Shear flow of Inelastic Maxwell particles

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STATISTICAL PHYSICS in EXTREMADURA

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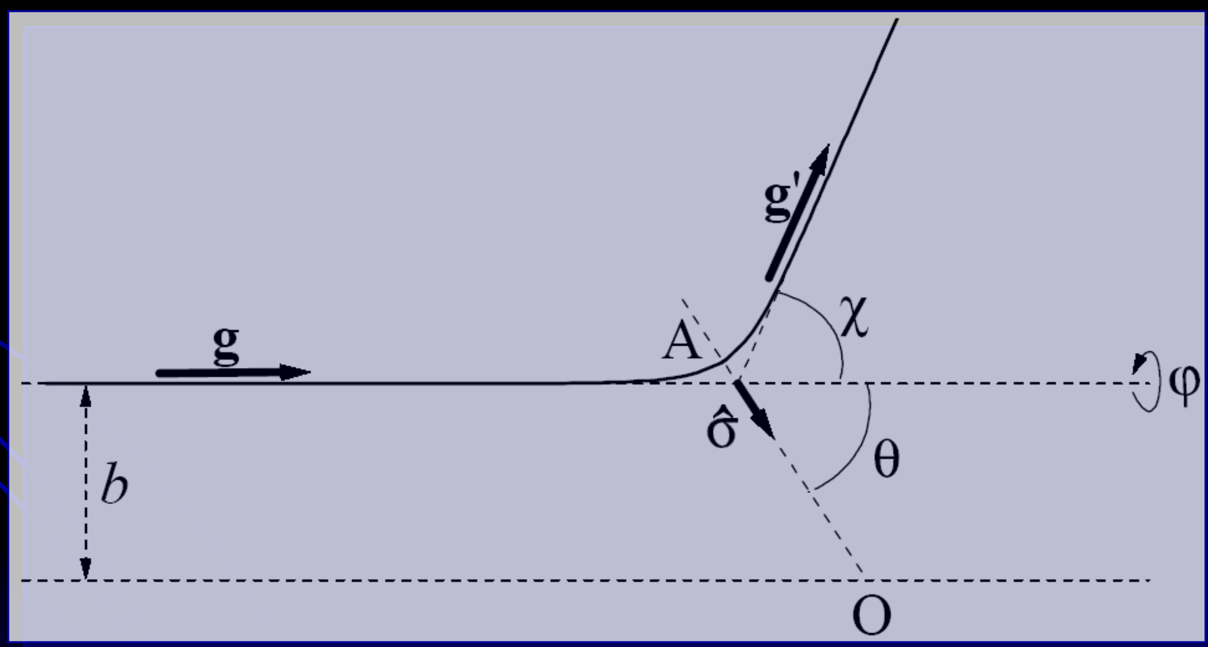
Boltzmann equation (elastic particles)

$$\partial_t f(\mathbf{v}) + \mathbf{v}\cdot\nabla f(\mathbf{v}) = J[\mathbf{v}|f,f]$$



$$J[\mathbf{v}|f,f] = \int d\mathbf{v}\int d\widehat{\boldsymbol{\sigma}}\,\mathcal{F}(g,\widehat{\mathbf{g}}\cdot\widehat{\boldsymbol{\sigma}}) \times\left[f(\mathbf{v}'')f(\mathbf{v}_1'') - f(\mathbf{v})f(\mathbf{v}_1)\right]$$

$\mathcal{F}(g,\widehat{\mathbf{g}}\cdot\widehat{\boldsymbol{\sigma}}) \sim$ Collision rate

$$\left.\begin{aligned} \mathbf{v}'' &= \mathbf{v} - (\mathbf{g}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}} \\ \mathbf{v}_1'' &= \mathbf{v}_1 + (\mathbf{g}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}} \end{aligned}\right\}$$ Restituting velocities

Collision models

Hard spheres: $\mathcal{F}(g, \hat{\mathbf{g}} \cdot \hat{\boldsymbol{\sigma}}) = \sigma^2 g \Theta(\hat{\mathbf{g}} \cdot \hat{\boldsymbol{\sigma}})(\hat{\mathbf{g}} \cdot \hat{\boldsymbol{\sigma}}) \propto g$

Maxwell models: $\mathcal{F}(g, \hat{\mathbf{g}} \cdot \hat{\boldsymbol{\sigma}}) = \frac{\nu_0}{n} \Phi(\hat{\mathbf{g}} \cdot \hat{\boldsymbol{\sigma}}) =$ g-independent

Maxwell models

Velocity moments: $M_r = \int d\mathbf{v} \mathcal{P}_r(\mathbf{v}) f(\mathbf{v})$

Collisional moments: $J_r = \int d\mathbf{v} \mathcal{P}_r(\mathbf{v}) J[\mathbf{v}|f, f]$

$$J_r = \sum_{s=0}^{r} C_{r,s} M_s M_{r-s}$$

Maxwell models

Ernst & Brito (2002): *"What harmonic oscillators are for quantum mechanics, and dumb-bells for polymer physics, that is what elastic and inelastic Maxwell models are for kinetic theory"*

- Exact derivation of Navier-Stokes and Burnett transport coefficients.
- Bobylev-Krook-Wu's (1976) exact solution of the homogeneous BE.
- Rheological properties of the uniform shear flow (Ikenberry &Truesdell,1956).
- Singular behavior of high-degree moments in the USF (Santos et al., 1993; Montanero et al., 1996).
- Fourier law in the nonlinear planar heat flow (Asmolov et al., 1979).
- Rheological properties of the planar Couette flow (Makashev & Nosik, 1981; Tij & Santos, 1995).
- Rheological properties of the gravity-driven Poiseuille flow (Tij et al., 1998).
- …
- Benchmarks to test Bird's Direct Simulation Monte Carlo method (Gallis et al., 2005).

Inelastic Maxwell Particles

(Bobylev et al., 2000 ; Krapivsky & Ben-Naim, 2000; Ernst & Brito, 2002)

- What if collisions are inelastic?
- A new parameter: the coefficient of normal restitution $\alpha \leq 1$.

$$\left.\begin{aligned} \mathbf{v}'' &= \mathbf{v} - \frac{1+\alpha^{-1}}{2}(\mathbf{g}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}} \\ \mathbf{v}_1'' &= \mathbf{v}_1 + \frac{1+\alpha^{-1}}{2}(\mathbf{g}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}} \end{aligned}\right\} \text{Restituting velocities}$$

$$J[\mathbf{v}|f,f] = \frac{5\nu_0}{8\pi n}\int d\mathbf{v}\int d\widehat{\boldsymbol{\sigma}} \times \left[\alpha^{-1} f(\mathbf{v}'')f(\mathbf{v}_1'') - f(\mathbf{v})f(\mathbf{v}_1)\right]$$

Inelastic “cooling”

“Granular” temperature: $T = \frac{m}{3n}\int d\mathbf{v} V^2 f(\mathbf{v})$

$$\frac{m}{3n}\int d\mathbf{v} V^2 J[\mathbf{v}|f,f] = -\zeta(\alpha) T$$

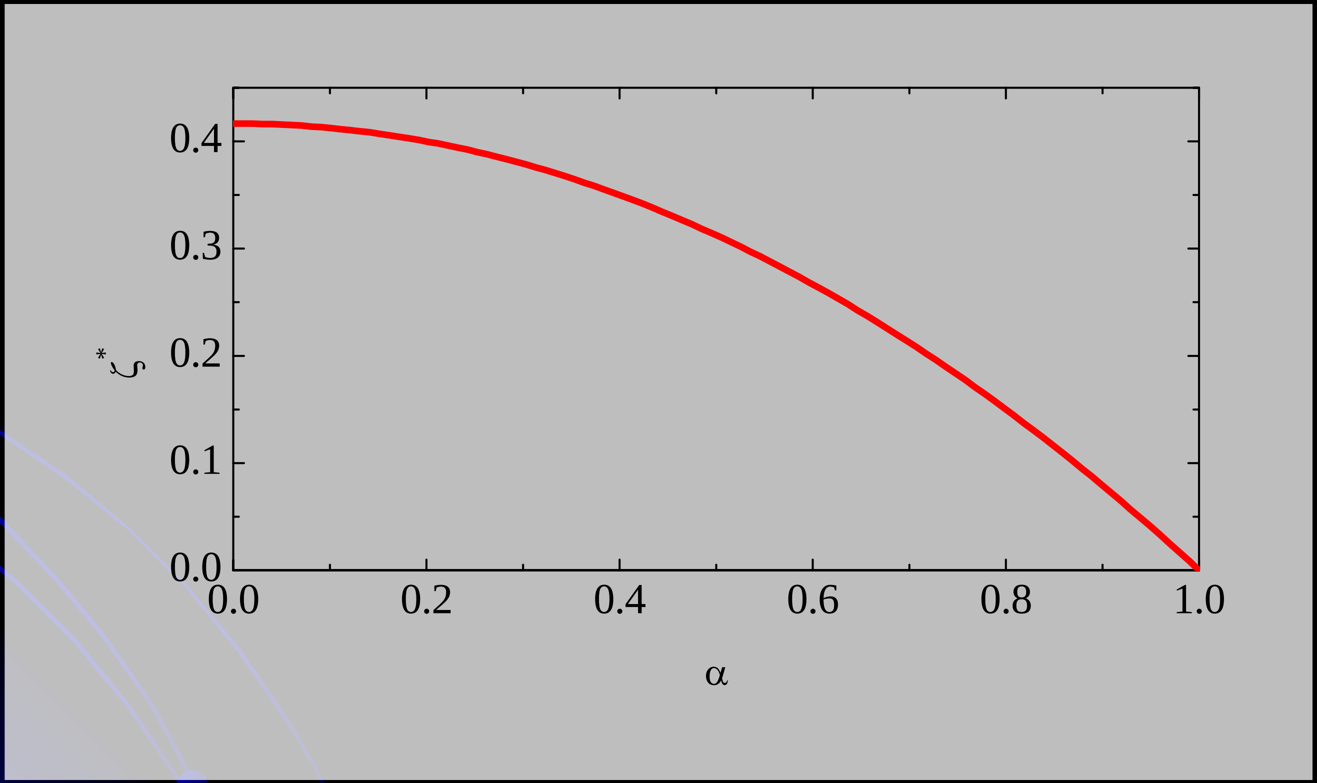
$\zeta(\alpha)$: “Cooling” rate

$$J_r = \sum_{s=0}^{r} C_{r,s}(\alpha) M_s M_{r-s}$$

- We have evaluated the coefficients $C_{r,s}(\alpha)$ associated with the moments through 4th degree.
- 2nd degree: 2 linear coeffs. (cooling rate and momentum transfer rate).
- 3rd degree: 2 linear coeffs. (energy transfer rate and an extra relaxation rate).
- 4th degree: 3 linear coeffs. (relaxation rates) plus 5 nonlinear coeffs.

