insert[tabTE, headTE, 1] // TableForm
```

Out[102]//TableForm=

step	t	$U_{T\&E}(\pi/2,t)$	$u_{exact}(\pi/2,t)$	error
0.	0.	1.	1.	0.
1.	9.53674×10^{-9}	0.991549	0.989207	0.00234278
2.	1.62125×10^{-7}	0.981127	0.978308	0.00281937
3.	1.38283×10^{-6}	0.967293	0.963452	0.0038404
4.	6.26564×10^{-6}	0.952201	0.947501	0.00470047
5.	0.0000257969	0.932896	0.926751	0.00614552
6.	0.0000648594	0.916253	0.909393	0.00685951
7.	0.000142984	0.899393	0.891647	0.00774581
8.	0.000299234	0.8812	0.872395	0.00880473
9.	0.000611734	0.861092	0.851089	0.0100029
10.	0.00123673	0.83874	0.82745	0.0112906
11.	0.00248673	0.814008	0.801421	0.0125865
12.	0.00498673	0.787011	0.773252	0.0137593
13.	0.00998673	0.758302	0.743689	0.0146131
14.	0.0199867	0.72925	0.71436	0.01489
15.	0.0399867	0.702753	0.688438	0.0143147
16.	0.0799867	0.683816	0.671154	0.0126623
17.	0.239987	0.661216	0.654857	0.0063594
18.	0.319987	0.632109	0.62727	0.00483828
19.	0.399987	0.58529	0.581321	0.00396916
20.	0.559987	0.467897	0.464176	0.00372055
21.	0.639987	0.419491	0.415372	0.00411808
22.	0.719987	0.3911	0.386641	0.00445895
23.	0.799987	0.387595	0.383011	0.00458403
24.	0.879987	0.407943	0.40356	0.00438328
25.	1.03999	0.489996	0.486212	0.00378341
26.	1.11999	0.527535	0.525077	0.00245781
27.	1.19999	0.548814	0.547341	0.00147288
28.	1.27999	0.547167	0.546368	0.000799038
29.	1.35999	0.521864	0.52135	0.000514589
30.	1.43999	0.478133	0.477494	0.000639275
31.	1.59999	0.37743	0.375247	0.00218321
32.	1.67999	0.3431	0.340465	0.00263494
33.	1.75999	0.331269	0.328215	0.00305387
34.	1.83999	0.343977	0.34079	0.00318733
35.	1.91999	0.37733	0.374376	0.00295412
36.	2.07999	0.467305	0.465721	0.00158431
37.	2.15999	0.500392	0.499627	0.0007656
38.	2.23999	0.512828	0.512736	0.0000925927
39.	2.31999	0.50096	0.501238	-0.000277301
40.	2.39999	0.467168	0.467432	-0.000263527

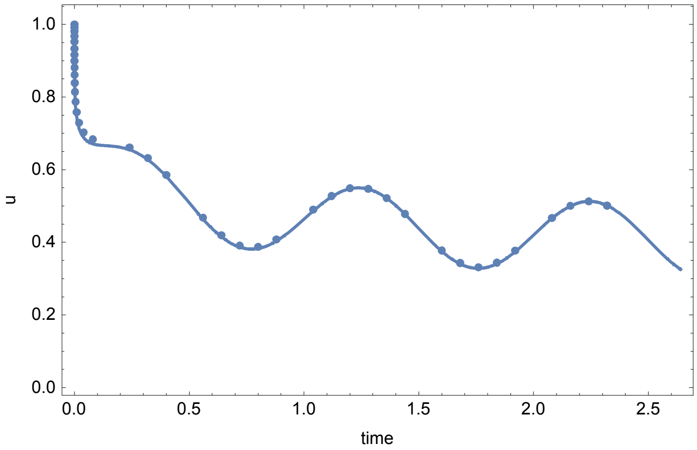
A plot comparing the numerical and the exact solution at position x=π/2 for times t with $0 \le t \le t_{numSteps}$

In[103]:=
```
Print["T&E method"];
Print["Numerical solution (points) and exact solution
   (line) at position x=π/2 vs. time"]; FigNum = ListPlot[
  Table[{tabTE[[n, 2]], tabTE[[n, 3]]}, {n, 1, numSteps}], PlotRange → All];
FigExact = Plot[uSol[Pi / 2, time, gammak],
   {time, 0, 1.1 * tabTE[[numSteps + 1, 2]]}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"time", "u"}, PlotRange → All]
```

T&E method

Numerical solution (points) and exact solution (line) at position x=π/2 vs. time

Out[105]=



■ Numerical results with the *predictive* method for tolerance τ=0.001

uMs is a matrix that contains the full solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$ for m=0,1,2,....,numSteps.

In[106]:=
```
tolerance = 0.001;
numSteps = 40;
CPUtime = Timing[uMs = uMsolutionPredictive[numSteps, tolerance]];
Print["Example 2. Predictive method. Case with τ=", tolerance,
  " for γ=", gammak, ", Np=", Np, ", Δ₀=", Delta0Predictive, ", Δmax=" ,
  DeltatcapPredictive , ",ω=", omegaParameter, ", θ=", thetaParameter ];
Print["CPU time=", CPUtime[[1]], " seconds"];
Print["The solution is saved in the matrix uMs"];
```

Example 2. Predictive method. Case with τ=0.001
 for γ=0.25, Np=40, Δ_0=0.01, Δ_{max}=1. × 10^6, ω=0.5, θ=1.5

CPU time=6.72364 seconds

The solution is saved in the matrix uMs

The solution for the m-th timestep is just the m+1 element of this matrix:
uMs[[m+1]]=$\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$

In[112]:=
```
mStep = 5;
Print["The numerical solution at time ", tt[mStep], " is:"];
uMs[[mStep]]
```

The numerical solution at time 0.0000273439 is:

Out[113]= $\{-1.18893\times10^{-16}$, 0.0743466, 0.148235, 0.221209, 0.29282, 0.362625, 0.430194, 0.495111, 0.556976, 0.615407, 0.670043, 0.720549, 0.766612, 0.807948, 0.844304, 0.875454, 0.901206, 0.921402, 0.935918, 0.944663, 0.947584, 0.944663, 0.935918, 0.921402, 0.901206, 0.875454, 0.844304, 0.807948, 0.766612, 0.720549, 0.670043, 0.615407, 0.556976, 0.495111, 0.430194, 0.362625, 0.29282, 0.221209, 0.148235, 0.0743466, 0.$\}$

The [[m+1,j+1]] element of this matrix is the numerical solution for time t_m and position x_j: uMs[[m+1,j+1]]=$U_j^{(m)}$

In[114]:=
```
mStep = 5; jPos = Nx + 1;
Print["The numerical solution at time t=",
 tt[mStep], " and position x= " , jPos * deltax , " is:"];
uMs[[mStep, jPos]]
```

The numerical solution at time t=0.0000273439 and position x= 1.64934 is:

Out[115]= 0.947584

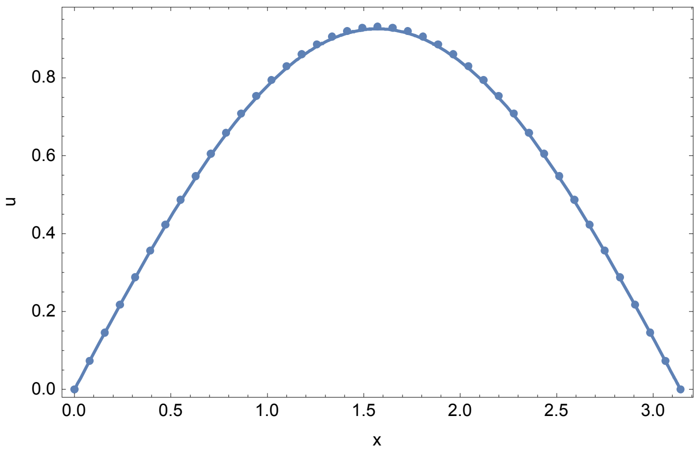
A plot comparing the numerical and the exact solution at time t_{mStep}

In[116]:=
```
mStep = 5;
Print["Predictive method"];
Print["Numerical solution (points) and exact solution (line)  at time t=",
  tt[mStep]];
FigNum = ListPlot[Table[{xj = j * deltax, uMs[[mStep + 1, j + 1]]}, {j, 0, Np}],
  PlotRange → All];
FigExact = Plot[uSol[x, tt[mStep], gammak], {x, 0, Pi}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"x", "u"}]
```

Predictive method

Numerical solution (points) and exact solution (line) at time t=0.0000273439

Out[119]=



Here we give a table that compares the numerical and exact solution at the middle point, x=π/2, for the first numSteps timesteps.

