Combined heat and momentum transport in a dilute gas

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The infinite hierarchy of moment equations derived from the Boltzmann equation for Maxwell molecules is analyzed in the case of steady planar Couette flow. It is proved that a solution exists that is consistent with the following hydrodynamic profiles: $p=nk_BT=\text{const}$, $T\partial u_x/\partial y=\text{const}$, $(T\partial/\partial y)^2T=\text{const}$. In general, the velocity moments of order k are polynomials of degree $k-2$ in a scaled space variable $s \propto \int T^{-1}dy$. The momentum and energy transport are described by a nonlinear shear viscosity $\eta(a)=\eta(0)F_\eta(a)$ and a nonlinear thermal conductivity $\kappa(a)=\kappa(0)F_\kappa(a)$, respectively, where $a\equiv\partial u_x/\partial s$ is the (constant) reduced shear rate. By performing a perturbation expansion in powers of a, it is found that $F_\eta(a)=1-1.472a^2+\mathcal{O}(a^4)$ and $F_\kappa(a)=1-3.226a^2+\mathcal{O}(a^4)$. These numerical values are compared with those obtained from the BGK and the Liu kinetic models. © *1995 American Institute of Physics.*

I. INTRODUCTION

The steady planar Couette flow is one of the most interesting states to analyze transport phenomena. It corresponds to a fluid enclosed between two infinite parallel plates maintained in relative motion. The plates can be kept at the same temperature or at two different temperatures. In either case, a temperature profile is expected to exist across the system, in addition to a velocity profile and a density profile.

Let x and y denote the coordinate parallel to the flow and the coordinate normal to the plates, respectively. The steady-state hydrodynamic balance equations are

$$\frac{\partial}{\partial y} P_{xy}=\frac{\partial}{\partial y} P_{yy}=0, \tag{1a}$$

$$\frac{\partial}{\partial y} q_y+P_{xy}\frac{\partial}{\partial y} u_x=0, \tag{1b}$$

where $\mathbf{u}=u_x\hat{x}$ is the flow velocity, P is the pressure tensor, and $\mathbf{q}=q_x\hat{x}+q_y\hat{y}$ is the heat flux. Equation (1b) shows that the existence of a velocity field induces the presence of a heat flux. Consequently, a thermal gradient $\partial T/\partial y$ is present, even though both plates are kept at the same temperature. For small gradients (Navier–Stokes order), the constitutive equations are

$$q_y=-\kappa_0(n,T)\frac{\partial}{\partial y} T, \tag{2a}$$

$$P_{xy}=-\eta_0(n,T)\frac{\partial}{\partial y} u_x, \tag{2b}$$

where κ_0 and η_0 are the thermal conductivity and shear viscosity, respectively. They are functions of the local density n and temperature T, which are connected by the (local) equilibrium equation of state $p\equiv\frac{1}{3}\,\text{tr}\,\mathsf{P}=p_{\text{eq}}(n,T)$ and the condition

$$p=\text{const}. \tag{3a}$$

Substitution of Eqs. (2) into Eqs. (1) yields the velocity and temperature profiles at Navier–Stokes order:

$$\eta_0(n,T)\frac{\partial}{\partial y} u_x=\text{const}, \tag{3b}$$

$$\frac{\partial}{\partial y}\left[\kappa_0(n,T)\frac{\partial}{\partial y} T\right]=-\eta_0(n,T)\left(\frac{\partial}{\partial y} u_x\right)^2. \tag{3c}$$

The planar Couette flow described here must not be confused with the so-called uniform shear flow. In the latter, $\partial u_x/\partial y=\text{const}$ is the only non-zero gradient and the uniformity of the temperature is achieved at the expense of viscous heating, which can be controlled by the application of a nonconservative thermostat force.[1,2] Both kinds of shear flow are generated in computer simulations by means of different boundary conditions[3] and give rise to different transport coefficients beyond the Navier–Stokes regime.[4]

As a prototype fluid to investigate transport phenomena beyond the scope of the Navier–Stokes order, it is useful to consider a monatomic low-density gas with short-range interactions. The state of the system is then specified by the one-particle velocity distribution function $f(\mathbf{r},\mathbf{v},t)$, which is the solution of the nonlinear Boltzmann equation[5]

$$\frac{\partial}{\partial t} f+\mathbf{v}\cdot\nabla f=\int d\mathbf{v}_1\int d\Omega g\sigma(g,\cos\theta)[f'f_1'-ff_1] \equiv J[f,f] \tag{4}$$

with appropriate initial and boundary conditions. The hydrodynamic quantities and their fluxes can be expressed in terms of moments of f:

$$n=\int d\mathbf{v}f, \tag{5}$$

$$n\mathbf{u}=\int d\mathbf{v}\mathbf{v}f, \tag{6}$$

$$\tfrac{3}{2}\,nk_BT=\frac{m}{2}\int d\mathbf{v}V^2f, \tag{7}$$

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$$\mathsf{P}=m\int d\mathbf{v}\mathbf{V}\mathbf{V}f, \tag{8}$$

$$\mathbf{q}=\frac{m}{2}\int d\mathbf{v}V^2\mathbf{V}f. \tag{9}$$

In Eqs. (7)–(9), $\mathbf{V}\equiv\mathbf{v}-\mathbf{u}$ is the peculiar velocity, m is the mass of a particle, and k_B is the Boltzmann constant. The equation of state is that of an ideal gas, namely $p=nk_BT$.

Most of the known solutions to Eq. (4) for spatially inhomogeneous states correspond to Maxwell molecules (i.e., particles interacting via an r^{-4} repulsive potential), for which the collision rate $g\sigma(g,\cos\theta)=\sigma_0(\cos\theta)$ is independent of the relative velocity g. About 40 years ago, Ikenberry and Truesdell[6] obtained the exact expression of the pressure tensor for a gas of Maxwell molecules under uniform shear flow. This solution has been recently extended to the fourth order moments.[7] In the case of steady planar Fourier flow, Asmolov *et al.*[8,9] analyzed Eq. (4) for Maxwell molecules. The planar Fourier flow can be seen as a particularization of the planar Couette flow to $\partial u_x/\partial y=0$, i.e., the plates have different temperatures but both are at rest. It was shown the existence of a self-consistent solution where the kth order moments are polynomials of degree $k-2$ in the temperature gradient. In particular, the heat flux is proportional to the temperature gradient, so that the Fourier law, Eq. (2a) holds exactly for arbitrarily large temperature gradients.

To the best of our knowledge, no solution of the Boltzmann equation for the planar Couette flow with arbitrary temperature and velocity gradients is known. Such a solution exists, however, in the case of the Bhatnagar–Gross–Krook (BGK) model kinetic equation.[10,11] In the BGK model,[12] the Boltzmann collision operator $J[f,f]$ is replaced by a single-time relaxation term:

$$J[f,f]\rightarrow-\nu(f-f_{\mathrm{LE}}), \tag{10}$$

where $\nu(n,T)$ is an effective collision frequency and

$$f_{\mathrm{LE}}(\mathbf{r},\mathbf{v},t)=n\left(\frac{m}{2\pi k_BT}\right)^{3/2}\exp\left[-m\,\frac{(\mathbf{v}-\mathbf{u})^2}{2k_BT}\right] \tag{11}$$

is the local equilibrium distribution function. The exact solution of the BGK model for the steady planar Couette flow[10] is characterized by the following hydrodynamic profiles:

$$p=\text{const}, \tag{12a}$$

$$\frac{1}{\nu(n,T)}\frac{\partial}{\partial y}u_x=a=\text{const}, \tag{12b}$$

$$\left[\frac{1}{\nu(n,T)}\frac{\partial}{\partial y}\right]^2T=-\frac{\eta_0(T)}{\kappa_0(T)}\frac{F_\eta(a)}{F_\kappa(a)}a^2, \tag{12c}$$