Cooling rate

$$\zeta(\alpha) = \nu_0 \frac{5}{12}(1-\alpha^2)$$



Homogeneous Cooling State

$$\partial_t f(\mathbf{v},t) = J[\mathbf{v}|f,f]$$

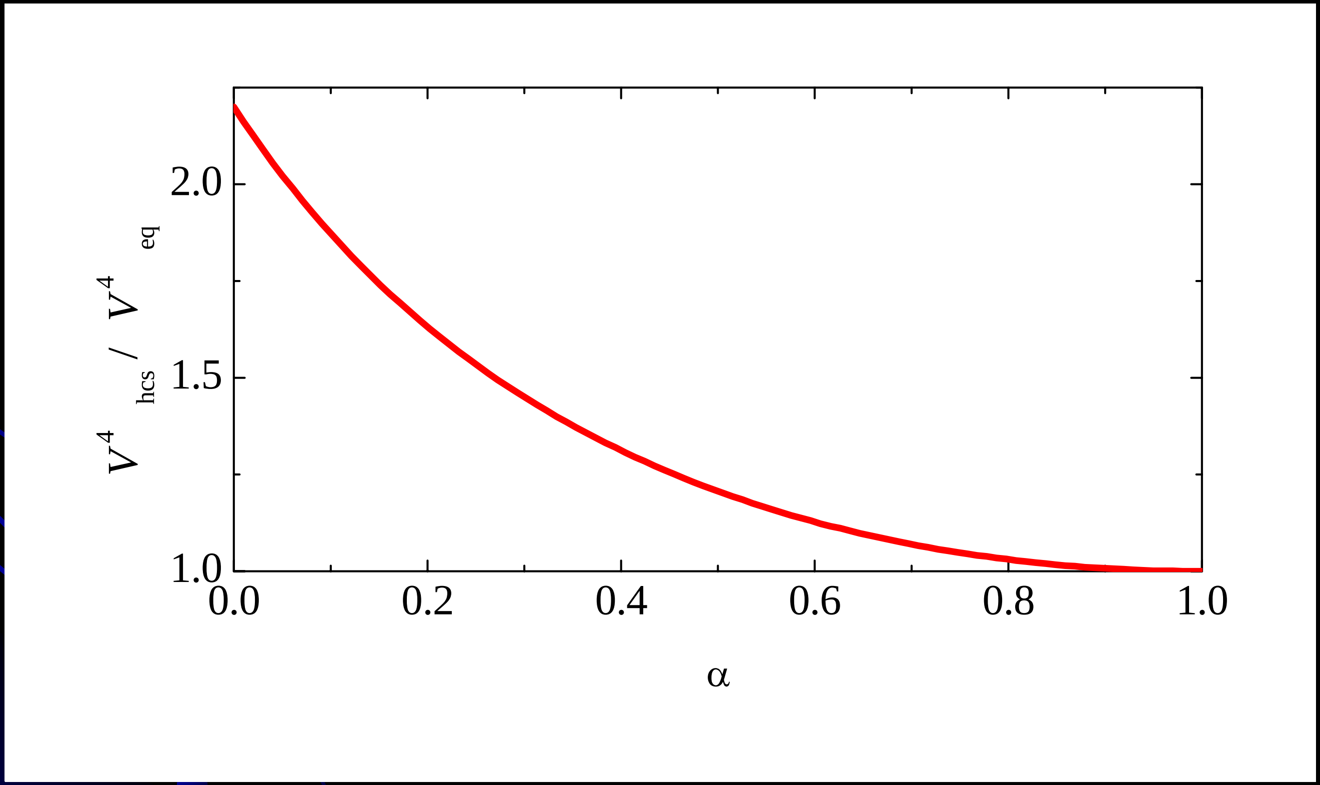
$$\partial_t T = -\zeta(\alpha) T$$

HCS $\Rightarrow$ Similarity solution:

$$f_{\mathsf{hcs}}(\mathbf{v},t) = [T(t)]^{-3/2} F\left(v/\sqrt{T(t)}\right)$$

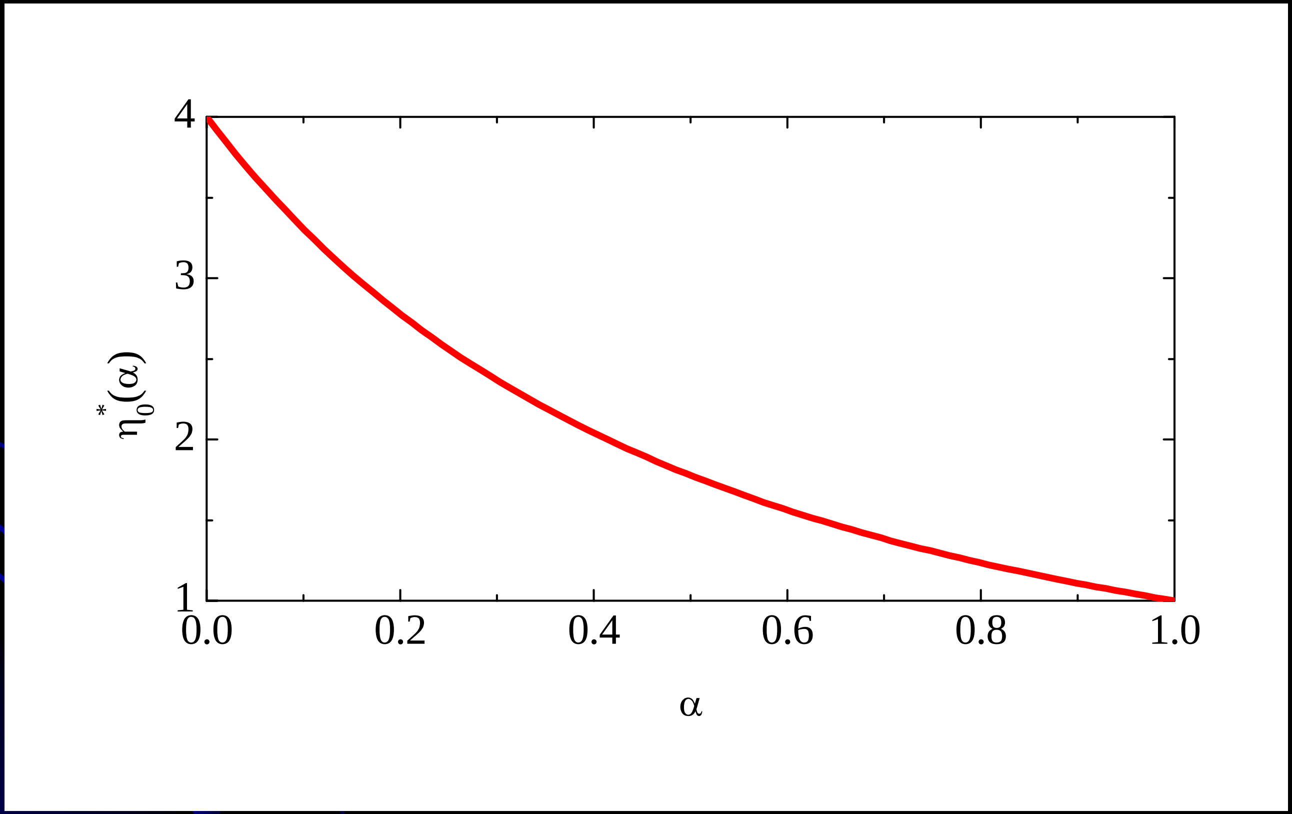
Homogeneous Cooling State

$$\langle V^4\rangle_{\mathrm{hcs}}/\langle V^4\rangle_0 = 1 + \frac{6(1-\alpha)^2}{5+3\alpha(2-\alpha)}$$