```
In[120]:= headPredictive = {"step", "t", "UPredic(π/2,t)", "uexact(π/2,t)", "error"};

In[121]:= tabPred = Table[{m, tt[m], uMs[[m + 1, Nx + 1]], uSol[Pi / 2, tt[m], gammak],
              uMs[[m + 1, Nx + 1]] - uSol[Pi / 2, tt[m], gammak]}, {m, 0, numSteps}] // N;
          Insert[tabPred, headPredictive, 1] // TableForm
```

Out[121]//TableForm=

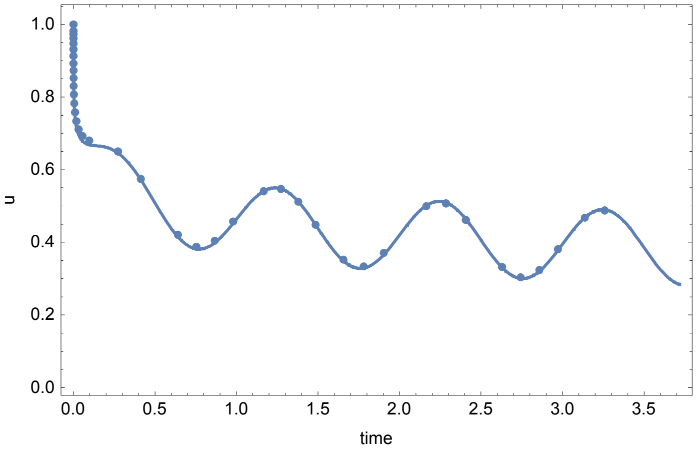
step	t	$U_{Predic}(\pi/2,t)$	$u_{exact}(\pi/2,t)$	error
0.	0.	1.	1.	0.
1.	1.99158×10^{-7}	0.982118	0.977187	0.00493095
2.	7.1136×10^{-7}	0.972555	0.968886	0.00366901
3.	2.62604×10^{-6}	0.96125	0.957349	0.00390109
4.	8.946×10^{-6}	0.947584	0.942874	0.00471027
5.	0.0000273439	0.931536	0.925752	0.00578428
6.	0.0000758795	0.91312	0.906093	0.00702719
7.	0.000194263	0.892378	0.883985	0.00839321
8.	0.000406444	0.873002	0.863601	0.00940085
9.	0.000812616	0.852408	0.841873	0.0105345
10.	0.00157782	0.830355	0.818656	0.0116998
11.	0.00298614	0.807004	0.794208	0.0127966
12.	0.00552596	0.782665	0.768945	0.0137202
13.	0.0100287	0.757858	0.743508	0.0143499
14.	0.0178981	0.733459	0.718905	0.0145547
15.	0.0315013	0.710898	0.696684	0.0142131
16.	0.0549534	0.692314	0.679074	0.0132394
17.	0.09674	0.680155	0.668611	0.0115435
18.	0.273203	0.650122	0.645778	0.00434415
19.	0.413491	0.574132	0.57213	0.00200182
20.	0.640445	0.420596	0.415143	0.00545229
21.	0.753619	0.387873	0.381969	0.00590351
22.	0.866793	0.404636	0.398726	0.00591068
23.	0.979967	0.457383	0.452591	0.00479263
24.	1.16661	0.540716	0.540616	0.0000999709
25.	1.2726	0.546954	0.547499	-0.000545438
26.	1.37859	0.51173	0.51253	-0.000799441
27.	1.48458	0.448299	0.448417	-0.000118356
28.	1.65621	0.35205	0.348707	0.0033427
29.	1.77996	0.33368	0.329076	0.00460421
30.	1.90371	0.370899	0.366216	0.00468354
31.	2.16382	0.499506	0.500782	-0.00127585
32.	2.2852	0.507091	0.509255	-0.00216421
33.	2.40657	0.461676	0.463866	-0.0021893
34.	2.62898	0.332253	0.330547	0.00170599
35.	2.74328	0.303576	0.300468	0.00310719
36.	2.85757	0.32357	0.319853	0.00371765
37.	2.97187	0.380892	0.377875	0.0030172
38.	3.13659	0.467615	0.467821	-0.000205541
39.	3.25798	0.487856	0.489704	-0.00184778
40.	3.37937	0.453953	0.456342	-0.00238814

A plot comparing the numerical and the exact solution at position x=π/2 for times t with $0 \le t \le t_{numSteps}$

In[122]:=
```
Print["Predictive method"];
Print["Numerical solution (points) and exact solution
    (line) at position x=π/2 vs. time"]; FigNum = ListPlot[
  Table[{tabPred[[n, 2]], tabPred[[n, 3]]}, {n, 1, numSteps}], PlotRange → All];
FigExact = Plot[uSol[Pi / 2, time, gammak],
    {time, 0, 1.1 * tabPred[[numSteps + 1, 2]]}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"time", "u"}, PlotRange → All]
```

```
Predictive method
Numerical solution (points) and exact solution (line) at position x=π/2 vs. time
```

Out[124]=



■ Numerical results with the method with *fixed timesteps* for Δ=0.01=htFT

uMs is a matrix that contains the full solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$ for m=0,1,2,....,numSteps.

In[125]:=
```
htFT = 0.01;
numSteps = 50;
CPUtime = Timing[uMs = uMsolutionFixedTimesteps[numSteps * htFT, htFT]];
Print["Example 1. Method with fixed timesteps of size ",
  htFT, " for γ=", gammak, ", Np=", Np ];
Print["CPU time=", CPUtime[[1]], " seconds"];
Print["The solution is saved in the matrix uMs"];
```

```
Example 1. Method with fixed timesteps of size 0.01 for γ=0.25, Np=40
CPU time=1.32601 seconds
The solution is saved in the matrix uMs
```

The solution for the m-th timestep is just the m+1 element of this matrix:
uMs[[m+1]]=$\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$

In[131]:=
```
mStep = 5;
Print["The numerical solution at time ", tt[mStep], " is:"];
uMs[[mStep]]
```

The numerical solution at time 0.05 is:

Out[132]= {1.47792×10^{-14}, 0.0556149, 0.110887, 0.165475, 0.219043, 0.271261, 0.321806, 0.370367, 0.416645, 0.460354, 0.501225, 0.539005, 0.573463, 0.604385, 0.63158, 0.654882, 0.674146, 0.689254, 0.700112, 0.706654, 0.708839, 0.706654, 0.700112, 0.689254, 0.674146, 0.654882, 0.63158, 0.604385, 0.573463, 0.539005, 0.501225, 0.460354, 0.416645, 0.370367, 0.321806, 0.271261, 0.219043, 0.165475, 0.110887, 0.0556149, 0.}

The [[m+1,j+1] element of this matrix is the numerical solution for time t_m and position x_j: uMs[[m+1,j+1]]=$U_j^{(m)}$

In[133]:=
```
mStep = 5; jPos = Nx + 1;
Print["The numerical solution at time t=",
 tt[mStep], " and position x= " , jPos * deltax , " is:"];
uMs[[mStep, jPos]]
```

The numerical solution at time t=0.05 and position x= 1.64934 is:

Out[134]= 0.708839

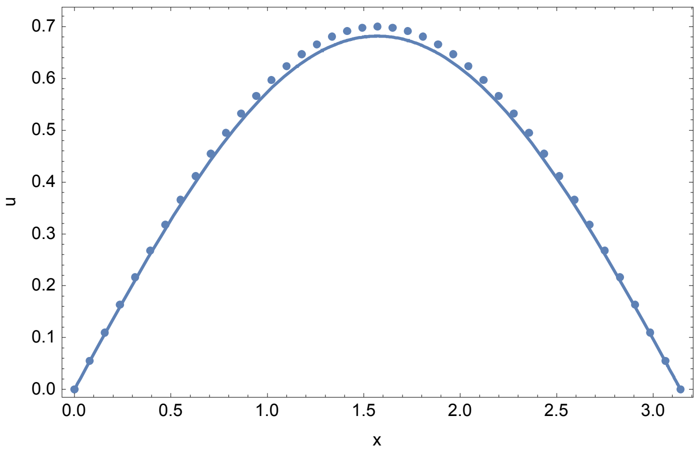
A plot comparing the numerical and the exact solution at time t_{mStep}

In[135]:=
```
mStep = 5;
Print["Method with fixed timesteps"];
Print["Numerical solution (points) and exact solution (line)  at time t=",
  tt[mStep]];
 FigNum = ListPlot[Table[{xj = j * deltax, uMs[[mStep + 1, j + 1]]}, {j, 0, Np}],
  PlotRange → All];
FigExact = Plot[uSol[x, tt[mStep], gammak], {x, 0, Pi}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"x", "u"}]
```

Method with fixed timesteps

Numerical solution (points) and exact solution (line) at time t=0.05

Out[138]=



Here we give a table that compares the numerical and exact solution at the middle point, x=π/2, for the first numSteps timesteps.