where $\eta_0=p/\nu$, $\kappa_0=5k_Bp/2m\nu$, and $F_\eta(a)$ and $F_\kappa(a)$ are nonlinear functions of the reduced shear rate a, whose expressions can be found in Ref. 10. These functions define a generalized thermal conductivity and a generalized shear viscosity through the relations

$$q_y=-\kappa_0(T)F_\kappa(a)\frac{\partial}{\partial y}T, \tag{13a}$$

$$P_{xy}=-\eta_0(T)F_\eta(a)\frac{\partial}{\partial y}u_x. \tag{13b}$$

For the purpose of this paper, it is convenient to give here the behaviors of F_κ and F_η for small a. They are

$$F_\kappa(a)=1+F_\kappa^{(2)}a^2+\mathcal{O}(a^4), \tag{14a}$$

$$F_\eta(a)=1+F_\eta^{(2)}a^2+\mathcal{O}(a^4), \tag{14b}$$

with $F_\kappa^{(2)}=-\frac{162}{25}$ and $F_\eta^{(2)}=-\frac{18}{5}$. It has been recently proved[13] that a solution consistent with the profiles (12) and (13) also holds for the Liu model,[14] an extension of the BGK model which introduces two independent collision frequencies (ν and ζ) and gives the correct value of the Prandtl number; in the case of the Liu model,[13] $F_\kappa^{(2)}=-\frac{98}{25}(\nu/\zeta)^2$ and $F_\eta^{(2)}=-\frac{16}{5}(\nu/\zeta)^2$. Notice that in the limit $a\rightarrow0$, Eqs. (12) become Eqs. (3) and Eqs. (13) become Eqs. (2).

The aim of this paper is to prove that the steady-state Boltzmann equation admits a solution characterized by Eqs. (12) and (13) in the particular case of Maxwell molecules. This generalizes the known solution for the Fourier flow.[8,9] The proof is carried out in Sec. II, where the hierarchy of moment equations is considered. This hierarchy consists of an infinite number of first-order differential equations with respect to the reduced thermal gradient. The solutions are polynomials in the thermal gradient with coefficients that are nonlinear functions of the shear rate and satisfy an infinite hierarchy of algebraic equations. The exact expressions for those coefficients cannot be found in closed form, so that a perturbation expansion in powers of the shear rate is performed in Sec. III. This allows us to obtain the (super-Burnett) coefficients $F_\kappa^{(2)}$ and $F_\eta^{(2)}$. The results are discussed in Sec. IV.

II. HIERARCHY OF MOMENT EQUATIONS

In the steady planar Couette flow the distribution function is expected to depend on space only through the coordinate y, i.e., $f=f(y,\mathbf{v})$. The Boltzmann equation, Eq. (4), then becomes

$$v_y\frac{\partial}{\partial y}f=J[f,f]. \tag{15}$$

The balance equations (1) can be easily obtained from Eq. (15). They are just the first few members of the infinite hierarchy of equations for the velocity moments of f. Let us introduce the moments

$$M_{k_1,k_2,k_3}(y)=\int d\mathbf{v}V_x^{k_1}V_y^{k_2}V_z^{k_3}f(y,\mathbf{v}). \tag{16}$$

From Eq. (15) one gets

$$\frac{\partial}{\partial y}M_{k_1,k_2+1,k_3}+\frac{\partial u_x}{\partial y}k_1M_{k_1-1,k_2+1,k_3}=J_{k_1,k_2,k_3}, \tag{17}$$

where

$$J_{k_1,k_2,k_3}(y)=\int d\mathbf{v}V_x^{k_1}V_y^{k_2}V_z^{k_3}J[f,f]. \tag{18}$$

In the sequel, we will use the roman boldface $\mathbf{k}$ to denote the triad $\{k_1,k_2,k_3\}$ and the italic lightface k to denote the sum $k_1+k_2+k_3$. Thus, $M_{\mathbf{k}}\equiv M_{k_1,k_2,k_3}$ is a moment of order $k\equiv k_1+k_2+k_3$.

In the special case of Maxwell molecules, the collision rate is independent of the velocity, i.e., $g\sigma(g,\cos\theta)=\sigma_0(\cos\theta)$, and $J_{\mathbf{k}}$ can be expressed as a bilinear combination of moments of order equal to or smaller than k:[15]

$$J_{\mathbf{k}}=\sum_{\mathbf{k}',\mathbf{k}''}{}^{\dagger}C^{\mathbf{k}}_{\mathbf{k}',\mathbf{k}''}M_{\mathbf{k}'}M_{\mathbf{k}''}, \tag{19}$$

where the dagger denotes the constraint $k'+k''=k$. The coefficients $C^{\mathbf{k}}_{\mathbf{k}',\mathbf{k}''}$ are linear combinations of the eigenvalues

$$\lambda_{r\ell}=\int d\Omega\,\sigma_0(\cos\theta)\left[1+\delta_{r0}\delta_{\ell 0}-\cos^{2r+\ell}\frac{\theta}{2}\,P_\ell\left(\cos\frac{\theta}{2}\right)-\sin^{2r+\ell}\frac{\theta}{2}\,P_\ell\left(\sin\frac{\theta}{2}\right)\right] \tag{20}$$

of the linearized collision operator.[15,16] The explicit expressions for $J_{\mathbf{k}}$ through order $k=4$ are given in Appendix A. The Navier–Stokes thermal conductivity and shear viscosity for Maxwell molecules are[5]

$$\kappa_0(T)=\frac{5k_B}{2m}\frac{p}{n\lambda_{11}}, \tag{21a}$$

$$\eta_0(T)=\frac{p}{n\lambda_{02}}, \tag{21b}$$

where $\lambda_{02}/\lambda_{11}=\frac{3}{2}$. Let us define an effective collision frequency as

$$\nu=n\lambda_{11} \tag{22}$$

and introduce the scaled space variable

$$s(y)=\int_0^y dy'\,\nu(y'). \tag{23}$$

Now we make the assumption, to be confirmed by consistency, that a solution to Eq. (15) or, equivalently, to Eq. (17) exists compatible with Eqs. (12), where now ν is given by Eq. (22) and $\eta_0/\kappa_0=4m/15k_B$. Equations (12b) and (12c) imply that the flow velocity is a linear function of s, while the temperature is a quadratic function of s. If we choose $y=0$ as the plane where the temperature $T(0)=T_0$ is a maximum, then

$$T(s)=T_0-\frac{m}{k_B}\gamma(a)s^2, \tag{24}$$

where

$$\gamma(a)\equiv\tfrac{2}{15}a^2\frac{F_\eta(a)}{F_\kappa(a)}. \tag{25}$$

From Eqs. (12a) and (22)–(24) one gets the relationship between s and y:

$$y=\frac{k_B}{\lambda_{11}p}s\left(T_0-\frac{m}{3k_B}\gamma s^2\right). \tag{26}$$

So far, the space dependence of the hydrodynamic fields is established but the coefficients $F_\eta(a)$ and $F_\kappa(a)$ remain unknown.