Navier-Stokes transport coefficients

Shear viscosity: $\eta_0(\alpha) = \frac{p}{\nu_0} \frac{4}{(1+\alpha)^2}$



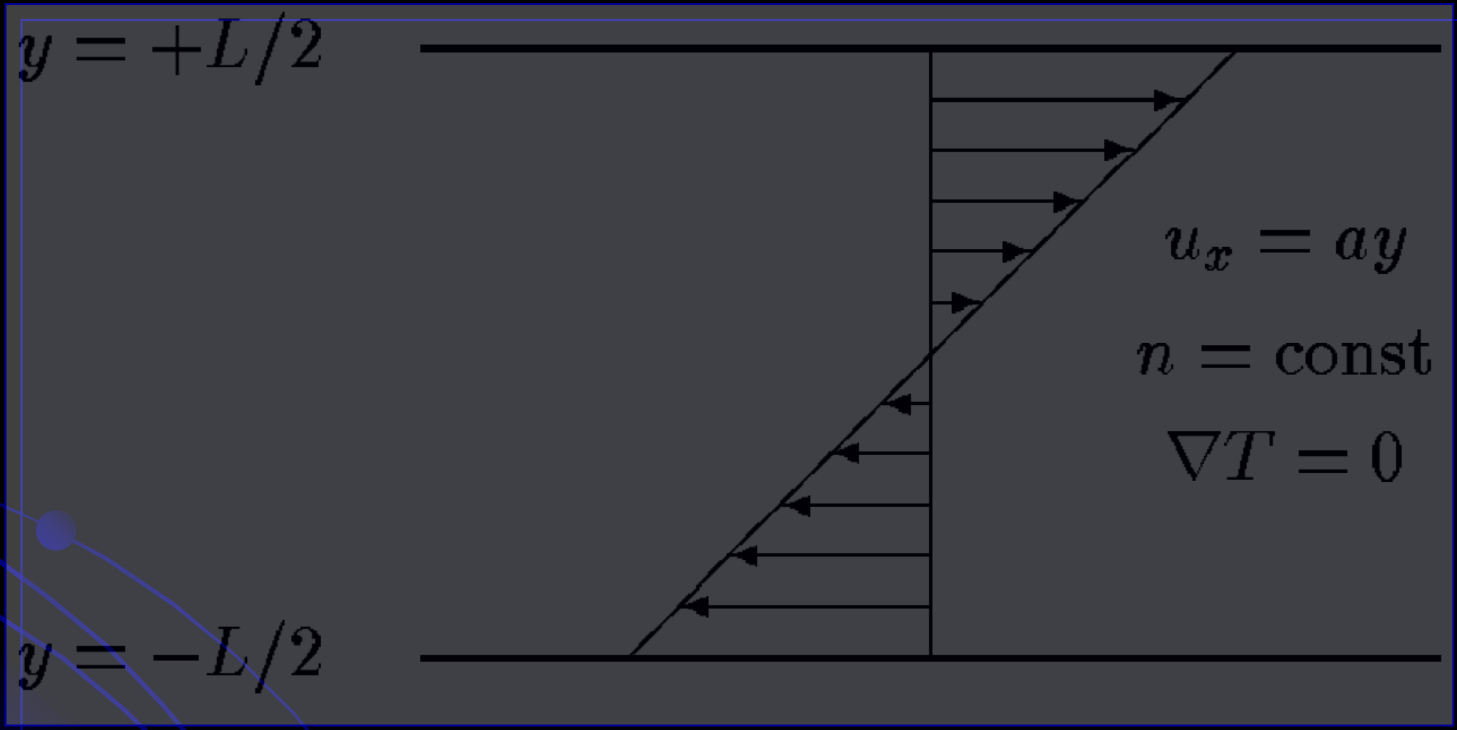
Burnett transport coefficients

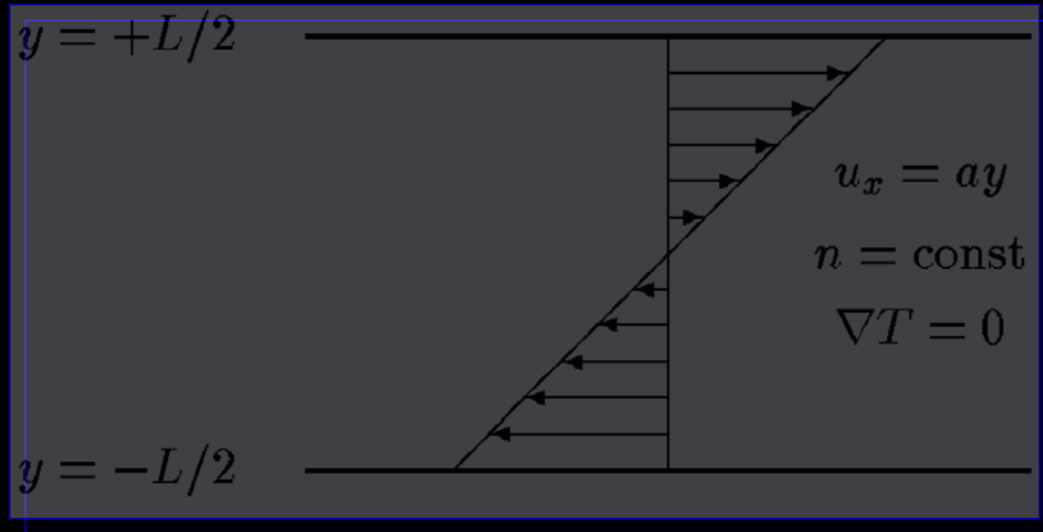
$$P_{xx}^{(2)} - P_{yy}^{(2)} = \varpi_2 \frac{[\eta_0(\alpha)]^2}{p} \left(\frac{\partial u_x}{\partial y}\right)^2 + \cdots$$

$$P_{zz}^{(2)} - P_{yy}^{(2)} = (4\varpi_2 - \varpi_6) \frac{[\eta_0(\alpha)]^2}{p} \left(\frac{\partial u_x}{\partial y}\right)^2 + \cdots$$

$$\varpi_2 = 2, \quad \varpi_6 = 8$$

Paradigmatic nonequilibrium state: Uniform Shear Flow (USF)