```
In[139]:= headFixedTimesteps = {"step", "t", "UFixed Δ(π/2,t)", "uexact(π/2,t)", "error"};

In[140]:= tabFT = Table[{m, tt[m], uMs[[m + 1, Nx + 1]], uSol[Pi / 2, tt[m], gammak],
              uMs[[m + 1, Nx + 1]] - uSol[Pi / 2, tt[m], gammak]}, {m, 0, numSteps}] // N;
          Insert[tabFT, headFixedTimesteps, 1] // TableForm
```

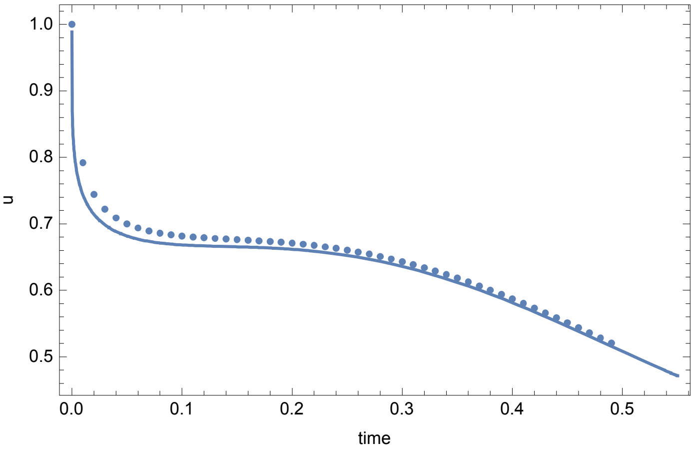
Out[140]//TableForm=

step	t	$U_{Fixed\,\Delta}(\pi/2,t)$	$u_{exact}(\pi/2,t)$	error
0.	0.	1.	1.	0.
1.	0.01	0.792004	0.743632	0.0483723
2.	0.02	0.744105	0.714333	0.0297718
3.	0.03	0.722049	0.698475	0.0235742
4.	0.04	0.708839	0.688428	0.0204115
5.	0.05	0.700017	0.681631	0.0183866
6.	0.06	0.693806	0.676888	0.0169178
7.	0.07	0.689304	0.673533	0.01577
8.	0.08	0.68598	0.671151	0.0148291
9.	0.09	0.683494	0.669462	0.0140322
10.	0.1	0.681607	0.668266	0.0133415
11.	0.11	0.680146	0.667413	0.0127324
12.	0.12	0.678977	0.666789	0.0121882
13.	0.13	0.677997	0.6663	0.0116969
14.	0.14	0.677122	0.665872	0.0112497
15.	0.15	0.676281	0.665441	0.0108398
16.	0.16	0.675417	0.664956	0.0104619
17.	0.17	0.674482	0.66437	0.010112
18.	0.18	0.673433	0.663646	0.00978654
19.	0.19	0.672234	0.662751	0.00948277
20.	0.2	0.670855	0.661657	0.00919835
21.	0.21	0.66927	0.660339	0.0089313
22.	0.22	0.667457	0.658777	0.00867994
23.	0.23	0.665396	0.656953	0.00844282
24.	0.24	0.663072	0.654854	0.00821868
25.	0.25	0.660473	0.652467	0.00800643
26.	0.26	0.657589	0.649784	0.00780509
27.	0.27	0.654413	0.646799	0.00761383
28.	0.28	0.650939	0.643507	0.00743186
29.	0.29	0.647165	0.639906	0.00725853
30.	0.3	0.64309	0.635997	0.00709321
31.	0.31	0.638717	0.631782	0.00693536
32.	0.32	0.634049	0.627264	0.00678449
33.	0.33	0.629091	0.622451	0.00664014
34.	0.34	0.623851	0.617349	0.00650191
35.	0.35	0.618337	0.611968	0.00636941
36.	0.36	0.612561	0.606319	0.00624231
37.	0.37	0.606535	0.600415	0.00612029
38.	0.38	0.600271	0.594268	0.00600306
39.	0.39	0.593786	0.587895	0.00589035
40.	0.4	0.587094	0.581312	0.00578192
41.	0.41	0.580214	0.574537	0.00567753
42.	0.42	0.573165	0.567588	0.00557698
43.	0.43	0.565964	0.560484	0.00548005
44.	0.44	0.558633	0.553247	0.00538657
45.	0.45	0.551193	0.545897	0.00529636
46.	0.46	0.543666	0.538456	0.00520926
47.	0.47	0.536073	0.530948	0.00512512
48.	0.48	0.528438	0.523394	0.00504379
49.	0.49	0.520784	0.515819	0.00496513
50.	0.5	0.513134	0.508245	0.00488903

A plot comparing the numerical and the exact solution at position x=π/2 for times t with $0 \le t \le t_{numSteps}$

In[141]:=
```
Print["Method with fixed timesteps"];
Print["Numerical solution (points) and exact solution
   (line) at position x=π/2 vs. time"]; FigNum = ListPlot[
  Table[{tabFT[[n, 2]], tabFT[[n, 3]]}, {n, 1, numSteps}], PlotRange → All];
FigExact = Plot[uSol[Pi / 2, time, gammak],
   {time, 0, 1.1 * tabFT[[numSteps + 1, 2]]}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"time", "u"}, PlotRange → All]
```