The thermal gradient is a function of space. In dimensionless form, it can be represented by the parameter

$$\epsilon=\left(\frac{2k_BT}{m}\right)^{1/2}\frac{\partial}{\partial s}\ln T. \tag{27}$$

This parameter measures the relative variation of temperature along the y direction over a distance equal to the (local) mean free path. The reduced shear rate a measures the variation of flow velocity relative to the thermal velocity over the same distance. Both dimensionless parameters are independent and describe how far the system is from equilibrium at each point. Far from the boundaries, the solution to Eq. (15) is expected to adopt the form

$$f(y,\mathbf{v})=n(y)\left[\frac{2k_BT(y)}{m}\right]^{-3/2}\phi(a,\epsilon(y);\boldsymbol{\xi}(y)), \tag{28a}$$

where

$$\boldsymbol{\xi}(y)=\left[\frac{2k_BT(y)}{m}\right]^{-1/2}[\mathbf{v}-\mathbf{u}(y)] \tag{28b}$$

is the peculiar velocity relative to the thermal velocity. According to the symmetry properties of the problem, the reduced distribution function ϕ must be invariant under the transformations

$$\xi_z\rightarrow-\xi_z, \tag{29a}$$

$$\xi_x\rightarrow-\xi_x,\xi_y\rightarrow-\xi_y,\epsilon\rightarrow-\epsilon, \tag{29b}$$

$$\xi_x\rightarrow-\xi_x,a\rightarrow-a. \tag{29c}$$

Equation (28a) represents a *normal* solution,[15] since it depends on space only through its dependence on the hydrodynamic fields and their gradients. Thus, it does not necessarily satisfy the specific boundary conditions of the problem. On the other hand, the solution (28a) should apply in a "bulk" domain, outside the boundary layers. From a technical point of view, it can be seen as a particular solution to Eq. (15) corresponding to homogeneous boundary conditions (half distributions vanishing at the boundaries).[11] These are (idealized) conditions for planar Couette flow between very cold walls.

Equation (28a) implies that the velocity moments, Eq. (16), have the form

$$M_{\mathbf{k}}(y)=n\left[\frac{2k_BT(y)}{m}\right]^{k/2}\mathscr{M}_{\mathbf{k}}(a,\epsilon(y)), \tag{30}$$

where

$$\mathscr{M}_{k_1,k_2,k_3}(a,\epsilon)=\int d\boldsymbol{\xi}\,\xi_x^{k_1}\xi_y^{k_2}\xi_z^{k_3}\phi(a,\epsilon;\boldsymbol{\xi}). \tag{31}$$

As a consequence,

$$\frac{\partial}{\partial s}M_{\mathbf{k}}=n\left(\frac{2k_BT}{m}\right)^{(k-1)/2}\left[\left(\frac{k}{2}-1\right)\epsilon\mathscr{M}_{\mathbf{k}}-\left(\frac{\epsilon^2}{2}+4\gamma\right)\frac{\partial}{\partial\epsilon}\mathscr{M}_{\mathbf{k}}\right], \tag{32}$$

where use has been made of Eqs. (12), (25), and (27). Then Eq. (17) becomes

$$\frac{k-1}{2}\epsilon\mathcal{M}_{k_1,k_2+1,k_3}-\left(\frac{\epsilon^2}{2}+4\gamma\right)\frac{\partial}{\partial\epsilon}\mathcal{M}_{k_1,k_2+1,k_3}$$

$$+k_1a\mathcal{M}_{k_1-1,k_2+1,k_3}$$

$$=\frac{1}{\lambda_{11}}\sum_{\mathbf{k}',\mathbf{k}''}{}^{\dagger}C^{\mathbf{k}}_{\mathbf{k}',\mathbf{k}''}\mathcal{M}_{\mathbf{k}'}\mathcal{M}_{\mathbf{k}''}. \tag{33}$$

The hierarchy (33) is equivalent to the hierarchy (17) (for Maxwell molecules), provided that the profiles (12) are self-consistent. The consistency conditions are

$$\mathcal{M}_{000}=1, \tag{34a}$$

$$\mathcal{M}_{100}=\mathcal{M}_{010}=\mathcal{M}_{001}=0, \tag{34b}$$

$$\mathcal{M}_{200}+\mathcal{M}_{020}+\mathcal{M}_{002}=\tfrac{3}{2}. \tag{34c}$$

Insertion into Eq. (33) yields

$$\frac{\partial}{\partial\epsilon}\mathcal{M}_{110}=\frac{\partial}{\partial\epsilon}\mathcal{M}_{00}\text{[illegible]}0, \tag{35a}$$

$$\left[\frac{\epsilon}{2}-\left(\frac{\epsilon^2}{2}+4\gamma\right)\frac{\partial}{\partial\epsilon}\right](\mathcal{M}_{210}+\mathcal{M}_{030}+\mathcal{M}_{012})$$

$$=-2a\mathcal{M}_{110}. \tag{35b}$$

Equations (35) are equivalent to Eqs. (1). Equation (35a) shows that $\mathcal{M}_{110}$ does not depend on ϵ. This is consistent with the generalized Newton law, Eq. (13b). The physically meaningful solution of Eq. (35b) is

$$\mathcal{M}_{210}+\mathcal{M}_{030}+\mathcal{M}_{012}=\frac{a}{2\gamma}\mathcal{M}_{110}\epsilon. \tag{36}$$

The left-hand side is proportional to the component q_y of the heat flux. The fact that it depends linearly on ϵ indicates that the generalized Fourier law, Eq. (13a), holds for *arbitrary* thermal gradients.

In order to complete the proof of the consistency of Eqs. (12), we need to show that the hierarchy (33), seen as an infinite set of first-order differential equations with respect to ϵ, admits a solution. By inspection, it is easy to realize that such a solution does exist. In it, the moments of order k are *polynomials* in ϵ of degree $k-2$, the coefficients being functions of the shear rate a. More specifically,

$$\mathcal{M}_{\mathbf{k}}(a,\epsilon)=\sum_{\ell=0}^{k-2}\mu_{\mathbf{k}}^{(\ell)}(a)\epsilon^{\ell}. \tag{37}$$

The symmetry properties (29a) and (29b) imply that $\mu_{\mathbf{k}}^{(\ell)}=0$ if k_3 or $k_1+k_2+\ell$ are odd numbers. The key point is that, according to Eq. (37), $(k-2-\epsilon\partial/\partial\epsilon)\mathcal{M}_{\mathbf{k}}$ is a polynomial of degree $k-4$, so that both sides of Eq. (33) are polynomials of degree $k-2$. By equating the coefficients of the same degree, one has

$$\frac{k-\ell}{2}\mu^{(\ell-1)}_{k_1,k_2+1,k_3}-4(\ell+1)\gamma\mu^{(\ell+1)}_{k_1,k_2+1,k_3}+k_1a\mu^{(\ell)}_{k_1-1,k_2+1,k_3}$$

$$=\frac{1}{\lambda_{11}}\sum_{\mathbf{k}',\mathbf{k}''}{}^{\dagger}C^{\mathbf{k}}_{\mathbf{k}',\mathbf{k}''}\sum_{\ell'=0}^{k'}\mu_{\mathbf{k}'}^{(\ell')}\mu_{\mathbf{k}''}^{(\ell-\ell')}$$

$$\equiv\chi^{(\ell)}_{k_1,k_2,k_3} \tag{38}$$

with the convention that $\mu_{\mathbf{k}}^{(\ell)}=0$ if $\ell<0$ or $\ell>k-2$, except $\mu_{000}^{(\ell)}=\delta_{\ell 0}$, on account of Eq. (34a).