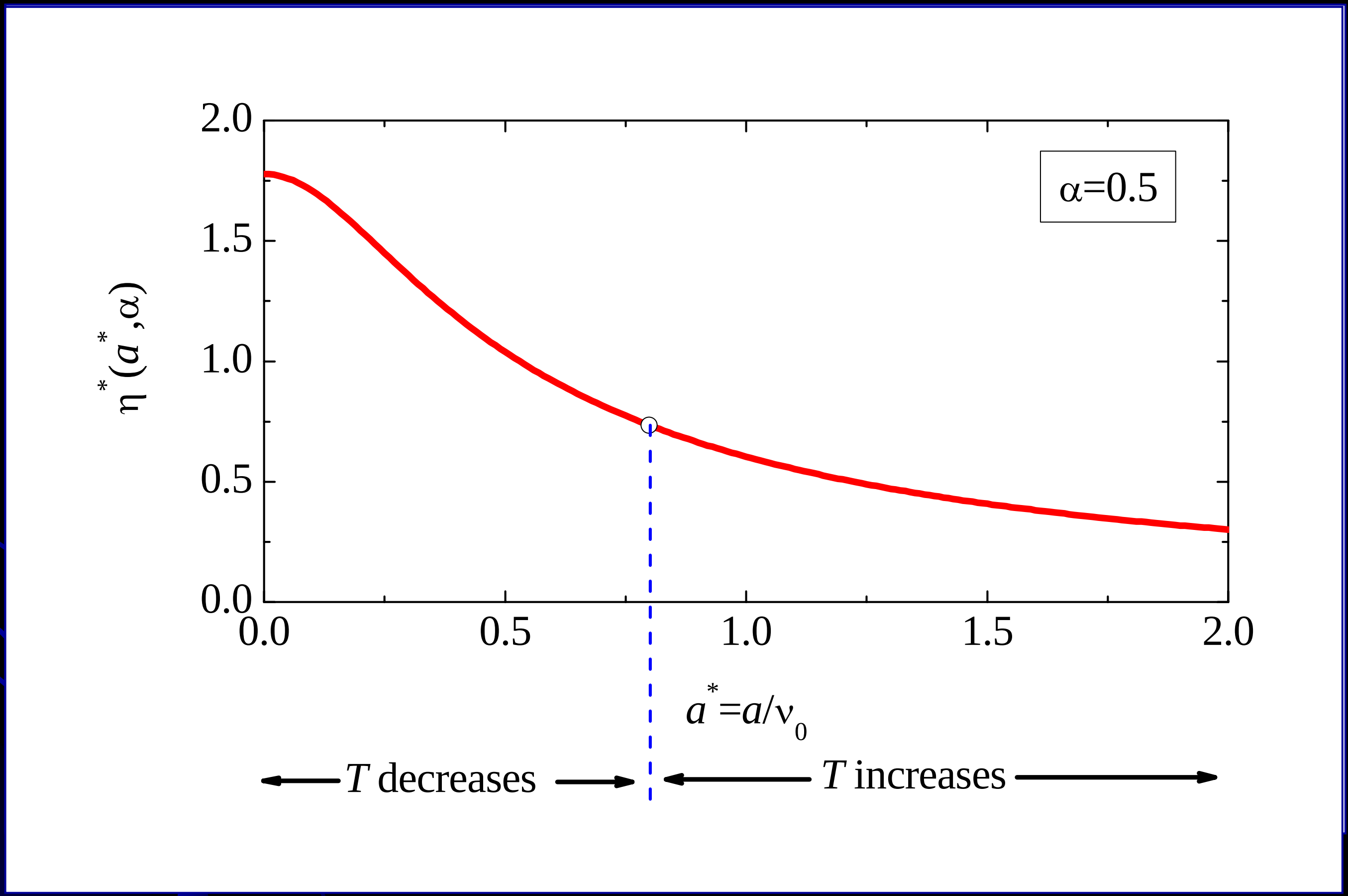




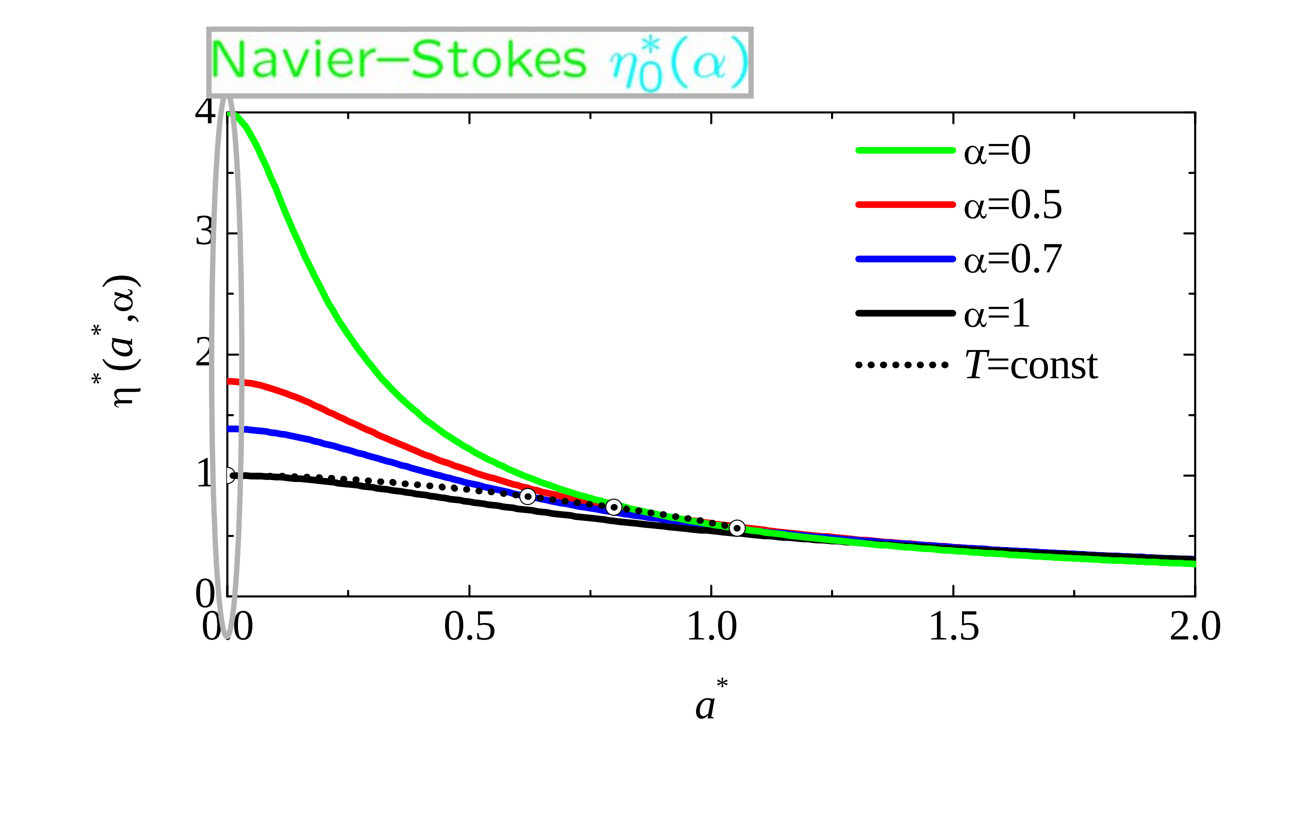
$$\partial_t T = \underbrace{\frac{2}{3} a |P_{xy}|}_{\text{Viscous heating}} - \underbrace{\zeta(\alpha) T}_{\text{Inelastic cooling}}$$

Independent parameters $\begin{cases} \alpha \\ a^* \equiv a/\nu_0 = \mathrm{const} \end{cases}$

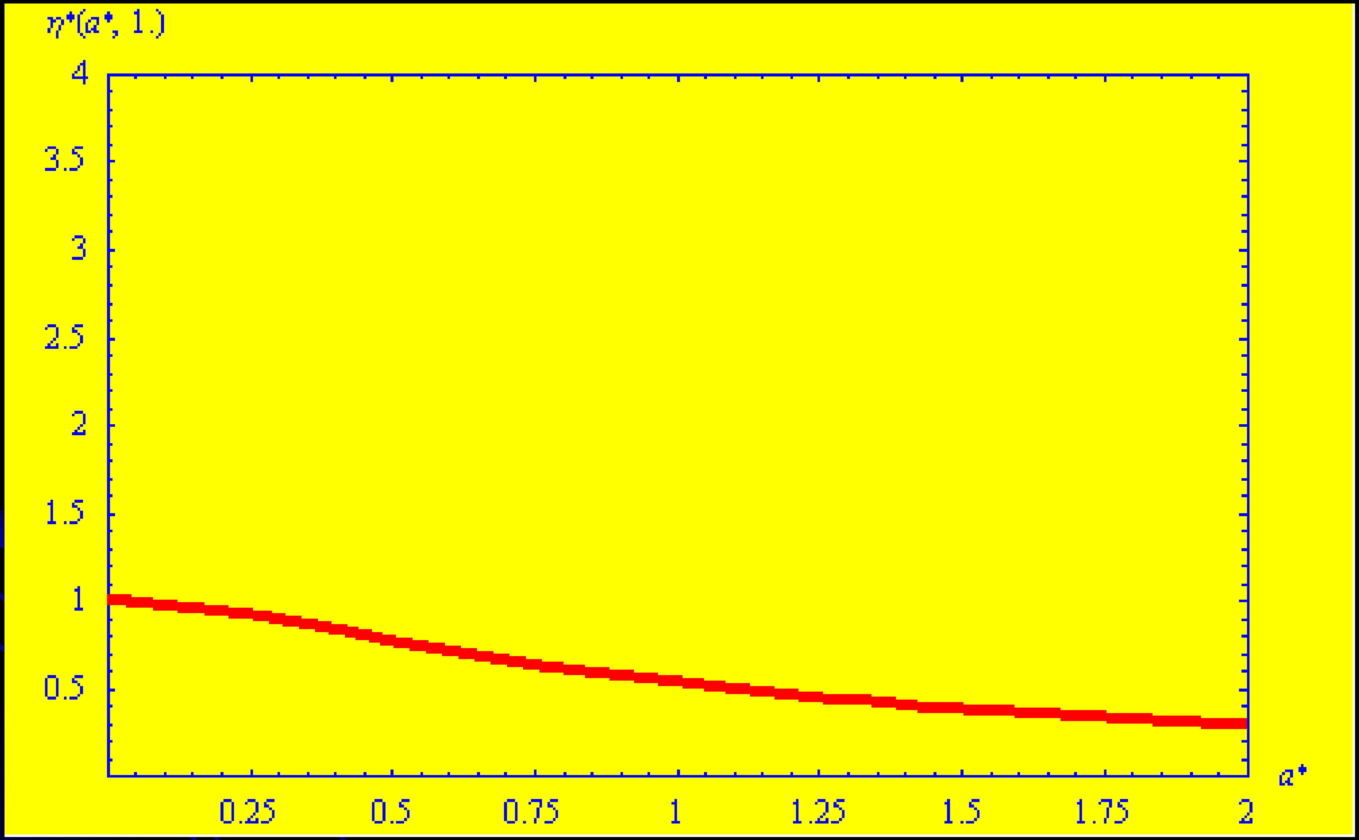
2nd-degree moments: Rheological properties *Nonlinear Shear Viscosity*



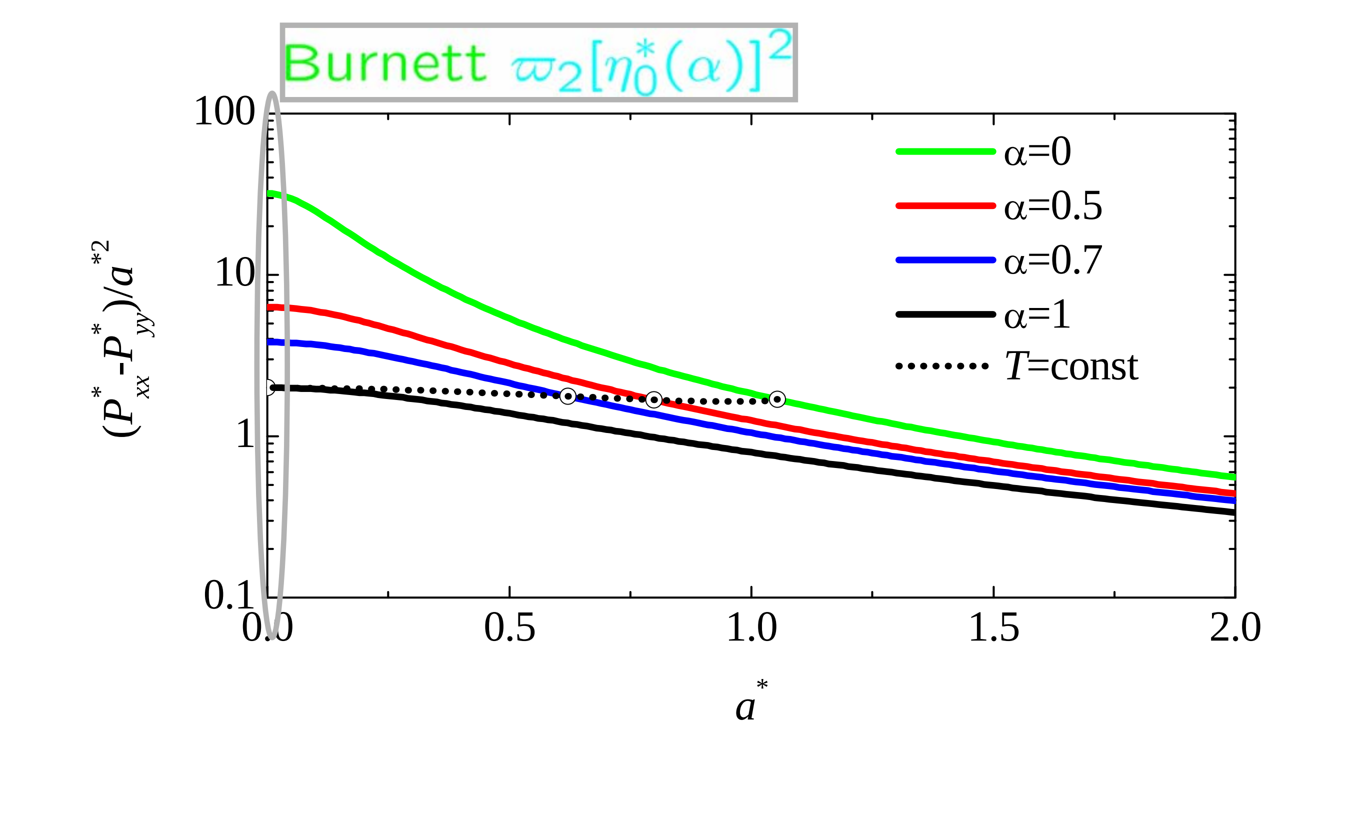
2nd-degree moments: Rheological properties *Nonlinear Shear Viscosity*