```
Method with fixed timesteps
Numerical solution (points) and exact solution (line) at position x=π/2 vs. time
```

Out[143]=


Example 3

$$\frac{\partial^\gamma u}{\partial t^\gamma} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L,$$
$$u(x=0,t) = u_0,$$
$$u(x=L,t) = 0,$$
$$u(x,0) = 0$$

Here we provide the parameters of the problem and its exact solution

Parameters: K_γ=Kg=1, u_0=u0val=1, L=LL=4, M=Mvalue=8

In[144]:=
```
Kg = 1.; u0value = 1.; LL = 4.; Mvalue = 8;
```

Exact solution [Eq. (26)]

$$u(x,t) = u_0 \sum_{m=0}^{M} H_{10}^{11}\left[mz_c + z \left| \begin{matrix} 1, \gamma/2 \\ 0, 1 \end{matrix} \right. \right] - u_0 \sum_{m=1}^{M} H_{10}^{11}\left[mz_c - z \left| \begin{matrix} 1, \gamma/2 \\ 0, 1 \end{matrix} \right. \right]$$

where $z=x/(K\,t^\gamma)^{1/2}$ and $z_c = 2L\,/\,(K\,t^\gamma)^{1/2}$ and $M \to \infty$

Fox H functions $H^{11}_{10}\left[z \,\middle|\, \begin{matrix} 1, \gamma/2 \\ 0, 1 \end{matrix}\right]$ =H1011[z,γ] for γ=1/4, 1/3, 1/2, 2/3, 3/4 and 1:

In[145]:= H1011[z_, 1/4] :=

$$-\frac{1}{\text{Gamma}\left[\frac{7}{8}\right]} z \,\text{HypergeometricPFQ}\left[\left\{\frac{1}{8}\right\}, \left\{\frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, \frac{9}{8}\right\}, -\frac{z^8}{16\,777\,216}\right] + \frac{1}{5040}\left(5040 + \frac{1}{\text{Gamma}\left[\frac{3}{4}\right]} 2520 z^2 \,\text{HypergeometricPFQ}\left[\left\{\frac{1}{4}\right\}, \left\{\frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, \frac{9}{8}, \frac{5}{4}\right\}, -\frac{z^8}{16\,777\,216}\right] - \frac{1}{\text{Gamma}\left[\frac{5}{8}\right]} 840 z^3 \,\text{HypergeometricPFQ}\left[\left\{\frac{3}{8}\right\}, \left\{\frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, \frac{9}{8}, \frac{5}{4}, \frac{11}{8}\right\}, -\frac{z^8}{16\,777\,216}\right] + \frac{1}{\sqrt{\pi}} 210 z^4 \,\text{HypergeometricPFQ}\left[\left\{\frac{1}{2}\right\}, \left\{\frac{5}{8}, \frac{3}{4}, \frac{7}{8}, \frac{9}{8}, \frac{5}{4}, \frac{11}{8}, \frac{3}{2}\right\}, -\frac{z^8}{16\,777\,216}\right] - \frac{1}{\text{Gamma}\left[\frac{3}{8}\right]} 42 z^5 \,\text{HypergeometricPFQ}\left[\left\{\frac{5}{8}\right\}, \left\{\frac{3}{4}, \frac{7}{8}, \frac{9}{8}, \frac{5}{4}, \frac{11}{8}, \frac{3}{2}, \frac{13}{8}\right\}, -\frac{z^8}{16\,777\,216}\right] + \frac{1}{\text{Gamma}\left[\frac{1}{4}\right]} 7 z^6 \,\text{HypergeometricPFQ}\left[\left\{\frac{3}{4}\right\}, \left\{\frac{7}{8}, \frac{9}{8}, \frac{5}{4}, \frac{11}{8}, \frac{3}{2}, \frac{13}{8}, \frac{7}{4}\right\}, -\frac{z^8}{16\,777\,216}\right] - \frac{1}{\text{Gamma}\left[\frac{1}{8}\right]} z^7 \,\text{HypergeometricPFQ}\left[\left\{\frac{7}{8}\right\}, \left\{\frac{9}{8}, \frac{5}{4}, \frac{11}{8}, \frac{3}{2}, \frac{13}{8}, \frac{7}{4}, \frac{15}{8}\right\}, -\frac{z^8}{16\,777\,216}\right]\right)$$

In[146]:= H1011[z_, 1/3] :=

$$1 + \frac{1}{120} z \left(-\frac{1}{\text{Gamma}\left[\frac{5}{6}\right]} 120 \,\text{HypergeometricPFQ}\left[\left\{\frac{1}{6}\right\}, \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \frac{7}{6}\right\}, -\frac{z^6}{46\,656}\right] + z\left(\frac{1}{\text{Gamma}\left[\frac{2}{3}\right]} 60 \,\text{HypergeometricPFQ}\left[\left\{\frac{1}{3}\right\}, \left\{\frac{1}{2}, \frac{2}{3}, \frac{5}{6}, \frac{7}{6}, \frac{4}{3}\right\}, -\frac{z^6}{46\,656}\right] + z\left(-\frac{1}{\sqrt{\pi}} 20 \,\text{HypergeometricPFQ}\left[\left\{\frac{1}{2}\right\}, \left\{\frac{2}{3}, \frac{5}{6}, \frac{7}{6}, \frac{4}{3}, \frac{3}{2}\right\}, -\frac{z^6}{46\,656}\right] + z\left(\frac{1}{\text{Gamma}\left[\frac{1}{3}\right]} 5 \,\text{HypergeometricPFQ}\left[\left\{\frac{2}{3}\right\}, \left\{\frac{5}{6}, \frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}\right\}, -\frac{z^6}{46\,656}\right] - \frac{1}{\text{Gamma}\left[\frac{1}{6}\right]} z \,\text{HypergeometricPFQ}\left[\left\{\frac{5}{6}\right\}, \left\{\frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{11}{6}\right\}, -\frac{z^6}{46\,656}\right]\right)\right)\right)\right)$$

In[147]:= H1011[z_, 1/2] :=

$$1 - \left(z \,\text{HypergeometricPFQ}\left[\left\{\frac{1}{4}\right\}, \left\{\frac{1}{2}, \frac{3}{4}, \frac{5}{4}\right\}, -\frac{z^4}{256}\right]\right) \Big/ \text{Gamma}\left[\frac{3}{4}\right] + \frac{1}{6} z^2 \left(\frac{1}{\sqrt{\pi}} 3 \,\text{HypergeometricPFQ}\left[\left\{\frac{1}{2}\right\}, \left\{\frac{3}{4}, \frac{5}{4}, \frac{3}{2}\right\}, -\frac{z^4}{256}\right] - \left(z \,\text{HypergeometricPFQ}\left[\left\{\frac{3}{4}\right\}, \left\{\frac{5}{4}, \frac{3}{2}, \frac{7}{4}\right\}, -\frac{z^4}{256}\right]\right) \Big/ \text{Gamma}\left[\frac{1}{4}\right]\right)$$

In[148]:= H1011[z_, 2/3] :=

$$1 - \left(z \,\text{HypergeometricPFQ}\left[\left\{\frac{1}{3}\right\}, \left\{\frac{2}{3}, \frac{4}{3}\right\}, \frac{z^3}{27}\right]\right) \Big/ \text{Gamma}\left[\frac{2}{3}\right] + \left(z^2 \,\text{HypergeometricPFQ}\left[\left\{\frac{2}{3}\right\}, \left\{\frac{4}{3}, \frac{5}{3}\right\}, \frac{z^3}{27}\right]\right) \Big/ \left(2 \,\text{Gamma}\left[\frac{1}{3}\right]\right)$$

In[149]:=
```
H1011[z_, 3/4] := -1/Gamma[5/8] z HypergeometricPFQ[{1/8, 11/24, 19/24}, {1/4, 3/8, 1/2, 5/8, 3/4, 7/8, 9/8}, -27 z^8/16777216] +
  1/5040 (5040 + 1/Gamma[1/4] 2520 z^2 HypergeometricPFQ[{1/4, 7/12, 11/12}, {3/8, 1/2, 5/8, 3/4, 7/8, 9/8, 5/4}, -27 z^8/16777216] -
    1/Gamma[-1/8] 840 z^3 HypergeometricPFQ[{3/8, 17/24, 25/24}, {1/2, 5/8, 3/4, 7/8, 9/8, 5/4, 11/8}, -27 z^8/16777216] -
    1/Sqrt[π] 105 z^4 HypergeometricPFQ[{1/2, 5/6, 7/6}, {5/8, 3/4, 7/8, 9/8, 5/4, 11/8, 3/2}, -27 z^8/16777216] -
    1/Gamma[-7/8] 42 z^5 HypergeometricPFQ[{5/8, 23/24, 31/24}, {3/4, 7/8, 9/8, 5/4, 11/8, 3/2, 13/8}, -27 z^8/16777216] +
    1/Gamma[-5/4] 7 z^6 HypergeometricPFQ[{3/4, 13/12, 17/12}, {7/8, 9/8, 5/4, 11/8, 3/2, 13/8, 7/4}, -27 z^8/16777216] -
    1/Gamma[-13/8] z^7 HypergeometricPFQ[{7/8, 29/24, 37/24}, {9/8, 5/4, 11/8, 3/2, 13/8, 7/4, 15/8}, -27 z^8/16777216])
```

In[150]:=
```
H1011[z_, 1] := Erfc[z/2]
```

Auxiliary functions for evaluating the exact solution

In[151]:=
```
HEsca[z_, ga_] := Which[z > 50, 0, z ≤ 50, H1011[SetPrecision[z, 60], ga]]
 (*for z larger than 50 we approximate this function by zero*)
```

In[152]:=
```
uEscalon[x_, t_, ga_, LL_, nterm_] := (
  z = x / (Kg * t) ^ (ga / 2);
  zc = (2 * LL) / (Kg * t) ^ (ga / 2);
  HEsca[z, ga] - Sum[HEsca[n * zc - z, ga] - HEsca[n * zc + z, ga], {n, 1, nterm}]
 ) (*for very small times, so that z>50,
 this function is approximated by zero*)
```

Exact solution (an approximation retaining Mvalue terms in the series expansion)

In[153]:=
```
uSol[x_, 0, ga_] := 0 /; x > 0
```

In[154]:=
```
uSol[x_, t_, ga_] := u0value * uEscalon[x, t, ga, LL, Mvalue]
```

Source term. In this example is zero.

In[155]:=
```
funSource[x_, t_, ga_] := 0.
```

Order of the fractional derivative: γ=gammak. Here we choose γ=1/4

In[156]:=
```
gammak = 0.25;
gammaVal = Rationalize[gammak];
```

Spatial discretization: Δx≡deltax=Pi/Np where, in this case, we choose Np=2Nx=40

In[158]:=
```
Nx = 20; Np = 2 * Nx; deltax = LL / (2. * Nx);
```

We set the initial condition: $U^{(0)}$=UmatrixIni

In[159]:=
```