The coefficients $F_\kappa(a)$ and $F_\eta(a)$, defined by Eqs. (13), are expressed as

$$F_\kappa(a)=-\tfrac{4}{5}[\mu_{210}^{(1)}(a)+\mu_{030}^{(1)}(a)+\mu_{012}^{(1)}(a)], \tag{39a}$$

$$F_\eta(a)=-3\frac{\mu_{110}^{(0)}(a)}{a}. \tag{39b}$$

In addition to the shear-rate dependent thermal conductivity and shear viscosity, other transport coefficients are also important. The viscometric functions $\Psi_1(a)$ and $\Psi_2(a)$ measure normal stress effects. They are defined as

$$P_{xx}=p\left\{1-\tfrac{1}{3}[2\Psi_1(a)+\Psi_2(a)]\left(\frac{\eta_0}{p}\frac{\partial u_x}{\partial y}\right)^2\right\}, \tag{40a}$$

$$P_{yy}=p\left\{1+\tfrac{1}{3}[\Psi_1(a)-\Psi_2(a)]\left(\frac{\eta_0}{p}\frac{\partial u_x}{\partial y}\right)^2\right\}, \tag{40b}$$

$$P_{zz}=p\left\{1+\tfrac{1}{3}[\Psi_1(a)+2\Psi_2(a)]\left(\frac{\eta_0}{p}\frac{\partial u_x}{\partial y}\right)^2\right\}. \tag{40c}$$

In terms of the coefficients $\mu_{\mathbf{k}}^{(\ell)}$, the viscometric functions are

$$\Psi_1(a)=\frac{9}{2a^2}[\mu_{020}^{(0)}(a)-\mu_{200}^{(0)}(a)], \tag{41a}$$

$$\Psi_2(a)=\frac{9}{2a^2}[\mu_{002}^{(0)}(a)-\mu_{020}^{(0)}(a)]. \tag{41b}$$

Although the temperature gradient is directed along the y direction, the presence of a profile of the x component of the flow velocity induces a non-zero x component of the heat flux. It is characterized by a transport coefficient $\Phi(a)$ defined by

$$q_x=\frac{\eta_0^2}{mnT}\Phi(a)\frac{\partial T}{\partial y}\frac{\partial u_x}{\partial y}. \tag{42}$$

Consequently,

$$\Phi(a)=\frac{9}{2a}[\mu_{300}^{(1)}(a)+\mu_{120}^{(1)}(a)+\mu_{102}^{(1)}(a)]. \tag{43}$$

Thus, we have proved in this section that the Boltzmann equation (15) for Maxwell molecules admits a solution with the *space* dependence expressed by Eqs. (12) and, more generally, Eq. (37). In fact, Eqs. (24), (27), (30), and (37) show that the moments $M_{\mathbf{k}}$ are polynomials in s of degree $k-2$. By inverting the relationship (26) one can get the dependence on y. On the other hand, the numerical coefficients $\gamma(a)$ and $\mu_{\mathbf{k}}^{(\ell)}(a)$ are still unknown. They obey the algebraic hierarchy (38), which cannot be solved in a recursive way, since the coefficients of order k and degree ℓ are coupled to those of order $k+1$ and degree $\ell+1$. However, the hierarchy can be solved step by step if one performs a perturbation expansion in powers of the shear rate.

III. EXPANSION IN POWERS OF THE SHEAR RATE

We assume now that the shear rate a is small and expand the coefficients $\mu_{\mathbf{k}}^{(\ell)}$ in powers of a:

$$\mu_{\mathbf{k}}^{(\ell)}(a)=\mu_{\mathbf{k}}^{(\ell;0)}+\mu_{\mathbf{k}}^{(\ell;1)}a+\mu_{\mathbf{k}}^{(\ell;2)}a^2+\cdots. \tag{44}$$

On account of the last identity in Eq. (38), one also has

$$\chi_{\mathbf{k}}^{(\ell)}(a)=\chi_{\mathbf{k}}^{(\ell;0)}+\chi_{\mathbf{k}}^{(\ell;1)}a+\chi_{\mathbf{k}}^{(\ell;2)}a^2+\cdots. \tag{45}$$

According to the invariance (29c) $\mu_{\mathbf{k}}^{(\ell;j)}=\chi_{\mathbf{k}}^{(\ell;j)}=0$ if $k_1+j=$odd. Notice that $\ell+j$ indicates the hydrodynamic order. Thus, $\mu_{\mathbf{k}}^{(\ell;j)}$ is a Navier–Stokes coefficient if $\ell+j=1$, a Burnett coefficient if $\ell+j=2$, a super-Burnett coefficient if $\ell+j=3$, and so on. Insertion of the expansion (44) into Eq. (38) gives rise to a set of hierarchies, one for each power of a. The first four hierarchies are

$$\frac{k-\ell}{2}\,\mu_{k_1,k_2+1,k_3}^{(\ell-1;0)}=\chi_{k_1,k_2,k_3}^{(\ell;0)}, \tag{46}$$

$$\frac{k-\ell}{2}\,\mu_{k_1,k_2+1,k_3}^{(\ell-1;1)}+k_1\mu_{k_1-1,k_2+1,k_3}^{(\ell;0)}=\chi_{k_1,k_2,k_3}^{(\ell;1)}, \tag{47}$$

$$\frac{k-\ell}{2}\,\mu_{k_1,k_2+1,k_3}^{(\ell-1;2)}-\tfrac{8}{15}(\ell+1)\mu_{k_1,k_2+1,k_3}^{(\ell+1;0)}+k_1\mu_{k_1-1,k_2+1,k_3}^{(\ell;1)}=\chi_{k_1,k_2,k_3}^{(\ell;2)}, \tag{48}$$

$$\frac{k-\ell}{2}\,\mu_{k_1,k_2+1,k_3}^{(\ell-1;3)}-\tfrac{8}{15}(\ell+1)\mu_{k_1,k_2+1,k_3}^{(\ell+1;1)}+k_1\mu_{k_1-1,k_2+1,k_3}^{(\ell;2)}=\chi_{k_1,k_2,k_3}^{(\ell;3)}. \tag{49}$$

In these equations we have taken into account that, according to Eq. (25), $\gamma(a)=\frac{2}{15}a^2+\mathcal{O}(a^4)$. Equation (46) is the hierarchy corresponding to the case of pure heat flow ($a=0$, ϵ arbitrary). It can be solved recursively[8,9] by following the scheme

$$\{\mu_{\mathbf{k}'}^{(\ell';0)},k'\leqslant k+1,k'+\ell'\leqslant k+\ell\}\to\{\mu_{\mathbf{k}}^{(\ell;0)}\}, \tag{50}$$

i.e., the knowledge of $\mu_{\mathbf{k}}^{(\ell;0)}$ requires the previous knowledge of the coefficients $\mu_{\mathbf{k}'}^{(\ell';0)}$ with $k'\leqslant k+1$ and $k'+\ell'\leqslant k+\ell$. The explicit expressions for the moments at $a=0$ through order $k=5$ are[9]

$$\mathcal{M}_{200}(\epsilon,0)=\mathcal{M}_{002}(\epsilon,0)=\tfrac{1}{2}, \tag{51a}$$

$$\mathcal{M}_{020}(\epsilon,0)=\tfrac{1}{2}, \tag{51b}$$

$$\mathcal{M}_{210}(\epsilon,0)=\mathcal{M}_{012}(\epsilon,0)=-\tfrac{1}{4}\epsilon, \tag{52a}$$

$$\mathcal{M}_{030}(\epsilon,0)=-\tfrac{3}{4}\epsilon, \tag{52b}$$

$$\mathcal{M}_{400}(\epsilon,0)=\mathcal{M}_{004}(\epsilon,0)=3\mathcal{M}_{202}(\epsilon,0)=\tfrac{3}{4}+\tfrac{33}{28}\epsilon^2, \tag{53a}$$

$$\mathcal{M}_{220}(\epsilon,0)=\mathcal{M}_{022}(\epsilon,0)=\tfrac{1}{4}+\tfrac{19}{28}\epsilon^2, \tag{53b}$$

$$\mathcal{M}_{040}(\epsilon,0)=\tfrac{3}{4}+\tfrac{81}{28}\epsilon^2, \tag{53c}$$

$$\mathcal{M}_{410}(\epsilon,0)=\mathcal{M}_{014}(\epsilon,0)=3\mathcal{M}_{212}(\epsilon,0)=-\frac{3}{4}\,\epsilon-\frac{9}{10}\left[\frac{1163}{378}+\frac{2}{3\omega'}-\frac{1}{\omega''}\left(\frac{8}{7}+\frac{1}{\omega'}\right)\right]\epsilon^3, \tag{54a}$$

$$\mathcal{M}_{230}(\epsilon,0)=\mathcal{M}_{032}(\epsilon,0)=-\frac{3}{4}\,\epsilon-\frac{9}{10}\left[\frac{1163}{378}+\frac{2}{3\omega'}-\frac{1}{6\omega''}\left(\frac{8}{7}+\frac{1}{\omega'}\right)\right]\epsilon^3, \tag{54b}$$

$$\mathcal{M}_{050}(\epsilon,0)=-\frac{15}{4}\,\epsilon-\frac{9}{2}\left[\frac{1163}{378}+\frac{2}{3\omega'}+\frac{2}{3\omega''}\left(\frac{8}{7}+\frac{1}{\omega'}\right)\right]\epsilon^3. \tag{54c}$$

In Eqs. (54), $\omega'\equiv\lambda_{22}/\lambda_{11}\simeq2.0133$ and $\omega''\equiv\lambda_{13}/\lambda_{11}\simeq2.3555$.