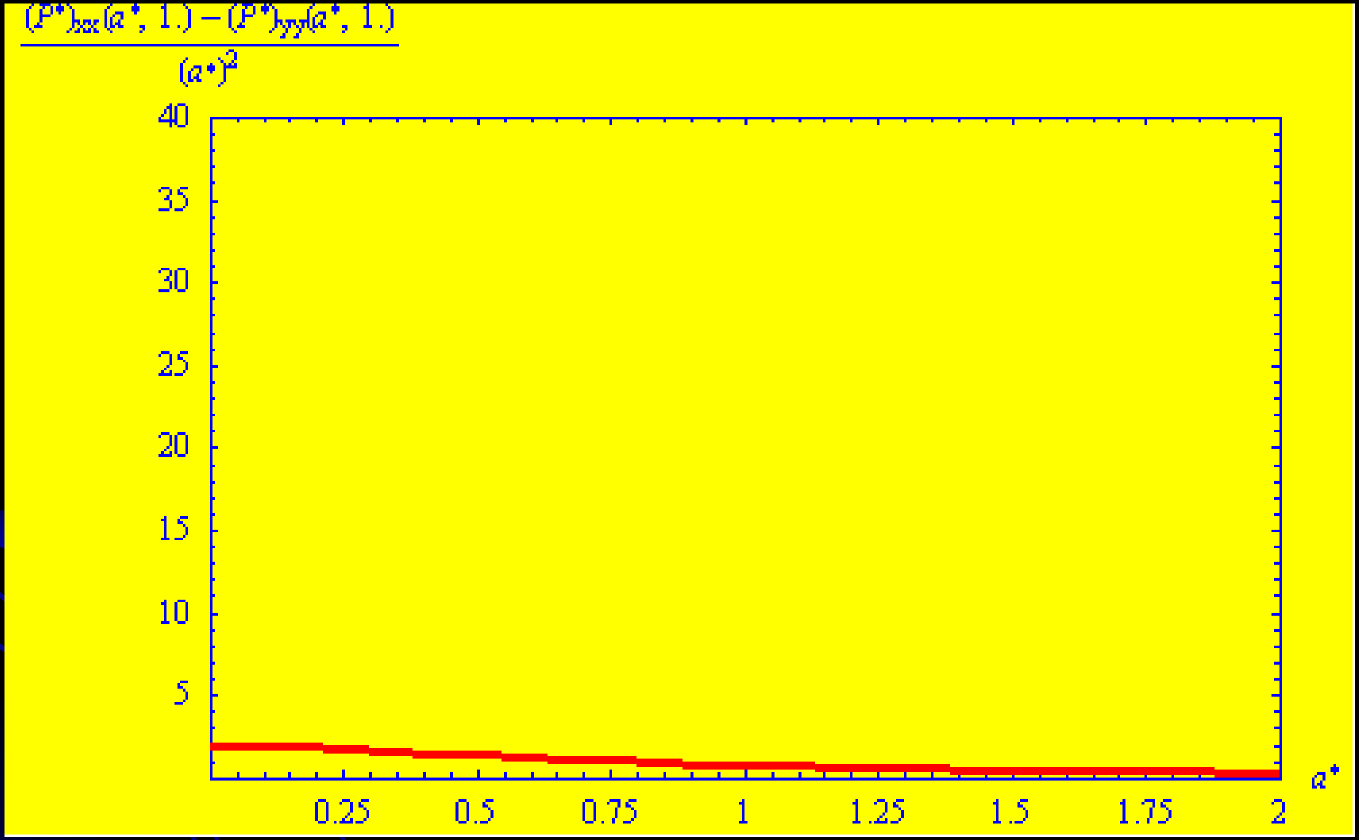
2nd-degree moments: Rheological properties *Nonlinear Shear Viscosity*



2nd-degree moments: Rheological properties *Normal Stress Difference*



2nd-degree moments: Rheological properties *Normal Stress Difference*



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Singular Behavior of Shear Flow Far from Equilibrium

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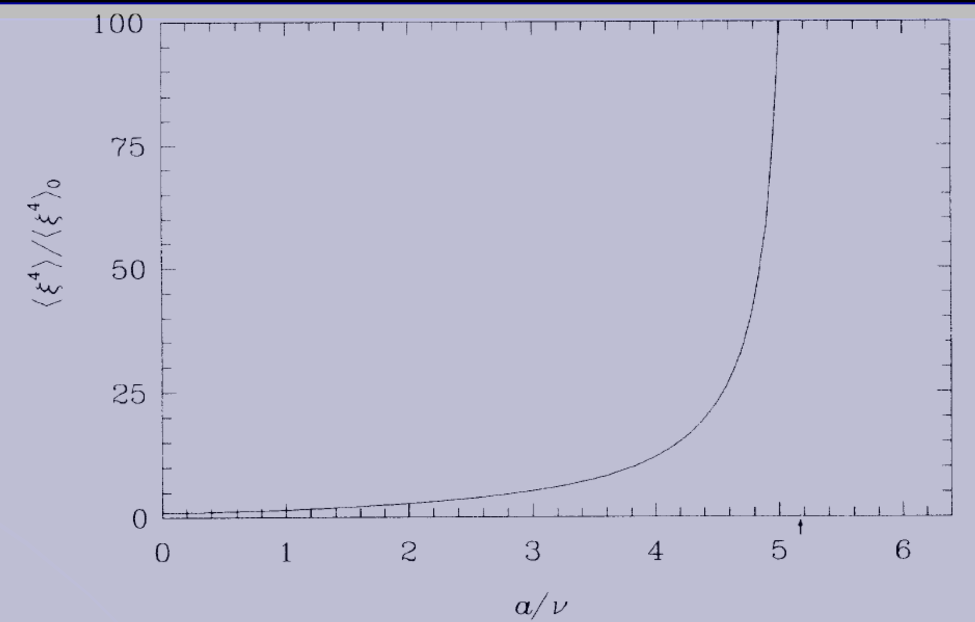


FIG. 2. Shear rate dependence of the steady state value of the moment $\langle\xi^4\rangle$ relative to its Maxwell-Boltzmann value $\langle\xi^4\rangle_0$. The arrow indicates the location of the critical shear rate a_c.

Elastic case

4th-degree ***reduced*** moments: *Time evolution*

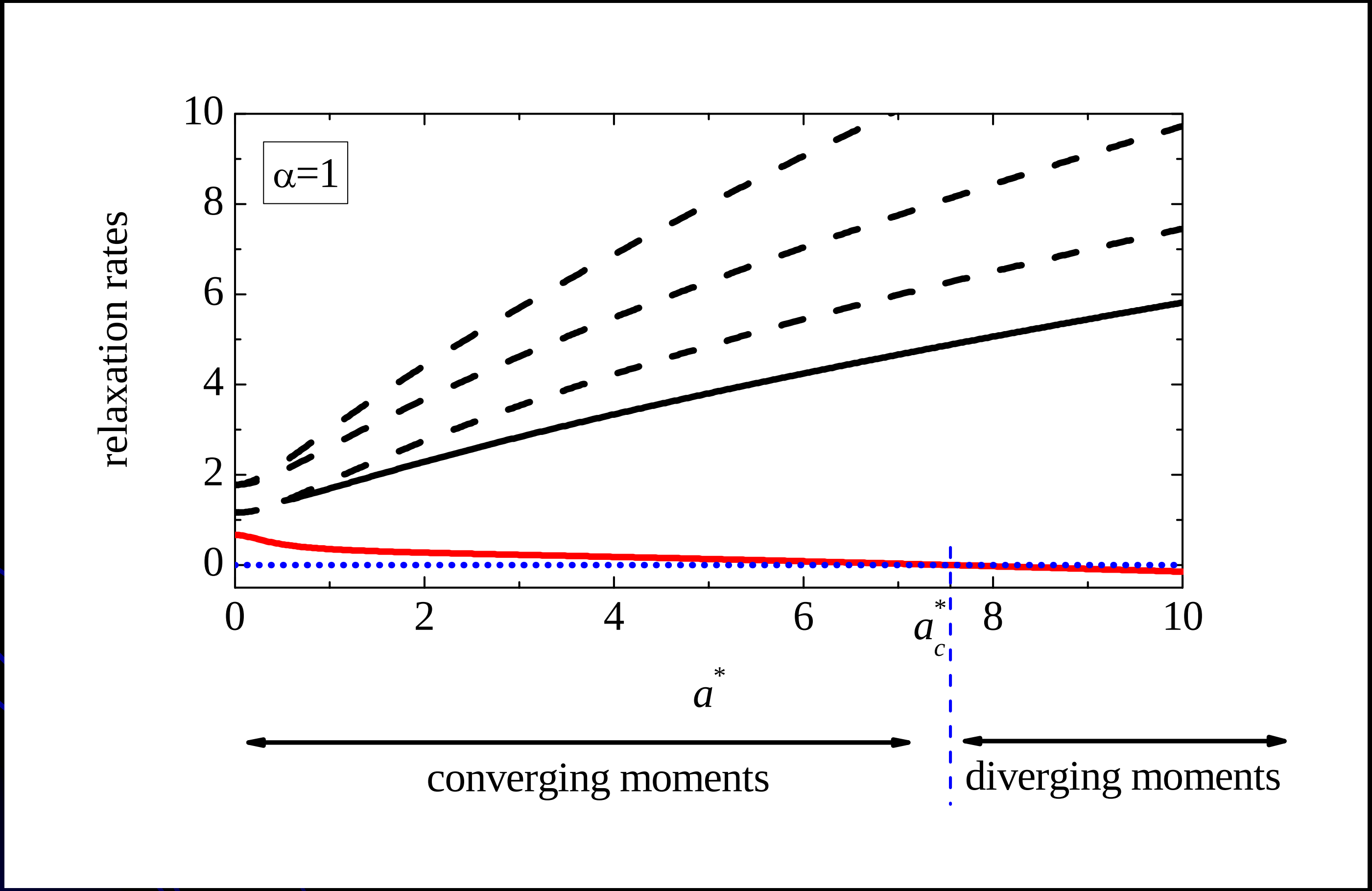
There are 8 relevant 4th-degree moments $\{\mathcal{M}_i\}$ which obey a coupled set of linear, inhomogeneous differential equations. In matrix form,

$$\frac{1}{\nu_0}\partial_t \mathcal{M}_i + \mathcal{L}_{ij}(a^*,\alpha)\mathcal{M}_j = \mathcal{C}_i$$