UmatrixIni = Table[ Which[j == 0, u0value, j > 0, 0.] , {j, 0, Np}]
```

Out[159]=
```
{1., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.,
 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.}
```

Boundary conditions: u(x=0,t)=uL[t]=u0val=1; u(x=π,t)=uR[t]=0.

In[160]:=
```
uL[t_] = u0value;
uR[t_] = 0.;
```

■ Numerical results with the T&E method for tolerance τ=0.001

uMs is a matrix that contains the full solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$ for m=0,1,2,....,numSteps.

In[162]:=
```
tolerance = 0.001;
numSteps = 50;
CPUtime = Timing[uMs = uMsolutionTE[numSteps, tolerance]];
Print["Example 3. T&E method. Case with Kg=",
  Kg, ", u₀=", u0value, ", L=", LL, ", M=", Mvalue];
Print["Numerical T&E solution for tolerance τ=", tolerance, " γ=",
  gammak, ", Np=", Np, ", Δ₀=", Delta0TE, ", Δmax=" , DeltatcapTE ];
Print["CPU time=", CPUtime[[1]], " seconds"];
Print["The solution is saved in the matrix uMs"];
```

Example 3. T&E method. Case with Kg=1., u_0=1., L=4., M=8

Numerical T&E solution for tolerance τ=0.001 γ=0.25, Np=40, Δ_0=0.01, $\Delta_{max}=1.\times 10^6$

CPU time=17.5345 seconds

The solution is saved in the matrix uMs

The solution for the m-th timestep is just the m+1 element of this matrix:
uMs[[m+1]]=$\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$

In[169]:=
```
mStep = 10;
Print["The numerical solution at time ", tt[mStep], " is:"];
uMs[[mStep]]
```

The numerical solution at time 1.82366×10^{-11} is:

Out[170]= $\{1., 0.141665, 0.0190509, 0.00246172, 0.000308289, 0.0000376529,$
$4.50636\times 10^{-6}, 5.30434\times 10^{-7}, 6.15842\times 10^{-8}, 7.06872\times 10^{-9}, 8.03625\times 10^{-10},$
$9.06288\times 10^{-11}, 1.01513\times 10^{-11}, 1.1305\times 10^{-12}, 1.25281\times 10^{-13}, 1.38254\times 10^{-14},$
$1.52025\times 10^{-15}, 1.66656\times 10^{-16}, 1.82214\times 10^{-17}, 1.98774\times 10^{-18}, 2.16417\times 10^{-19},$
$2.35232\times 10^{-20}, 2.55312\times 10^{-21}, 2.76761\times 10^{-22}, 2.99689\times 10^{-23}, 3.24215\times 10^{-24},$
$3.50465\times 10^{-25}, 3.78576\times 10^{-26}, 4.08697\times 10^{-27}, 4.40983\times 10^{-28}, 4.75605\times 10^{-29},$
$5.12745\times 10^{-30}, 5.52597\times 10^{-31}, 5.95372\times 10^{-32}, 6.41295\times 10^{-33}, 6.90607\times 10^{-34},$
$7.43568\times 10^{-35}, 8.00458\times 10^{-36}, 8.61462\times 10^{-37}, 9.16511\times 10^{-38}, 0.\}$

The [[m+1,j+1]] element of this matrix is the numerical solution for time t_m and position x_j:
uMs[[m+1,j+1]]=$U_j^{(m)}$

In[171]:=
```
mStep = 10; jPos = Nx + 1;
Print["The numerical solution at time t=",
 tt[mStep], " and position x= " , jPos * deltax , " is:"];
uMs[[mStep, jPos]]
```

The numerical solution at time t=1.82366×10^{-11} and position x= 2.1 is:

Out[172]= 2.16417×10^{-19}

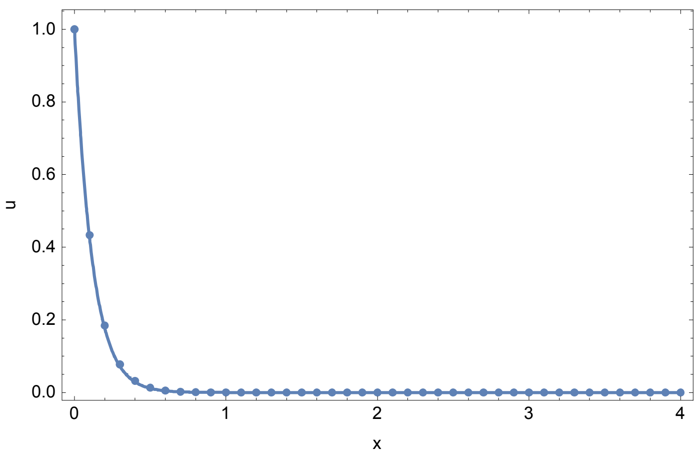
A plot comparing the numerical and the exact solution at time t_{mStep}

In[173]:=
```
mStep = 20;
Print["T&E method"];
Print["Numerical solution (points) and exact solution (line)  at time t=",
  tt[mStep]];
 FigNum = ListPlot[Table[{xj = j * deltax, uMs[[mStep + 1, j + 1]]}, {j, 0, Np}],
  PlotRange → All];
FigExact = Plot[uSol[x, tt[mStep], gammaVal], {x, 0, LL}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"x", "u"}, PlotRange → All]
```

T&E method

Numerical solution (points) and exact solution (line) at time t=1.90731×10^{-8}

Out[176]=



Here we give a table that compares the numerical and exact solution at the point xpos=1, for the first numSteps timesteps.

In[177]:=
```
headTE = {"step", "t", "UT&E(1,t)", "uexact(1,t)", "error"};
```

In[178]:=
```
xPos = 1.;
jPos = Round[xPos / deltax];
tabTE = Table[{m, tt[m], uMs[[m + 1, jPos + 1]], uSol[xPos, tt[m], gammaVal],
    uMs[[m + 1, jPos + 1]] - uSol[xPos, tt[m], gammaVal]}, {m, 0, numSteps}] // N;
Insert[tabTE, headTE, 1] // TableForm
```

Out[179]//TableForm=

step	t	$U_{T\&E}(1,t)$	$u_{exact}(1,t)$	error
0.	0.	0.	0.	0.
1.	7.10543×10^{-17}	1.54898×10^{-21}	0.	1.54898×10^{-21}
2.	1.20792×10^{-15}	1.69242×10^{-18}	0.	1.69242×10^{-18}
3.	1.03029×10^{-14}	2.86039×10^{-16}	0.	2.86039×10^{-16}
4.	4.66827×10^{-14}	9.37365×10^{-15}	3.38655×10^{-24}	9.37365×10^{-15}
5.	1.92202×10^{-13}	2.38698×10^{-13}	5.23569×10^{-20}	2.38698×10^{-13}
6.	7.74278×10^{-13}	5.24616×10^{-12}	1.26175×10^{-16}	5.24603×10^{-12}
7.	1.93843×10^{-12}	3.54283×10^{-11}	9.80888×10^{-15}	3.54185×10^{-11}
8.	4.26674×10^{-12}	1.80862×10^{-10}	2.70854×10^{-13}	1.80591×10^{-10}
9.	8.92335×10^{-12}	8.03625×10^{-10}	4.39719×10^{-12}	7.99227×10^{-10}
10.	1.82366×10^{-11}	3.26293×10^{-9}	5.03586×10^{-11}	3.21257×10^{-9}
11.	3.6863×10^{-11}	1.23474×10^{-8}	4.4261×10^{-10}	1.19047×10^{-8}
12.	7.41159×10^{-11}	4.38913×10^{-8}	3.12777×10^{-9}	4.07635×10^{-8}
13.	1.48622×10^{-10}	1.46949×10^{-7}	1.83116×10^{-8}	1.28637×10^{-7}
14.	2.97633×10^{-10}	4.63551×10^{-7}	9.0783×10^{-8}	3.72768×10^{-7}
15.	5.95657×10^{-10}	1.37717×10^{-6}	3.87855×10^{-7}	9.89313×10^{-7}
16.	1.1917×10^{-9}	3.85144×10^{-6}	1.44927×10^{-6}	2.40217×10^{-6}
17.	2.3838×10^{-9}	0.0000101375	4.79808×10^{-6}	5.33944×10^{-6}
18.	4.76798×10^{-9}	0.0000251215	0.0000142369	0.0000108846
19.	9.53635×10^{-9}	0.0000586588	0.000038254	0.0000204048
20.	1.90731×10^{-8}	0.000129236	0.0000939481	0.0000352881
21.	3.81466×10^{-8}	0.000269148	0.000212666	0.0000564823
22.	7.62936×10^{-8}	0.000531026	0.000447099	0.0000839268
23.	1.52588×10^{-7}	0.000995056	0.000879004	0.000116052
24.	3.05175×10^{-7}	0.0017757	0.00162615	0.000149552
25.	6.10351×10^{-7}	0.00302637	0.00284679	0.000179573
26.	1.2207×10^{-6}	0.00494048	0.00474016	0.00020032
27.	2.44141×10^{-6}	0.00774776	0.00754179	0.000205968
28.	4.88281×10^{-6}	0.0117054	0.0115138	0.000191649
29.	9.76562×10^{-6}	0.0170845	0.0169303	0.000154275
30.	0.0000195312	0.0241535	0.0240605	0.0000930352
31.	0.0000390625	0.0331597	0.0331502	9.48647×10^{-6}
32.	0.000078125	0.0443124	0.0444051	-0.0000927377
33.	0.00015625	0.0577681	0.0579766	-0.000208477
34.	0.0003125	0.0736199	0.0739517	-0.000331754
35.	0.000625	0.0918923	0.0923487	-0.000456404
36.	0.00125	0.11254	0.113116	-0.000576704
37.	0.0025	0.135451	0.136139	-0.000687733
38.	0.005	0.160457	0.161243	-0.000786152
39.	0.01	0.187338	0.188207	-0.000869664
40.	0.02	0.215839	0.216773	-0.000934434
41.	0.04	0.24567	0.246655	-0.000984525
42.	0.08	0.276531	0.277549	-0.00101787
43.	0.16	0.308102	0.30914	-0.00103871
44.	0.32	0.340081	0.341105	-0.00102428
45.	0.64	0.372089	0.373116	-0.00102666
46.	1.28	0.403885	0.404844	-0.000958511
47.	2.56	0.435067	0.435963	-0.00089603
48.	5.12	0.465321	0.46616	-0.000839249
49.	10.24	0.494386	0.495138	-0.000752173
50.	20.48	0.521952	0.522637	-0.00068441

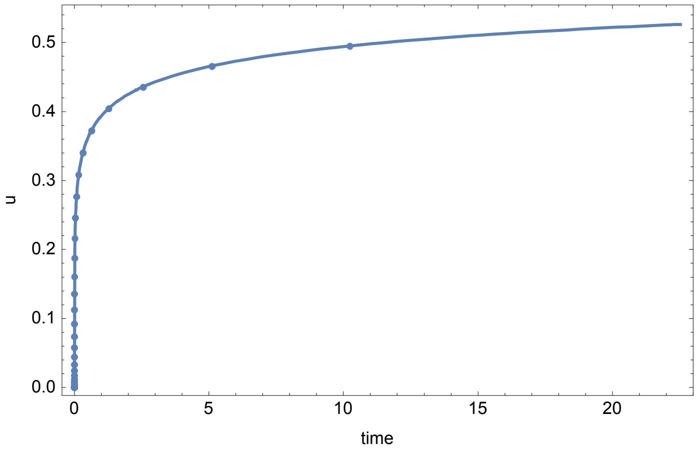
A plot comparing the numerical and the exact solution at position x=xpos for times t with 0≤t≤

$t_{numSteps}$

In[180]:=
```
Print["T&E method"];
Print["Numerical solution (points) and exact solution (line) at position x=",
 xPos, " vs. time"]; FigNum = ListPlot[
  Table[{tabTE[[n, 2]], tabTE[[n, 3]]}, {n, 1, numSteps}], PlotRange → All];
FigExact = Plot[uSol[xPos, time, gammaVal],
   {time, 0, 1.1 * tabTE[[numSteps + 1, 2]]}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"time", "u"}, PlotRange → All]
```