Once the coefficients $\mu_{\mathbf{k}}^{(\ell;0)}$ are known, the remaining coefficients $\mu_{\mathbf{k}}^{(\ell;j)}$ are obtained from Eqs. (47), (48), and (49) if $j=1$, 2, and 3, respectively, by generalizing the scheme (50) to

$$\{\mu_{\mathbf{k}'}^{(\ell';j')},k'\leqslant k+1,k'+\ell'+j'\leqslant k+\ell+j\}\to\{\mu_{\mathbf{k}}^{(\ell;j)}\}. \tag{55}$$

According to Eqs. (39), the coefficients $F_\kappa^{(2)}$ and $F_\eta^{(2)}$ defined in Eqs. (14) are

$$F_\kappa^{(2)}=-\tfrac{4}{5}(\mu_{210}^{(1;2)}+\mu_{030}^{(1;2)}+\mu_{012}^{(1;2)}), \tag{56a}$$

$$F_\eta^{(2)}=-3\mu_{110}^{(0;3)}. \tag{56b}$$

The calculations leading to these coefficients are rather straightforward but tedious and are done in Appendix B. Insertion of Eqs. (B10) and (B14) into Eqs. (56) yields

$$F_\kappa^{(2)}=-(\tfrac{2182}{675}-\tfrac{8}{3675}\omega)\simeq-3.226, \tag{57a}$$

$$F_\eta^{(2)}=-\tfrac{596}{405}, \tag{57b}$$

where $\omega\equiv\lambda_{04}/\lambda_{11}\simeq2.8097$. These are super-Burnett coefficients that, according to Eqs. (13) and (14), measure how the thermal conductivity and the shear viscosity tend to decrease with respect to their Navier–Stokes values as the shear rate a increases. The parameter $\gamma(a)$ defined in Eqs. (24) and (25) behaves as

$$\gamma(a)=\tfrac{2}{15}a^2[1+(F_\eta^{(2)}-F_\kappa^{(2)})a^2+\mathcal{O}(a^4)]. \tag{58}$$

Since $F_\eta^{(2)}>F_\kappa^{(2)}$, γ tends to increase with a.

The results of Appendix B allow us also to obtain the viscometric functions, Eqs. (41), and the function Φ, Eq. (43) in the limit $a\to0$. From Eqs. (B6) and (B7) one gets $\Psi_1(0)=-\frac{14}{5}$, $\Psi_2(0)=\frac{4}{5}$, and $\Phi(0)=\frac{105}{8}$. These are Burnett coefficients and coincide with the ones obtained from the Burnett constitutive equations for Maxwell molecules[5] if one particularizes to the profiles (12). In this respect, when making such a comparison, it is important to realize that, according to Eqs. (12), $\partial^2T/\partial y^2+T^{-1}(\partial T/\partial y)^2=-(4m/15k_B)\times(\partial u_x/\partial y)^2$ and $\partial^2u_x/\partial y^2=-(\partial u_x/\partial y)(\partial\ \ln T/\partial y)$.

IV. DISCUSSION

We have considered in this paper the state of a dilute gas subject to steady planar Couette flow. The gas is enclosed between two infinite parallel plates kept in relative motion and, in general, at different temperatures. In the particular case of plates at rest but with different temperatures, the state becomes that of steady planar Fourier flow. By analyzing the exact infinite hierarchy of moment equations derived from the Boltzmann equation for Maxwell molecules, we have proved that the solution has a quite simple spatial structure. More specifically, the hydrodynamic profiles are $p=\text{const}$, $\partial u_x/\partial s=a=\text{const}$, $\partial^2 T/\partial s^2=-(2m/k_B)\gamma(a)=\text{const}$, where s is a scaled space variable, Eqs. (23) and (26). In general, the velocity moments of order k are polynomials of degree $k-2$ in s. In dimensionless form, the moments of order k are polynomials of degree $k-2$ in the reduced thermal gradient $\epsilon(s)=[2k_BT(s)/m]^{1/2}\partial \ln T/\partial s$, the coefficients being nonlinear functions of the reduced shear rate a. The two independent parameters ϵ and a are arbitrary and characterize the departure of the system from equilibrium. In the solution found here, the pressure tensor, and hence the nonlinear shear viscosity $\eta(a)=\eta(0)F_\eta(a)$, is independent of ϵ, while the heat flux is proportional to ϵ. The latter allows one to define a nonlinear thermal conductivity $\kappa(a)=\kappa(0)F_\kappa(a)$. The coefficients $F_\eta(a)$ and $F_\kappa(a)$ are related to the curvature of the temperature profile through the identity $\gamma(a)=2a^2F_\eta(a)/15F_\kappa(a)$.

In the particular case of the Fourier flow ($a=0$, ϵ arbitrary), the exact solution of Asmolov *et al.*[8] is recovered. For arbitrary shear rates, the coefficients such as $F_\eta(a)$ and $F_\kappa(a)$ obey an infinite hierarchy of coupled *algebraic* equations. By performing a perturbation expansion in powers of a around the Fourier flow state, the hierarchy can be solved in a recursive way. In particular, $F_\eta(a)=1+F_\eta^{(2)}a^2+\mathcal{O}(a^4)$, $F_\kappa(a)=1+F_\kappa^{(2)}a^2+\mathcal{O}(a^4)$, where $F_\eta^{(2)}=-\frac{596}{405}$, $F_\kappa^{(2)}\simeq-3.226$ are super-Burnett coefficients. The BGK and Liu model kinetic equations admit a solution analogous to the one analyzed here but, obviously, with different values for the numerical coefficients. For instance, the BGK equation[10] yields $F_\eta^{(2)}=-\frac{18}{5}(\nu/\nu_{\text{BGK}})^2$, $F_\kappa^{(2)}=-\frac{162}{25}(\nu/\nu_{\text{BGK}})^2$, where ν_{BGK} is the only collision frequency contained in the BGK equation. It can be adjusted as to reproduce either the exact Navier–Stokes shear viscosity ($\nu_{\text{BGK}}=\frac{3}{2}\nu$) or the exact Navier–Stokes thermal conductivity ($\nu_{\text{BGK}}=\nu$). In the former case, the coefficients $F_\eta^{(2)}$ and $F_\eta^{(2)}$ agree within about 10% with the exact values. In the case of the Liu model, one has[13] $F_\eta^{(2)}=-\frac{16}{5}(\nu/\zeta)^2$, $F_\kappa^{(2)}=-\frac{98}{25}(\nu/\zeta)^2$, where ζ is an additional collision frequency and the exact Navier–Stokes transport coefficients are obtained with independence of the choice for the ratio ζ/ν.[14] If one chooses $\zeta=\nu$, the exact constitutive equation for the heat flux to Burnett order is obtained,[17] while that happens for the pressure tensor if one chooses $\zeta=\frac{3}{2}\nu$. In the first case, $F_\kappa^{(2)}$ is estimated with a deviation of 22%, while in the second case $F_\eta^{(2)}$ is estimated with a deviation of 4%. In either case the ratio $F_\kappa^{(2)}/F_\eta^{(2)}$ is underestimated in a 44% by the Liu model and, paradoxically, in an 18% by the BGK model. Higher order coefficients, such as $F_\kappa^{(4)}$ and $F_\eta^{(4)}$, can also be obtained, but the algebra involved becomes much less manageable.