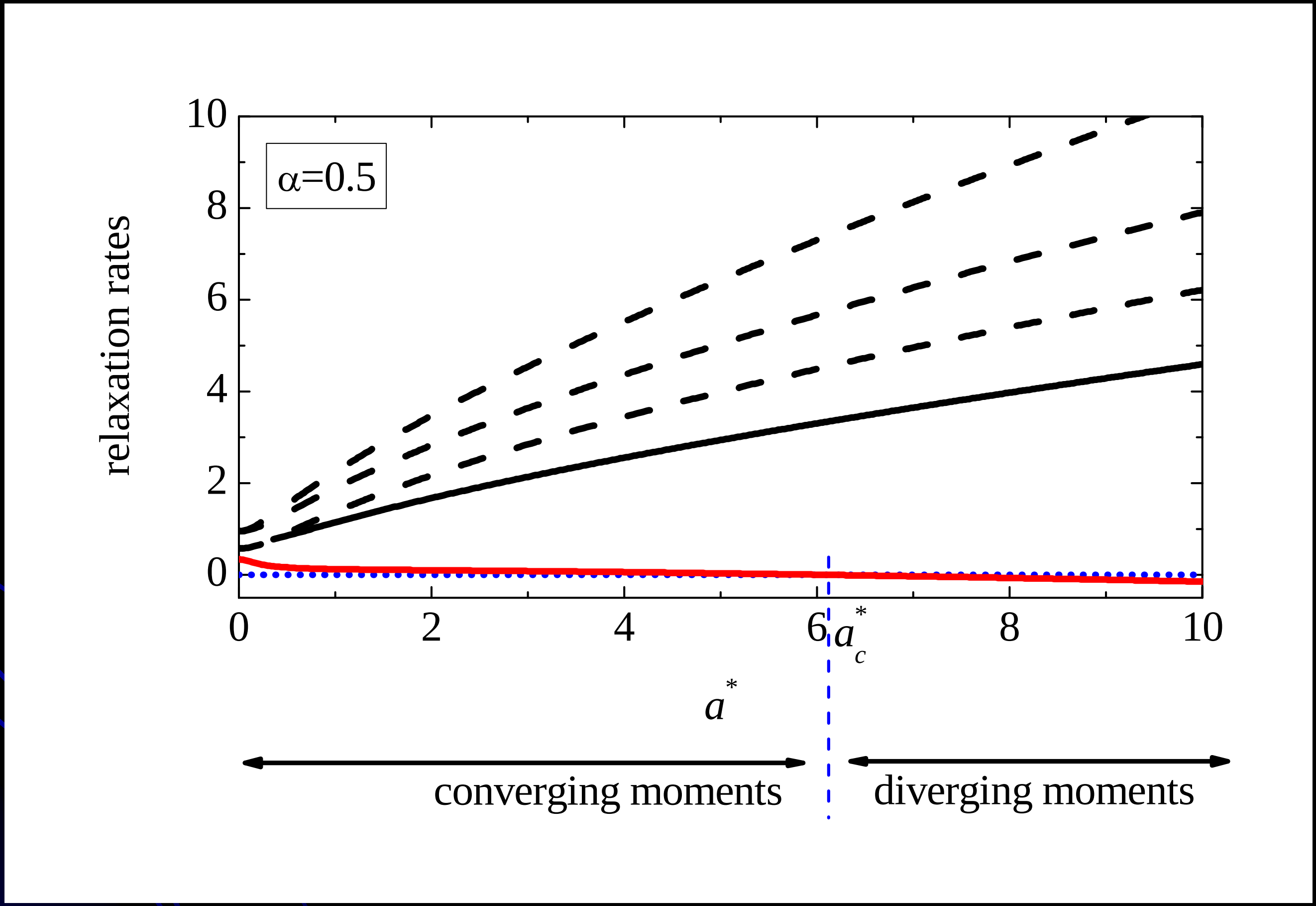
Combination of known 2nd-degree moments

The evolution of $\{\mathcal{M}_i\}$ is governed by the eigenvalues of the 8×8 matrix $\mathcal{L}_{ij}$

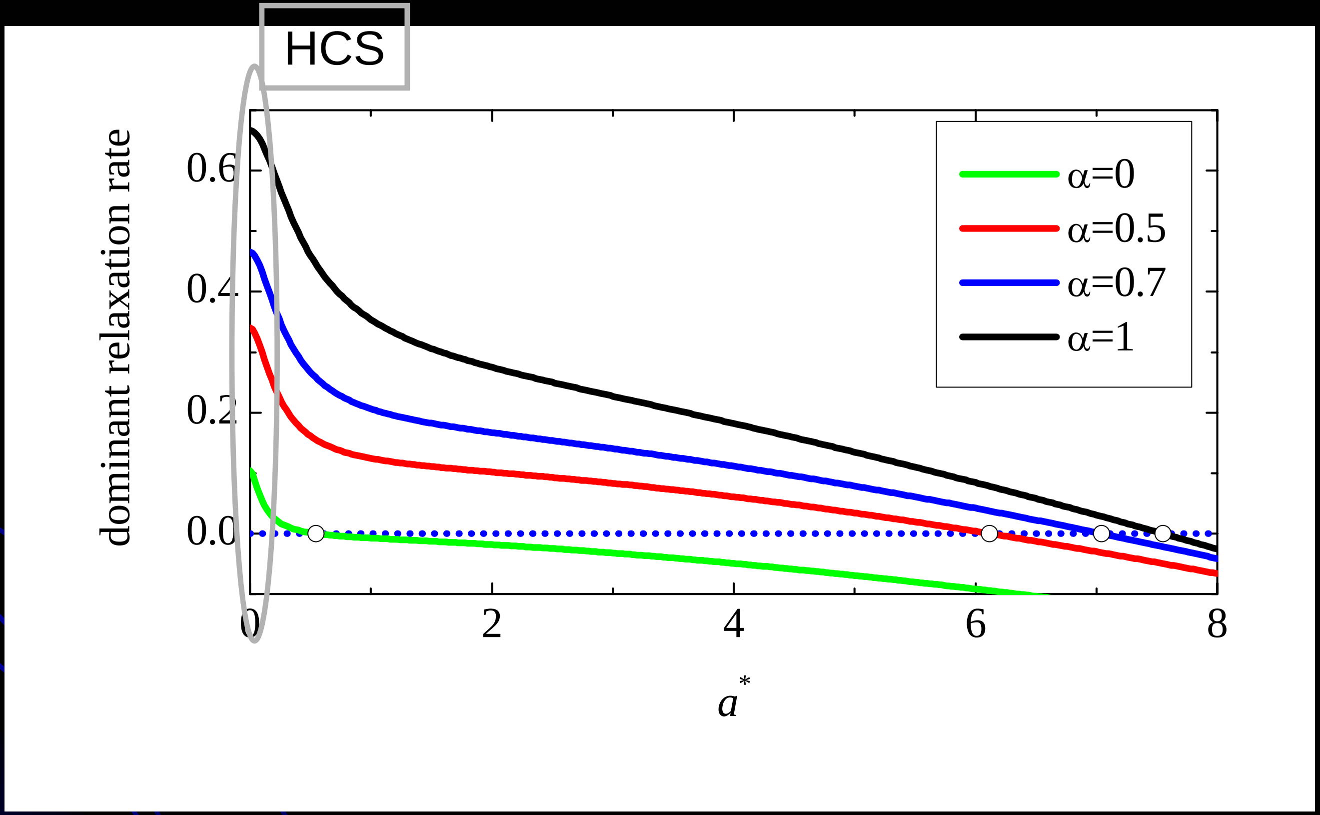
4th-degree *reduced* moments: *Time evolution*



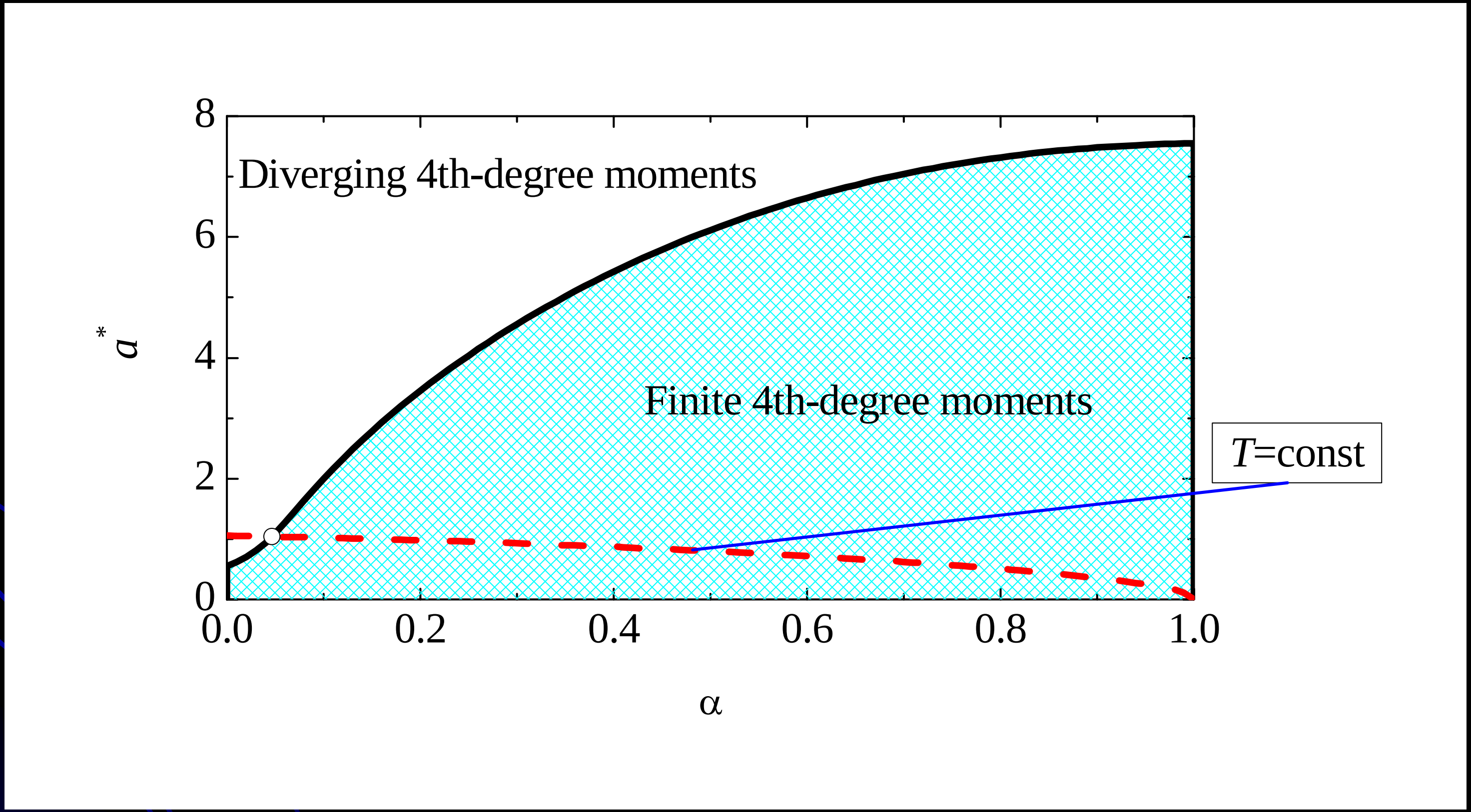
4th-degree ***reduced*** moments: *Time evolution*