```
T&E method
Numerical solution (points) and exact solution (line) at position x=1. vs. time
```

Out[182]=



■ Numerical results with the *predictive* method for tolerance τ=0.001

uMs is a matrix that contains the full solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$ for m=0,1,2,....,numSteps.

In[183]:=
```
tolerance = 0.001;
numSteps = 50;
CPUtime = Timing[uMs = uMsolutionPredictive[numSteps, tolerance]];
Print["Example 3. Predictive method. Case with Kg=",
  Kg, ", u0=", u0value, ", L=", LL, ", M=", Mvalue];
Print["Numerical Predictive solution for tolerance τ=", tolerance,
  " γ=", gammak, ", Np=", Np, ", Δ0=", Delta0Predictive, ", Δmax=" ,
  DeltatcapPredictive ",ω=", omegaParameter, ", θ=", thetaParameter ];
Print["CPU time=", CPUtime[[1]], " seconds"];
Print["The solution is saved in the matrix uMs"];
```

```
Example 3. Predictive method. Case with Kg=1., u0=1., L=4., M=8
Numerical Predictive solution for tolerance τ=
 0.001 γ=0.25, Np=40, Δ0=0.01, Δmax=1. × 10^6 ,ω=0.5, θ=1.5
CPU time=9.71886 seconds
The solution is saved in the matrix uMs
```

The solution for the m-th timestep is just the m+1 element of this matrix:
uMs[[m+1]]=$\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$

In[190]:=
```
mStep = 5;
Print["The numerical solution at time ", tt[mStep], " is:"];
uMs[[mStep]]
```

The numerical solution at time 2.72151×10^{-13} is:

Out[191]= $\{1., 0.0527855, 0.00260387, 0.00012275, 5.60758\times 10^{-6}, 2.50533\times 10^{-7}, 1.10154\times 10^{-8}, 4.78708\times 10^{-10}, 2.06265\times 10^{-11}, 8.83153\times 10^{-13}, 3.76365\times 10^{-14}, 1.59833\times 10^{-15}, 6.77\times 10^{-17}, 2.86197\times 10^{-18}, 1.20811\times 10^{-19}, 5.09412\times 10^{-21}, 2.14623\times 10^{-22}, 9.03674\times 10^{-24}, 3.80317\times 10^{-25}, 1.60003\times 10^{-26}, 6.72969\times 10^{-28}, 2.82993\times 10^{-29}, 1.18985\times 10^{-30}, 5.00216\times 10^{-32}, 2.10275\times 10^{-33}, 8.83872\times 10^{-35}, 3.7151\times 10^{-36}, 1.56148\times 10^{-37}, 6.5628\times 10^{-39}, 2.75825\times 10^{-40}, 1.15923\times 10^{-41}, 4.87196\times 10^{-43}, 2.04754\times 10^{-44}, 8.60513\times 10^{-46}, 3.61644\times 10^{-47}, 1.51986\times 10^{-48}, 6.3874\times 10^{-50}, 2.68438\times 10^{-51}, 1.12814\times 10^{-52}, 4.73275\times 10^{-54}, 0.\}$

The [[m+1,j+1] element of this matrix is the numerical solution for time t_m and position x_j:
uMs[[m+1,j+1]]=$U_j^{(m)}$

In[192]:=
```
mStep = 5; jPos = Nx + 1;
Print["The numerical solution at time t=",
 tt[mStep], " and position x= " , jPos * deltax , " is:"];
uMs[[mStep, jPos]]
```

The numerical solution at time t=2.72151×10^{-13} and position x= 2.1 is:

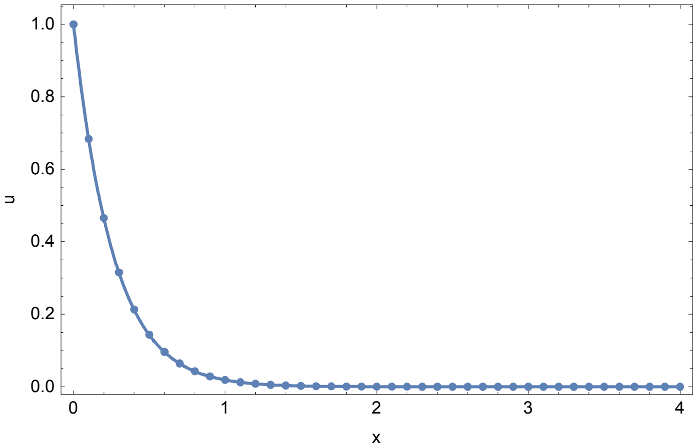
Out[193]= 6.72969×10^{-28}

A plot comparing the numerical and the exact solution at time t_{mStep}

In[194]:=
```
mStep = 30;
Print["Predictive method"];
Print["Numerical solution (points) and exact solution (line)  at time t=",
  tt[mStep]];
 FigNum = ListPlot[Table[{xj = j * deltax, uMs[[mStep + 1, j + 1]]}, {j, 0, Np}],
  PlotRange → All];
FigExact = Plot[uSol[x, tt[mStep], gammaVal], {x, 0, LL}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"x", "u"}, PlotRange → All]
```