It must be pointed out that the solution we have obtained for the Couette flow describes an infinite system (i.e., both moving plates are separated by an infinite distance), so that boundary effects are absent. Thus, the solution cannot be extrapolated inside the boundary layers and boundary effects (such as the velocity slip and the temperature jumps at the walls) may not be predicted. Instead, the solution corresponds to what is usually referred to as a "normal" solution to the Boltzmann equation.[15] This point, in the context of the BGK equation, is thoroughly discussed in Ref. 11. When the separation between the plates is measured in terms of the scaled variable s, it becomes finite, say $2s_w$. The normal solution can then be interpreted as the solution to the Boltzmann equation with boundary conditions at $s=\pm s_w$ corresponding to plates moving with velocities $\pm u_w=\pm a s_w$ and kept at *zero* temperature, $T_w=0$. The Boltzmann equation could be numerically solved by means of the DSMC method[18] with values of T_w sufficiently small to get the velocity moments, as well as the velocity distribution function itself. In order to minimize the boundary effects, one could apply boundary conditions corresponding to a gas enclosed between two baths described by the BGK solution of the steady planar Couette flow, following the same procedure as in the case of the Fourier flow.[19] Since we have proved here that the moments are polynomials in s, simple fits outside the boundary layers would provide transport coefficients such as $F_\eta(a)$ and $F_\kappa(a)$ for *arbitrary* values of the shear rate a.

Finally, it is worth remarking how important exact solutions of the Boltzmann equation are, especially in nonhomogeneous situations. They are useful to improve our understanding of nonequilibrium phenomena and also to assess the reliability of approximate methods or kinetic models. The solution we have dealt with in this paper has been inspired and suggested by a solution to the BGK equation having a similar structure. This shows the interest of using kinetic models as a means to explore the possibility of finding exact solutions of the Boltzmann equation.

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APPENDIX A: COLLISIONAL MOMENTS

In this appendix we list the collisional moments J_k, Eq. (19), for Maxwell molecules through order $k=4$. They are

$$J_{110}=-\tfrac{3}{2}\nu M_{110}, \tag{A1a}$$

$$J_{200}=-\tfrac{3}{2}\nu\left(M_{200}-\frac{p}{m}\right), \tag{A1b}$$

$$J_{111}=-\tfrac{9}{4}\nu M_{111}, \tag{A2a}$$

$$J_{210}=-2\nu[M_{210}-\tfrac{1}{8}(M_{012}+M_{030})], \tag{A2b}$$

$$J_{300}=-\tfrac{3}{2}\nu[M_{300}-\tfrac{1}{2}(M_{120}+M_{102})], \tag{A2c}$$

$$J_{112}=-(\tfrac{1}{4}\nu+\tfrac{6}{7}\nu')M_{112}+(\tfrac{1}{7}\nu'-\tfrac{1}{4}\nu)\times\left(M_{130}+M_{310}+3\frac{p}{mn}M_{110}\right)+(3\nu-\nu')\frac{1}{n}M_{110}M_{002}-(\nu-\tfrac{2}{7}\nu')\frac{1}{n}M_{110}\times(M_{200}+M_{020})+(5\nu-\tfrac{12}{7}\nu')\frac{1}{n}M_{101}M_{011}, \tag{A3a}$$

$$J_{220}=-(\tfrac{3}{10}\nu+\tfrac{27}{35}\nu')M_{220}+(\tfrac{1}{10}\nu-\tfrac{1}{35}\nu')\left[M_{004}-\frac{1}{n}\left(8M_{020}^2+8M_{200}^2-2M_{002}^2-9\frac{p^2}{m^2}+6M_{101}^2+6M_{011}^2\right)\right]+(\tfrac{3}{35}\nu'-\tfrac{1}{20}\nu)(M_{202}+M_{022})+(\tfrac{4}{35}\nu'-\tfrac{3}{20}\nu)(M_{400}+M_{040})+(3\nu-\nu')\frac{1}{n}M_{200}M_{020}+(\tfrac{3}{7}\nu'-\tfrac{3}{4}\nu)\frac{p}{mn}(M_{200}+M_{020})+(\tfrac{22}{5}\nu-\tfrac{54}{35}\nu')\frac{1}{n}M_{110}^2, \tag{A3b}$$

$$J_{310}=-(\tfrac{3}{4}\nu+\tfrac{4}{7}\nu')M_{310}+(\tfrac{3}{7}\nu'-\tfrac{3}{4}\nu)\times\left[M_{112}+M_{130}+3\frac{p}{mn}M_{110}\right]-(3\nu-\tfrac{6}{7}\nu')\frac{1}{n}(M_{020}M_{110}+M_{101}M_{011})-(\tfrac{15}{7}\nu'-6\nu)\frac{1}{n}M_{200}M_{110}, \tag{A3c}$$

$$J_{400}=-(\tfrac{6}{5}\nu+\tfrac{8}{35}\nu')M_{400}+(\tfrac{24}{35}\nu'-\tfrac{9}{10}\nu)(M_{220}+M_{202})+(\tfrac{3}{10}\nu-\tfrac{3}{35}\nu')\left[M_{040}+2M_{022}+M_{004}+\frac{1}{n}\left(2M_{020}^2+2M_{002}^2+9\frac{p^2}{m^2}+2M_{011}^2-16M_{110}^2-16M_{101}^2\right)\right]-(\tfrac{51}{35}\nu'-\tfrac{18}{5}\nu)\frac{1}{n}M_{200}^2+(\tfrac{18}{7}\nu'-\tfrac{9}{2}\nu)\frac{p}{mn}M_{200}. \tag{A3d}$$

In these equations, ν is given by Eq. (22) and $\nu'\equiv n\lambda_{04}$, where $\lambda_{04}/\lambda_{11}\equiv\omega\simeq 2.8097$.[16] The remaining collisional moments (such as J_{102} or J_{004}) can be easily obtained from Eqs. (A1)–(A3) by the adequate permutation of indices.

APPENDIX B: EVALUATION OF $\mu_{\mathbf{k}}^{(\ell;j)}$

In this appendix we are going to work out the first few solutions of Eqs. (47)–(49). First, we make use of Eqs. (A1)–(A3) to express $\chi_{\mathbf{k}}^{(\ell)}$, defined by the last identity in Eq. (38), in terms of $\mu_{\mathbf{k}}^{(\ell)}$:

$$\chi_{110}^{(0)}=-\tfrac{3}{2}\mu_{110}^{(0)}, \tag{B1a}$$

$$\chi_{200}^{(0)}=-\tfrac{3}{2}(\mu_{200}^{(0)}-\tfrac{1}{2}), \tag{B1b}$$

$$\chi_{210}^{(1)}=-2[\mu_{210}^{(1)}-\tfrac{1}{8}(\mu_{012}^{(1)}+\mu_{030}^{(1)})], \tag{B2a}$$

$$\chi_{300}^{(1)}=-\tfrac{3}{2}[\mu_{300}^{(1)}-\tfrac{1}{2}(\mu_{120}^{(1)}+\mu_{102}^{(1)})], \tag{B2b}$$

$$\chi_{112}^{(0)}=-(\tfrac{1}{4}+\tfrac{6}{7}\omega)\mu_{112}^{(0)}+(\tfrac{1}{7}\omega-\tfrac{1}{4})(\mu_{130}^{(0)}+\mu_{310}^{(0)}+\tfrac{3}{2}\mu_{110}^{(0)})+(3-\omega)\mu_{110}^{(0)}\mu_{002}^{(0)}-(1-\tfrac{2}{7}\omega)\mu_{110}^{(0)}(\mu_{200}^{(0)}+\mu_{020}^{(0)})+(5-\tfrac{2}{7}\omega)\mu_{101}^{(0)}\mu_{011}^{(0)}, \tag{B3a}$$