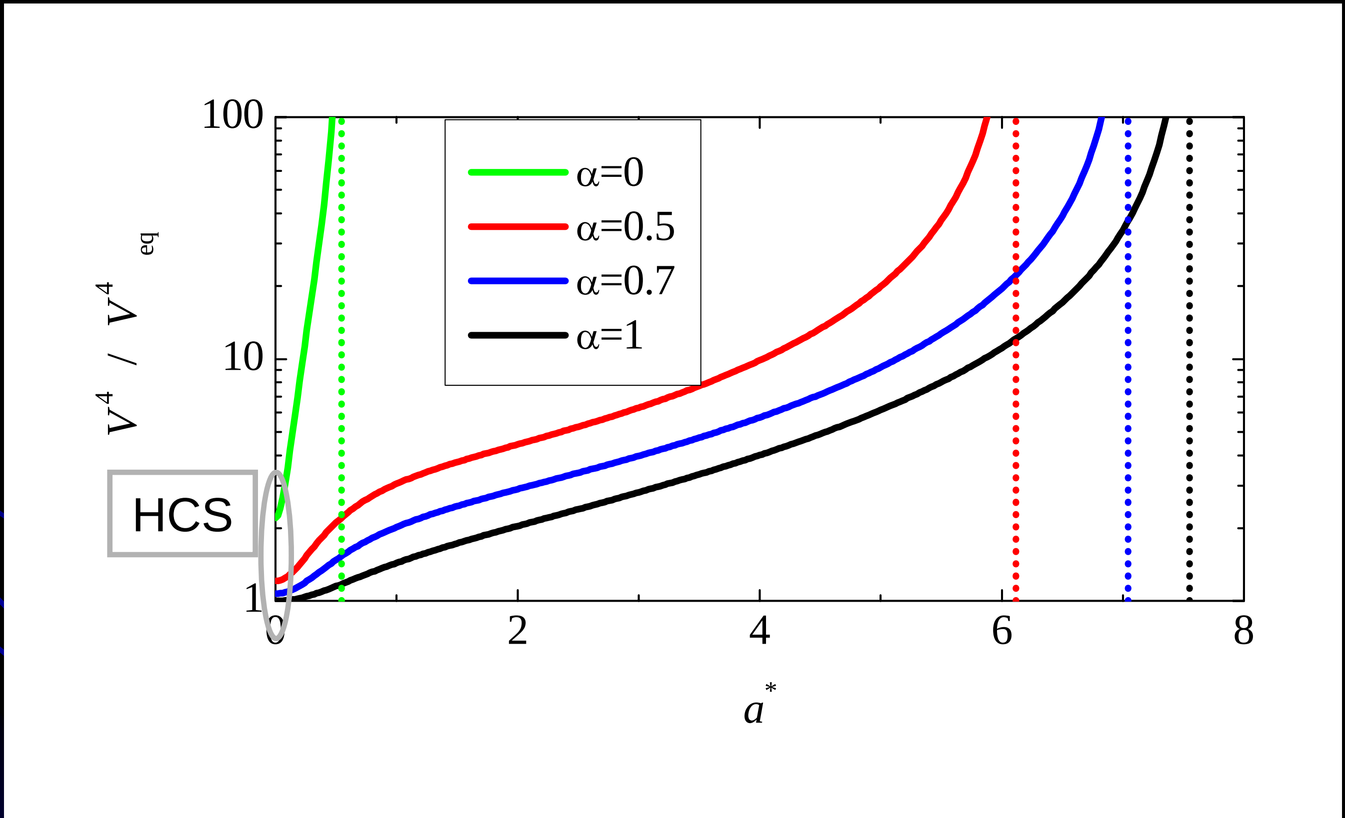
4th-degree ***reduced*** moments: *Time evolution*



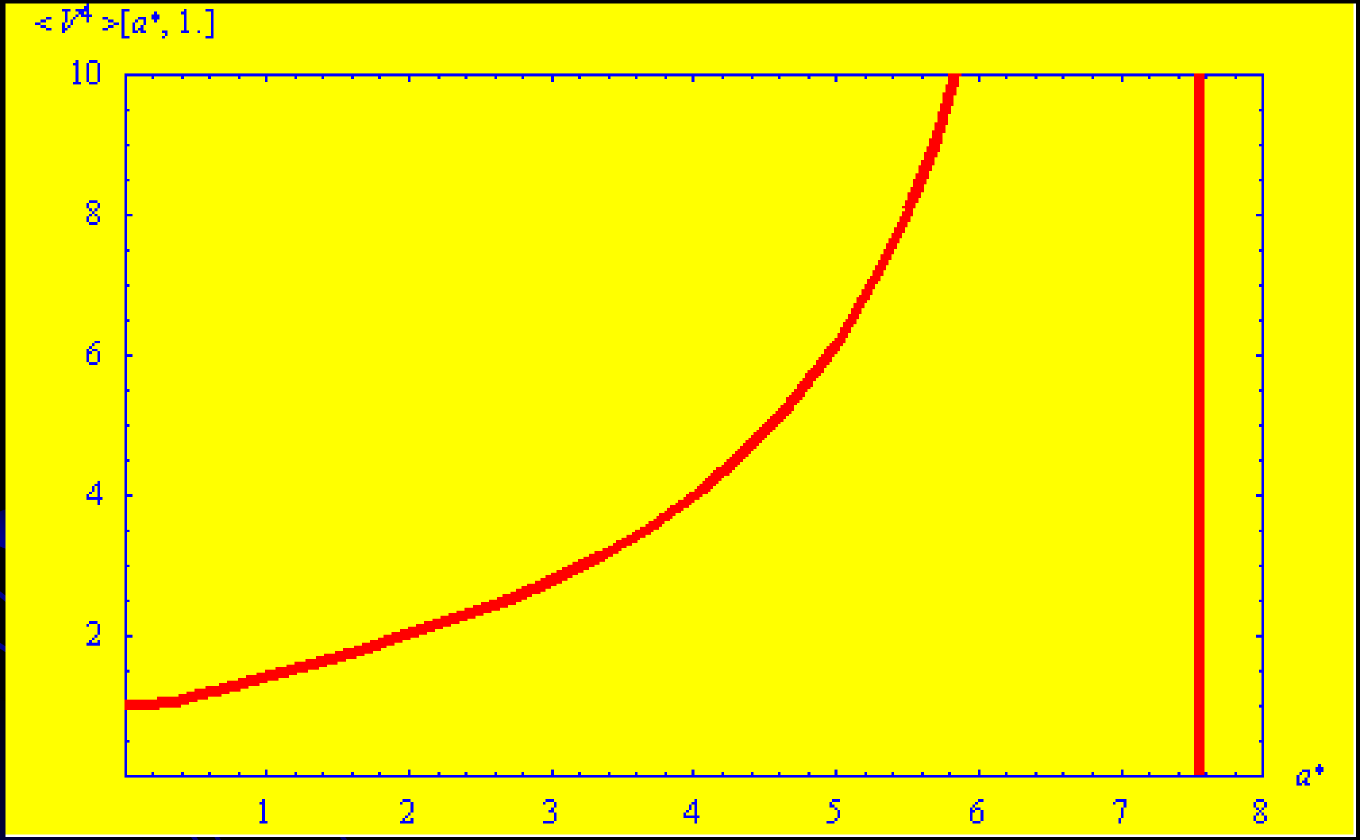
4th-degree ***reduced*** moments: *Phase diagram*



