J. Phys. A: Math. Theor. **44** (2011) 428001 (2pp) doi:10.1088/1751-8113/44/42/428001

COMMENT

Comment on 'A general integral identity'

Andrés Santos

Departamento de Física, Universidad de Extremadura, E-06071 Badajoz, Spain

E-mail: andres@unex.es

Received 13 May 2011, in final form 12 September 2011
Published 28 September 2011
Online at stacks.iop.org/JPhysA/44/428001

Abstract
A simple heuristic proof of an integral identity recently derived (Glasser M L 2011 *J. Phys. A: Math. Theor.* **44** 225202) is presented.

PACS numbers: 02.30.−f, 02.30.Gp

In a recent paper [1], Glasser has derived the integral identity

$$I(r) \equiv \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\pi/2} \mathrm{d}\phi \, \sin\theta F(r \sin\theta \sin\phi) = \frac{\pi}{2} \int_0^1 \mathrm{d}t F(rt), \tag{1}$$

where $F(u)$ is an arbitrary function, by following a relatively sophisticated method. At the end of his paper, Glasser states that 'the results derived here also follow from the invariance of an integral over the surface of an n-sphere under a permutation of the angular hyperspherical co-ordinates' and cites a private communication from the eminent physicist F J Dyson. The aim of this comment is to provide a simple proof of equation (1) based on symmetry arguments, thus fleshing out the details of the previous brief quotation.

First, we make the change of variables $\theta \to \pi - \theta$ to obtain

$$\begin{aligned} I(r) &= \int_{\pi/2}^{\pi} \mathrm{d}\theta \int_0^{\pi/2} \mathrm{d}\phi \, \sin\theta F(r \sin\theta \sin\phi) \\ &= \frac{1}{2} \int_0^{\pi} \mathrm{d}\theta \int_0^{\pi/2} \mathrm{d}\phi \, \sin\theta F(r \sin\theta \sin\phi). \end{aligned} \tag{2}$$

Next, we perform the analogous change $\phi \to \pi - \phi$. Thus,

$$I(r) = \frac{1}{4} \int_0^{\pi} \mathrm{d}\theta \int_0^{\pi} \mathrm{d}\phi \, \sin\theta F(r \sin\theta \sin\phi). \tag{3}$$

Now, let us assume that $r > 0$ (the case $r < 0$ can be treated separately in a similar way). In that case, the arguments of the function F in equation (1) are positive. This implies that, when proving equation (1), only $F(u)$ for $u > 0$ matters and thus we are free to extend $F(u)$

for $u < 0$. If we extend it as an even function, i.e. $F(-u) = F(u)$, and make the change $\phi \to \phi + \pi$ in equation (3), the result is

$$I(r) = \frac{1}{4}\int_0^{\pi} \mathrm{d}\theta \int_{\pi}^{2\pi} \mathrm{d}\phi \sin\theta F(r\sin\theta\sin\phi)$$
$$= \frac{1}{8}\int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \sin\theta F(r\sin\theta\sin\phi). \tag{4}$$

This can be recognized as an integration over all the spatial directions of the function F evaluated at $y = r\sin\theta\sin\phi$ in (three-dimensional) spherical coordinates, i.e.

$$I(r) = \frac{1}{8}\int \mathrm{d}\Omega F(y), \tag{5}$$

where $\mathrm{d}\Omega = \mathrm{d}\theta\,\mathrm{d}\phi \sin\theta$ is the elementary solid angle.

Thus far, only formal manipulations in the integrals defining $I(r)$ have been made. To conclude the proof of equation (1) we simply take into account that, by *symmetry*, $I(r)$ must be independent of the choice of Cartesian axes, so that we can freely replace $F(y)$ by $F(x)$ or $F(z)$ inside the integral. In particular, the latter choice yields

$$I(r) = \frac{1}{8}\int \mathrm{d}\Omega F(r\cos\theta)$$
$$= \frac{2\pi}{8}\int_0^{\pi} \mathrm{d}\theta \sin\theta F(r\cos\theta)$$
$$= \frac{\pi}{4}\int_{-1}^{1} \mathrm{d}t F(rt), \tag{6}$$

where the change $\theta \to t = \cos\theta$ has been made in the last step. Recalling the extension $F(-u) = F(u)$, equation (1) is finally obtained.

Acknowledgments

Support from the Ministerio de Ciencia e Innovación (Spain) through grant no FIS2010-16587 and the Junta de Extremadura (Spain) through grant no GR10158, partially financed by FEDER funds, is gratefully acknowledged.

References

[1] Glasser M L 2011 A general integral identity *J. Phys. A: Math. Theor.* **44** 225202