```
Predictive method
Numerical solution (points) and exact solution (line)  at time t=0.0000117433
```

Out[197]=



Here we give a table that compares the numerical and exact solution at the point xpos=1, for the first numSteps timesteps.

In[198]:=
```
headPredictive = {"step", "t", "UPredic(1,t)", "uexact(1,t)", "error"};
```

In[199]:=
```
xPos = 1.;
jPos = Round[xPos / deltax];
tabPred = Table[{m, tt[m], uMs[[m + 1, jPos + 1]], uSol[xPos, tt[m], gammaVal],
    uMs[[m + 1, jPos + 1]] - uSol[xPos, tt[m], gammaVal]}, {m, 0, numSteps}] // N;
Insert[tabPred, headPredictive, 1] // TableForm
```

Out[200]//TableForm=

step	t	$U_{Predic}(1,t)$	$u_{exact}(1,t)$	error
0.	0.	0.	0.	0.
1.	1.88907×10^{-15}	4.59444×10^{-18}	0.	4.59444×10^{-18}
2.	6.71556×10^{-15}	8.51621×10^{-17}	0.	8.51621×10^{-17}
3.	2.49647×10^{-14}	2.05524×10^{-15}	0.	2.05524×10^{-15}
4.	8.66039×10^{-14}	3.76365×10^{-14}	2.90135×10^{-22}	3.76365×10^{-14}
5.	2.72151×10^{-13}	5.03781×10^{-13}	4.23141×10^{-19}	5.0378×10^{-13}
6.	7.83539×10^{-13}	5.17085×10^{-12}	1.33988×10^{-16}	5.17072×10^{-12}
7.	2.09997×10^{-12}	4.25045×10^{-11}	1.3971×10^{-14}	4.24905×10^{-11}
8.	5.31417×10^{-12}	2.89587×10^{-10}	6.39864×10^{-13}	2.88948×10^{-10}
9.	1.11528×10^{-11}	1.25505×10^{-9}	9.6614×10^{-12}	1.24539×10^{-9}
10.	2.61509×10^{-11}	6.56361×10^{-9}	1.57453×10^{-10}	6.40615×10^{-9}
11.	5.25304×10^{-11}	2.36244×10^{-8}	1.22189×10^{-9}	2.24025×10^{-8}
12.	1.02792×10^{-10}	7.77859×10^{-8}	7.33353×10^{-9}	7.04523×10^{-8}
13.	1.98987×10^{-10}	2.39348×10^{-7}	3.65694×10^{-8}	2.02779×10^{-7}
14.	3.82306×10^{-10}	6.90761×10^{-7}	1.5582×10^{-7}	5.34941×10^{-7}
15.	7.30792×10^{-10}	1.87703×10^{-6}	5.79726×10^{-7}	1.2973×10^{-6}
16.	1.39329×10^{-9}	4.82105×10^{-6}	1.91753×10^{-6}	2.90352×10^{-6}
17.	2.65574×10^{-9}	0.0000117453	5.72377×10^{-6}	6.02155×10^{-6}
18.	5.78787×10^{-9}	0.0000320134	0.00001895	0.0000130633
19.	1.12522×10^{-8}	0.0000711915	0.0000477771	0.0000234144
20.	2.14996×10^{-8}	0.000147478	0.000108798	0.0000386799
21.	4.07158×10^{-8}	0.000288274	0.000228751	0.0000595232
22.	7.66657×10^{-8}	0.000535202	0.000449328	0.0000858738
23.	1.43906×10^{-7}	0.000948927	0.00083222	0.000116706
24.	2.69922×10^{-7}	0.00161427	0.0014644	0.000149872
25.	5.07034×10^{-7}	0.0026453	0.00246324	0.000182062
26.	9.55561×10^{-7}	0.00418928	0.00398031	0.000208968
27.	1.79269×10^{-6}	0.00639218	0.00616493	0.000227251
28.	3.35349×10^{-6}	0.00943211	0.00919906	0.000233042
29.	6.2701×10^{-6}	0.0135098	0.013287	0.000222811
30.	0.0000117433	0.0188431	0.0186493	0.000193824
31.	0.0000219651	0.0256041	0.0254564	0.00014774
32.	0.0000410202	0.033957	0.0338712	0.0000858139
33.	0.000076643	0.0440721	0.0440636	8.47193×10^{-6}
34.	0.000143185	0.0560558	0.0561357	-0.0000798865
35.	0.000267231	0.0699586	0.0701329	-0.000174288
36.	0.000499201	0.0858464	0.0861201	-0.000273713
37.	0.000931497	0.103651	0.104021	-0.000370297
38.	0.00173866	0.123341	0.123805	-0.000463823
39.	0.00324344	0.144786	0.145335	-0.00054925
40.	0.00605315	0.167877	0.168503	-0.000626582
41.	0.01129	0.192408	0.193099	-0.000691638
42.	0.0210581	0.218208	0.218954	-0.000745568
43.	0.0392947	0.245086	0.245874	-0.00078831
44.	0.0732909	0.272782	0.2736	-0.000817639
45.	0.136707	0.301089	0.301925	-0.000835567
46.	0.255115	0.329791	0.330634	-0.000842681
47.	0.476873	0.358702	0.359544	-0.000841972
48.	0.893181	0.387602	0.388429	-0.000826684
49.	1.68245	0.416397	0.417208	-0.000810971
50.	3.1883	0.444849	0.445639	-0.000790418

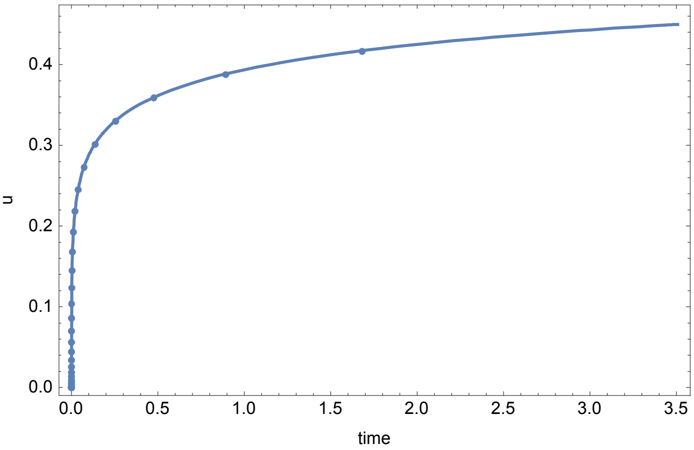
A plot comparing the numerical and the exact solution at position x=xpos for times t with 0≤t≤

t_{numSteps}

In[201]:=
```
Print["Predictive method"];
Print["Numerical solution (points) and exact solution (line) at position x=",
 xPos, " vs. time"]; FigNum = ListPlot[
  Table[{tabPred[[n, 2]], tabPred[[n, 3]]}, {n, 1, numSteps}], PlotRange → All];
FigExact = Plot[uSol[xPos, time, gammaVal],
   {time, 0, 1.1 * tabPred[[numSteps + 1, 2]]}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"time", "u"}, PlotRange → All]
```

```
Predictive method
Numerical solution (points) and exact solution (line) at position x=1. vs. time
```

Out[203]=



■ Numerical results with the method with *fixed timesteps* for Δ=0.01=htFT

uMs is a matrix that contains the full solution $\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$ for m=0,1,2,....,numSteps.

In[204]:=
```
htFT = 0.01;
numSteps = 50;
CPUtime = Timing[uMs = uMsolutionFixedTimesteps[numSteps * htFT, htFT]];
Print["Example 3. Method with fixed timesteps. Case with Kg=",
  Kg, ", u₀=", u0value, ", L=", LL];
Print["Numerical solution with fixed timesteps of size ",
  htFT, " for γ=", gammak, ", Np=", Np ];
Print["CPU time=", CPUtime[[1]], " seconds"];
Print["The solution is saved in the matrix uMs"];
```