4th-degree ***reduced*** moments: *Stationary values*



4th-degree ***reduced*** moments: *Stationary values*



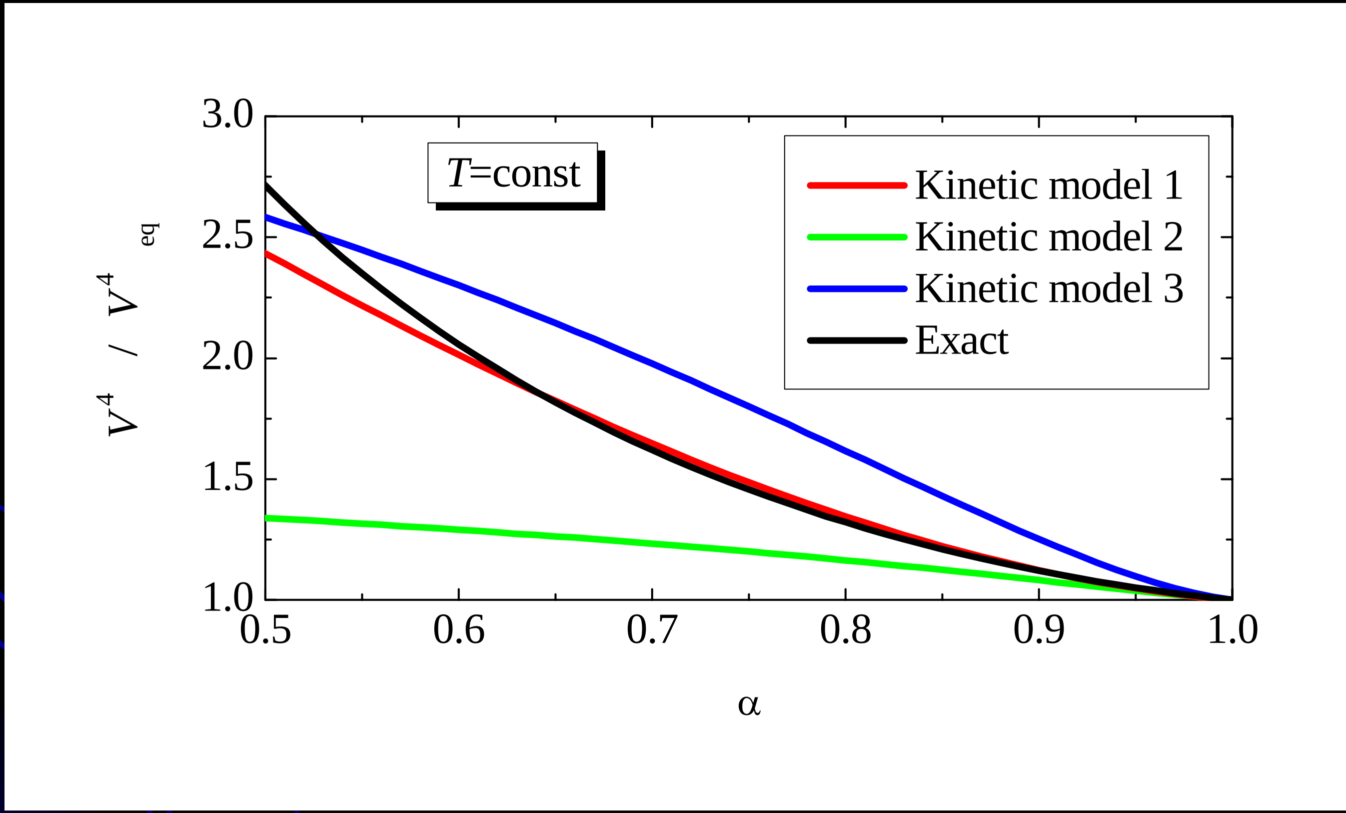
BGK-like kinetic models

- **Model 1** (Brey, Moreno, Dufty; 1996): two parameters fitted to reproduce $\zeta(\alpha)$ and $\eta_0(\alpha)$.
- **Model 2** (Brey, Dufty, Santos; 1999): two parameters fitted to reproduce $\zeta(\alpha)$ and $\eta_0(\alpha)$.
- **Model 3** (Dufty, Baskaran, Zogaib; 2004): three parameters fitted to reproduce $\zeta(\alpha)$, $\eta_0(\alpha)$, and $\kappa_0(\alpha)$.

BGK-like kinetic models

- All the models reproduce the exact rheological properties.
- None of them describe the singular behavior of the fourth-degree moments with increasing shear rate.
- For comparison, let's restrict ourselves to $\alpha \geq 0.5$ and the locus $T=\text{const}$.

BGK-like kinetic models



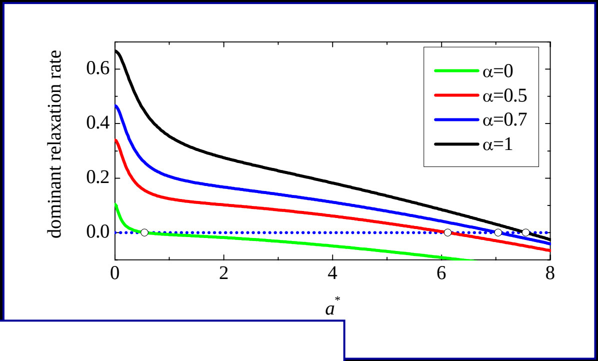
Conclusions (I)

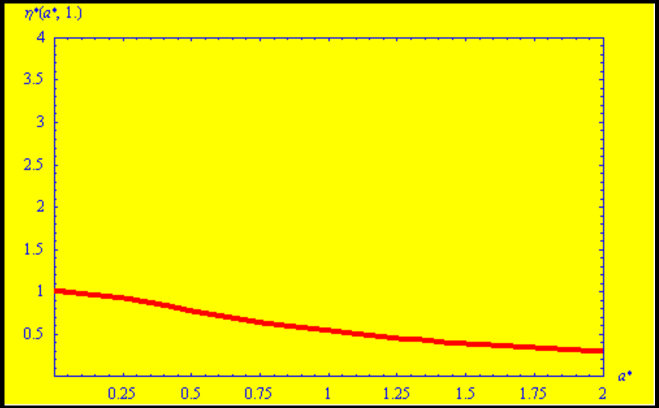
- The collisional moments through fourth degree have been exactly evaluated for inelastic Maxwell particles.
- This allows one to get the α-dependence of the cooling rate, as well as of the Navier-Stokes and Burnett transport coefficients.
- The rheological properties in the USF become less and less sensitive to α as the shear rate a^* increases.

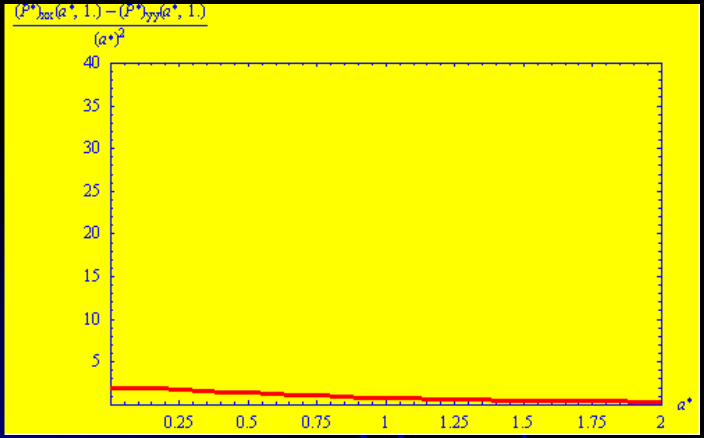
Conclusions (II)

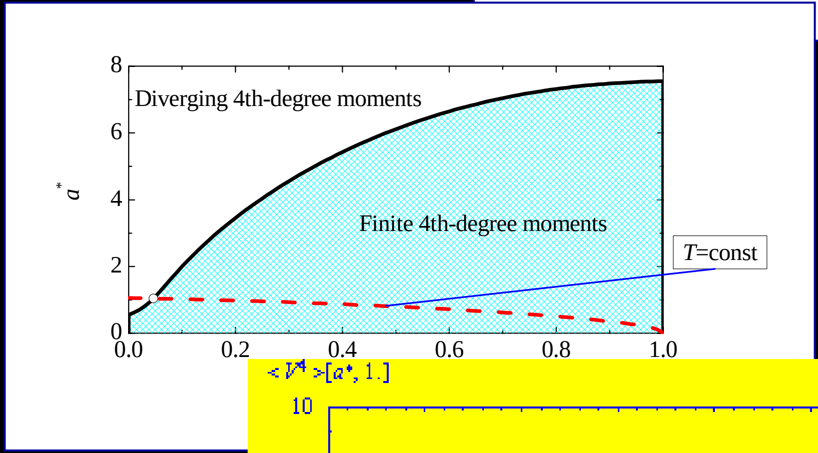
- On the other hand, the underlying velocity distribution is highly influenced by α, as exemplified by the fourth-degree moments.
- The latter diverge for shear rates larger than a critical value $a_c^*(\alpha)$, what indicates a slow algebraic decay of the velocity distribution.
- The Brey-Moreno-Dufty kinetic model reproduces reasonably well some of the properties of inelastic Maxwell particles (although not so well those of inelastic hard spheres).

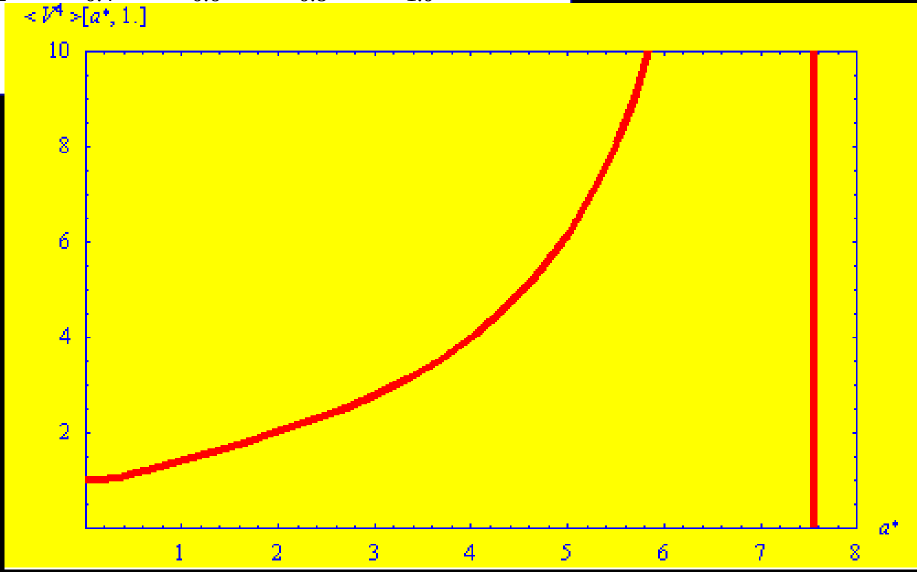
THANKS!


dominant relaxation rate
α=0
α=0.5
α=0.7
α=1
0.6
0.4
0.2
0.0
0
2
4
6
8
a^*


$\eta^*(a^*, 1.)$
4
3.5
3
2.5
2
1.5
1
0.5
0.25
0.5
0.75
1
1.25
1.5
1.75
2
a^*


$\frac{(P^*)_{xx}(a^*, 1.) - (P^*)_{yy}(a^*, 1.)}{(a^*)^2}$
40
35
30
25
20
15
10
5
0.25
0.5
0.75
1
1.25
1.5
1.75
2
a^*


Diverging 4th-degree moments
Finite 4th-degree moments
T=const
a^*
8
6
4
2
0
0.0
0.2
0.4
0.6
0.8
1.0


$< V^4 >[a^*, 1.]$
10
8
6
4
2
1
2
3
4
5
6
7
8
a^*