$$\chi_{220}^{(0)}=-(\tfrac{3}{10}+\tfrac{27}{35}\omega)\mu_{220}^{(0)}+(\tfrac{1}{10}-\tfrac{1}{35}\omega)(\mu_{004}^{(0)}-8\mu_{020}^{(0)2}-8\mu_{200}^{(0)2}+2\mu_{002}^{(0)2}+\tfrac{9}{4}-6\mu_{101}^{(0)2}-6\mu_{011}^{(0)2})+(\tfrac{3}{35}\omega-\tfrac{1}{20})(\mu_{202}^{(0)}+\mu_{022}^{(0)})+(\tfrac{4}{35}\omega-\tfrac{3}{20})(\mu_{400}^{(0)}+\mu_{040}^{(0)})+(3-\omega)\mu_{200}^{(0)}\mu_{020}^{(0)}+(\tfrac{3}{7}\omega-\tfrac{3}{4})\tfrac{1}{2}(\mu_{200}^{(0)}+\mu_{020}^{(0)})+(\tfrac{22}{5}-\tfrac{54}{35}\omega)\mu_{110}^{(0)2}, \tag{B3b}$$

$$\chi_{310}^{(0)}=-(\tfrac{3}{4}+\tfrac{4}{7}\omega)\mu_{310}^{(0)}+(\tfrac{3}{7}\omega-\tfrac{3}{4})(\mu_{112}^{(0)}+\mu_{130}^{(0)}+\tfrac{3}{2}\mu_{110}^{(0)})-(3-\tfrac{6}{7}\omega)(\mu_{020}^{(0)}\mu_{110}^{(0)}+\mu_{101}^{(0)}\mu_{011}^{(0)})-(\tfrac{15}{7}\omega-6)\mu_{200}^{(0)}\mu_{110}^{(0)}, \tag{B3c}$$

$$\chi_{400}^{(0)}=-(\tfrac{6}{5}+\tfrac{8}{35}\omega)\mu_{400}^{(0)}+(\tfrac{24}{35}\omega-\tfrac{9}{10})(\mu_{220}^{(0)}+\mu_{202}^{(0)})+(\tfrac{3}{10}-\tfrac{3}{35}\omega)(\mu_{040}^{(0)}+2\mu_{022}^{(0)}+\mu_{004}^{(0)}+2\mu_{020}^{(0)2}+2\mu_{002}^{(0)2}+\tfrac{9}{4}+2\mu_{011}^{(0)2}-16\mu_{110}^{(0)2}-16\mu_{101}^{(0)2})-(\tfrac{51}{35}\omega-\tfrac{18}{5})\mu_{200}^{(0)2}+(\tfrac{18}{7}\omega-\tfrac{9}{2})\tfrac{1}{2}\mu_{200}^{(0)}, \tag{B3d}$$

$$\chi_{112}^{(2)}=-(\tfrac{1}{4}+\tfrac{6}{7}\omega)\mu_{112}^{(2)}+(\tfrac{1}{7}\omega-\tfrac{1}{4})(\mu_{130}^{(2)}+\mu_{310}^{(2)}), \tag{B4a}$$

$$\chi_{220}^{(2)}=-(\tfrac{3}{10}+\tfrac{27}{35}\omega)\mu_{220}^{(2)}+(\tfrac{1}{10}-\tfrac{1}{35}\omega)\mu_{004}^{(2)}+(\tfrac{3}{35}\omega-\tfrac{1}{20})(\mu_{202}^{(2)}+\mu_{022}^{(2)})+(\tfrac{4}{35}\omega-\tfrac{3}{20})\times(\mu_{400}^{(2)}+\mu_{040}^{(2)}), \tag{B4b}$$

$$\chi_{310}^{(2)}=-(\tfrac{3}{4}+\tfrac{4}{7}\omega)\mu_{310}^{(2)}+(\tfrac{3}{7}\omega-\tfrac{3}{4})(\mu_{112}^{(2)}+\mu_{130}^{(2)}), \tag{B4c}$$

$$\chi_{400}^{(2)}=-(\tfrac{6}{5}+\tfrac{8}{35}\omega)\mu_{400}^{(2)}+(\tfrac{24}{35}\omega-\tfrac{9}{10})(\mu_{220}^{(2)}+\mu_{202}^{(2)})+(\tfrac{3}{10}-\tfrac{3}{35}\omega)(\mu_{040}^{(2)}+2\mu_{022}^{(2)}+\mu_{004}^{(2)}). \tag{B4d}$$

In the remainder of this appendix we are going to evaluate all the coefficients $\mu_{\mathbf{k}}^{(\ell;j)}$ such that $k\leq 5$, $j\leq 3$, and $k+\ell+j\leq 6$. We will proceed in a sequential way, so that at each step use will be made of results derived in previous steps, as well as of Eqs. (B1)–(B3). Of course, the coefficients with $j=0$ are given by Eqs. (51)–(54) and will be taken as a starting point. First, we take Eq. (47) with $\ell=0$ and $\mathbf{k}=\{110\}$:

$$\tfrac{1}{2}=-\tfrac{3}{2}\mu_{110}^{(0;1)}, \tag{B5}$$

so that $\mu_{110}^{(0;1)}=-\tfrac{1}{3}$. In fact, this is a well-known Navier–Stokes coefficient. Next, we take $\ell=0$ and $\mathbf{k}=\{002\}$, $\{020\}$, and $\{200\}$ in Eq. (48):

$$\tfrac{2}{15}=-\tfrac{3}{2}\mu_{002}^{(0;2)}, \tag{B6a}$$

$$\frac{2}{5} = -\frac{3}{2}\mu_{020}^{(0;2)}, \tag{B6b}$$

$$-\frac{8}{15} = -\frac{3}{2}\mu_{200}^{(0;2)}. \tag{B6c}$$

Thus, $\mu_{002}^{(0;2)} = -\frac{4}{45}$, $\mu_{020}^{(0;2)} = -\frac{4}{15}$, $\mu_{200}^{(0;2)} = \frac{16}{45}$. These are Burnett coefficients related to the viscometric functions.

Let us consider now Eq. (47) with $\ell = 0$ and $\mathbf{k} = \{310\}$, $\{130\}$, and $\{112\}$:

$$\frac{3}{4} = -\left(\frac{3}{4} + \frac{4}{7}\omega\right)\mu_{310}^{(0;1)} + \left(\frac{3}{7}\omega - \frac{3}{4}\right)\left(\mu_{112}^{(0;1)} + \mu_{130}^{(0;1)}\right) - \frac{1}{8}, \tag{B7a}$$

$$\frac{3}{4} = -\left(\frac{3}{4} + \frac{4}{7}\omega\right)\mu_{130}^{(0;1)} + \left(\frac{3}{7}\omega - \frac{3}{4}\right)\left(\mu_{112}^{(0;1)} + \mu_{310}^{(0;1)}\right) - \frac{1}{8}, \tag{B7b}$$

$$\frac{1}{4} = -\left(\frac{1}{4} + \frac{6}{7}\omega\right)\mu_{112}^{(0;1)} + \left(\frac{1}{7}\omega - \frac{1}{4}\right)\left(\mu_{130}^{(0;1)} + \mu_{310}^{(0;1)}\right) - \frac{1}{24}. \tag{B7c}$$

The solution of this set of equations is $\mu_{310}^{(0;1)} = \mu_{130}^{(0;1)} = 3\mu_{112}^{(0;1)} = -\frac{1}{2}$.