```
Example 3. Method with fixed timesteps. Case with Kg=1., u₀=1., L=4.
Numerical solution with fixed timesteps of size 0.01 for γ=0.25, Np=40
CPU time=1.26361 seconds
The solution is saved in the matrix uMs
```

The solution for the m-th timestep is just the m+1 element of this matrix:
uMs[[m+1]]=$\{U_0^{(m)}, U_1^{(m)}, U_2^{(m)}, \ldots U_{Np}^{(m)}\}$

In[211]:=
```
mStep = 5;
Print["The numerical solution at time ", tt[mStep], " is:"];
uMs[[mStep]]
```

```
The numerical solution at time 0.05 is:
```

Out[212]=
```
{1., 0.869442, 0.755468, 0.656046, 0.569377, 0.493878, 0.428152, 0.370973,
 0.32126, 0.278064, 0.240554, 0.208, 0.179763, 0.155283, 0.134073, 0.115705,
 0.0998049, 0.0860481, 0.0741508, 0.0638654, 0.0549769, 0.0472976,
 0.0406645, 0.0349358, 0.0299882, 0.0257146, 0.0220217, 0.0188285,
 0.0160645, 0.0136684, 0.0115864, 0.0097719, 0.00818373, 0.00678578,
 0.00554602, 0.00443591, 0.00342975, 0.00250411, 0.00163735, 0.000809146, 0.}
```

The [[m+1,j+1]] element of this matrix is the numerical solution for time t_m and position x_j: uMs[[m+1,j+1]]=$U_j^{(m)}$

In[213]:=
```
mStep = 5; jPos = Nx + 1;
Print["The numerical solution at time t=",
 tt[mStep], " and position x= " , jPos * deltax , " is:"];
uMs[[mStep, jPos]]
```

```
The numerical solution at time t=0.05 and position x= 2.1 is:
```

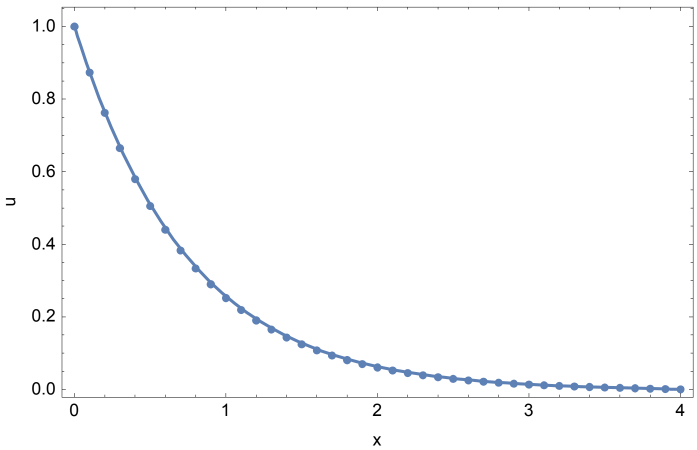
Out[214]= 0.0549769

A plot comparing the numerical and the exact solution at time t_{mStep}

In[215]:=
```
mStep = 5;
Print["Method with fixed timesteps"];
Print["Numerical solution (points) and exact solution (line)  at time t=",
  tt[mStep]];
FigNum = ListPlot[Table[{xj = j * deltax, uMs[[mStep + 1, j + 1]]}, {j, 0, Np}],
  PlotRange → All];
FigExact = Plot[uSol[x, tt[mStep], gammaVal], {x, 0, LL}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"x", "u"}]
```

```
Method with fixed timesteps
Numerical solution (points) and exact solution (line)  at time t=0.05
```

Out[218]=



Here we give a table that compares the numerical and exact solution at the middle point, x=xPos, for the first numSteps timesteps.

In[219]:=
```
headFixedTimesteps = {"step", "t", "U_FixedΔ(xPos,t)", "u_exact(xPos,t)", "error"};
```

In[220]:=
```
xPos = 1.;
jPos = Round[xPos / deltax];
Print["xPos=", xPos];
tabFT = Table[{m, tt[m], uMs[[m + 1, jPos + 1]], uSol[xPos, tt[m], gammaVal],
    uMs[[m + 1, jPos + 1]] - uSol[xPos, tt[m], gammaVal]}, {m, 0, numSteps}] // N;
Insert[tabFT, headFixedTimesteps, 1] // TableForm
```

xPos=1.

Out[222]//TableForm=

step	t	$U_{Fixed\,\Delta}$(xPos,t)	u_{exact}(xPos,t)	error
0.	0.	0.	0.	0.
1.	0.01	0.156877	0.188207	-0.0313305
2.	0.02	0.202973	0.216773	-0.0137997
3.	0.03	0.22562	0.234112	-0.00849165
4.	0.04	0.240554	0.246655	-0.00610114
5.	0.05	0.251757	0.256505	-0.00474831
6.	0.06	0.260747	0.264624	-0.0038773
7.	0.07	0.268264	0.271533	-0.00326925
8.	0.08	0.274729	0.277549	-0.00282059
9.	0.09	0.280401	0.282877	-0.0024759
10.	0.1	0.285456	0.287659	-0.00220283
11.	0.11	0.290015	0.291996	-0.00198117
12.	0.12	0.294167	0.295965	-0.0017977
13.	0.13	0.297979	0.299623	-0.00164334
14.	0.14	0.301504	0.303015	-0.00151171
15.	0.15	0.30478	0.306178	-0.00139815
16.	0.16	0.307841	0.30914	-0.00129919
17.	0.17	0.310714	0.311926	-0.0012122
18.	0.18	0.313419	0.314554	-0.00113515
19.	0.19	0.315977	0.317043	-0.00106643
20.	0.2	0.318401	0.319405	-0.00100478
21.	0.21	0.320705	0.321654	-0.000949157
22.	0.22	0.3229	0.323799	-0.000898736
23.	0.23	0.324997	0.32585	-0.000852821
24.	0.24	0.327003	0.327814	-0.000810841
25.	0.25	0.328926	0.329699	-0.000772315
26.	0.26	0.330773	0.33151	-0.000736839
27.	0.27	0.332549	0.333253	-0.000704067
28.	0.28	0.33426	0.334933	-0.000673705
29.	0.29	0.335909	0.336555	-0.0006455
30.	0.3	0.337502	0.338122	-0.000619233
31.	0.31	0.339043	0.339637	-0.000594714
32.	0.32	0.340533	0.341105	-0.000571775
33.	0.33	0.341977	0.342527	-0.00055027
34.	0.34	0.343377	0.343908	-0.000530072
35.	0.35	0.344737	0.345248	-0.000511066
36.	0.36	0.346057	0.34655	-0.000493152
37.	0.37	0.347341	0.347817	-0.000476239
38.	0.38	0.348589	0.34905	-0.000460247
39.	0.39	0.349805	0.350251	-0.000445104
40.	0.4	0.35099	0.351421	-0.000430746
41.	0.41	0.352145	0.352562	-0.000417113
42.	0.42	0.353272	0.353676	-0.000404154
43.	0.43	0.354372	0.354763	-0.000391821
44.	0.44	0.355446	0.355826	-0.000380069
45.	0.45	0.356495	0.356864	-0.000368861
46.	0.46	0.357521	0.35788	-0.000358159
47.	0.47	0.358525	0.358873	-0.00034793
48.	0.48	0.359507	0.359846	-0.000338146
49.	0.49	0.360469	0.360798	-0.000328777
50.	0.5	0.361411	0.361731	-0.000319799

A plot comparing the numerical and the exact solution at position x=xPos for times t with $0 \leq t \leq t_{numSteps}$

In[223]:=
```
Print["Method with fixed timesteps"];
Print["Numerical solution (points) and exact solution (line) at position x=",
  xPos, " vs. time"]; FigNum = ListPlot[
   Table[{tabFT[[n, 2]], tabFT[[n, 3]]}, {n, 1, numSteps}], PlotRange → All];
FigExact = Plot[uSol[xPos, time, gammaVal],
    {time, 0, 1.1 * tabFT[[numSteps + 1, 2]]}, PlotRange → All];
Show[FigNum, FigExact, Frame → True, FrameLabel → {"time", "u"}, PlotRange → All]
```

```
Method with fixed timesteps
Numerical solution (points) and exact solution (line) at position x=1. vs. time
```

Out[225]=