Next, we make $\ell = 1$ and $\mathbf{k} = \{300\}$, $\{120\}$, and $\{102\}$ in Eq. (47):

$$-\frac{5}{4} = -\frac{3}{2}\mu_{300}^{(1;1)} + \frac{3}{4}\left(\mu_{120}^{(1;1)} + \mu_{102}^{(1;1)}\right), \tag{B8a}$$

$$-\frac{5}{4} = -2\mu_{120}^{(1;1)} + \frac{1}{4}\left(\mu_{102}^{(1;1)} + \mu_{300}^{(1;1)}\right), \tag{B8b}$$

$$-\frac{5}{12} = -2\mu_{102}^{(1;1)} + \frac{1}{4}\left(\mu_{120}^{(1;1)} + \mu_{300}^{(1;1)}\right). \tag{B8c}$$

The solution is

$$\mu_{300}^{(1;1)} = \frac{55}{36}, \tag{B9a}$$

$$\mu_{120}^{(1;1)} = \frac{95}{108}, \tag{B9b}$$

$$\mu_{102}^{(1;1)} = \frac{55}{108}. \tag{B9c}$$

The super-Burnett coefficient $\mu_{110}^{(0;3)}$ is now easily obtained from Eq. (49) with $\ell = 0$ and $\mathbf{k} = \{110\}$:

$$\mu_{110}^{(0;3)} = \frac{596}{1215}. \tag{B10}$$

We are going now to determine the coefficients leading to the heat flux. Let us take Eq. (48) with $\ell = 0$ and $\mathbf{k} = \{400\}$, $\{040\}$, $\{004\}$, $\{220\}$, $\{202\}$, and $\{022\}$:

$$-\frac{8}{5} = -\left(\frac{6}{5} + \frac{8}{35}\omega\right)\mu_{400}^{(0;2)} + \left(\frac{24}{35}\omega - \frac{9}{10}\right)\left(\mu_{220}^{(0;2)} + \mu_{202}^{(0;2)}\right) + \left(\frac{3}{10} - \frac{3}{35}\omega\right)\left(\mu_{040}^{(0;2)} + 2\mu_{022}^{(0;2)} + \mu_{004}^{(0;2)}\right) + \left(\frac{16}{105}\omega - \frac{4}{15}\right), \tag{B11a}$$

$$2 = -\left(\frac{6}{5} + \frac{8}{35}\omega\right)\mu_{040}^{(0;2)} + \left(\frac{24}{35}\omega - \frac{9}{10}\right)\left(\mu_{220}^{(0;2)} + \mu_{022}^{(0;2)}\right) + \left(\frac{3}{10} - \frac{3}{35}\omega\right)\left(\mu_{400}^{(0;2)} + 2\mu_{202}^{(0;2)} + \mu_{004}^{(0;2)}\right) - \left(\frac{11}{15} - \frac{16}{105}\omega\right), \tag{B11b}$$

$$\frac{2}{5} = -\left(\frac{6}{5} + \frac{8}{35}\omega\right)\mu_{004}^{(0;2)} + \left(\frac{24}{35}\omega - \frac{9}{10}\right)\left(\mu_{202}^{(0;2)} + \mu_{022}^{(0;2)}\right) + \left(\frac{3}{10} - \frac{3}{35}\omega\right)\left(\mu_{400}^{(0;2)} + 2\mu_{220}^{(0;2)} + \mu_{040}^{(0;2)}\right) - \frac{2}{105}\omega, \tag{B11c}$$

$$-\frac{3}{5} = -\left(\frac{3}{10} + \frac{27}{35}\omega\right)\mu_{220}^{(0;2)} + \left(\frac{1}{10} - \frac{1}{35}\omega\right)\mu_{004}^{(0;2)} + \left(\frac{3}{35}\omega - \frac{1}{20}\right)\times\left(\mu_{202}^{(0;2)} + \mu_{022}^{(0;2)}\right) + \left(\frac{4}{35}\omega - \frac{3}{20}\right)\left(\mu_{400}^{(0;2)} + \mu_{040}^{(0;2)}\right) + \left(\frac{1}{2} - \frac{6}{35}\omega\right), \tag{B11d}$$

$$-\frac{1}{5} = -\left(\frac{3}{10} + \frac{27}{35}\omega\right)\mu_{202}^{(0;2)} + \left(\frac{1}{10} - \frac{1}{35}\omega\right)\mu_{040}^{(0;2)} + \left(\frac{3}{35}\omega - \frac{1}{20}\right)\times\left(\mu_{220}^{(0;2)} + \mu_{022}^{(0;2)}\right) + \left(\frac{4}{35}\omega - \frac{3}{20}\right)\left(\mu_{400}^{(0;2)} + \mu_{004}^{(0;2)}\right) + \left(\frac{2}{105}\omega - \frac{1}{30}\right), \tag{B11e}$$

$$\frac{2}{5} = -\left(\frac{3}{10} + \frac{27}{35}\omega\right)\mu_{022}^{(0;2)} + \left(\frac{1}{10} - \frac{1}{35}\omega\right)\mu_{400}^{(0;2)} + \left(\frac{3}{35}\omega - \frac{1}{20}\right)\times\left(\mu_{220}^{(0;2)} + \mu_{202}^{(0;2)}\right) + \left(\frac{4}{35}\omega - \frac{3}{20}\right)\left(\mu_{040}^{(0;2)} + \mu_{004}^{(0;2)}\right) + \left(\frac{2}{105}\omega - \frac{1}{9}\right). \tag{B11f}$$

The solution of this set of six coupled equations is

$$\mu_{400}^{(0;2)} = \frac{18\omega^2 + 36449\omega - 23667}{25725\omega} \simeq 1.091, \tag{B12a}$$

$$\mu_{040}^{(0;2)} = -\frac{-6\omega^2 + 7777\omega + 7889}{8575\omega} \simeq -1.232, \tag{B12b}$$

$$\mu_{004}^{(0;2)} = -\frac{-258\omega^2 + 10521\omega - 5488}{25725\omega} \simeq -0.305, \tag{B12c}$$

$$\mu_{220}^{(0;2)} = \frac{18\omega^2 - 10591\omega + 79233}{77175\omega} \simeq 0.229, \tag{B12d}$$

$$\mu_{202}^{(0;2)} = \frac{138\omega^2 + 12719\omega - 8232}{77175\omega} \simeq 0.132, \tag{B12e}$$

$$\mu_{022}^{(0;2)} = -\frac{-138\omega^2 + 17171\omega + 8232}{77175\omega} \simeq -0.255. \tag{B12f}$$

Finally, from Eq. (48) with $\ell = 1$ and $\mathbf{k} = \{210\}$, $\{030\}$, and $\{012\}$, one gets

$$\mu_{220}^{(0;2)} + \frac{1957}{1890} = -2\mu_{210}^{(1;2)} + \frac{1}{4}\left(\mu_{012}^{(1;2)} + \mu_{030}^{(1;2)}\right), \tag{B13a}$$

$$\mu_{040}^{(0;2)} - \frac{108}{35} = -\frac{3}{2}\mu_{030}^{(1;2)} + \frac{3}{4}\left(\mu_{210}^{(1;2)} + \mu_{012}^{(1;2)}\right), \tag{B13b}$$

$$\mu_{022}^{(0;2)} - \frac{76}{105} = -2\mu_{012}^{(1;2)} + \frac{1}{4}\left(\mu_{210}^{(1;2)} + \mu_{030}^{(1;2)}\right). \tag{B13c}$$

The solution is

$$\mu_{210}^{(1;2)} = -\frac{1692\omega^2 - 207389\omega + 1901592}{4167450\omega} \simeq -0.114, \tag{B14a}$$

$$\mu_{030}^{(1;2)} = \frac{-1692\omega^2 + 4336129\omega + 568008}{1389150\omega} \simeq 3.264, \tag{B14b}$$

$$\mu_{012}^{(1;2)} = \frac{-4572\omega^2 + 3623809\omega + 197568}{4167450\omega} \simeq 0.883, \tag{B14c}$$

where use has been made of Eqs. (B12b), (B12d), and (B12f).

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