Nonlinear Poiseuille flow in a gas

Mohamed Tij and Mohamed Sabbane
Département de Physique, Université Moulay Ismaïl, Meknès, Morocco

Andrés Santos
Departamento de Física, Universidad de Extremadura, E-06071 Badajoz, Spain

The nonlinear Boltzmann equation for the steady planar Poiseuille flow generated by an external field **g** is exactly solved through order g^2. It is shown that the pressure and temperature profiles, as well as the momentum and heat fluxes, are in qualitative disagreement with the Navier–Stokes predictions. For instance, the temperature has a local minimum at the middle layer instead of a maximum. Also, a longitudinal component of the heat flux exists in the absence of gradients along that direction and normal stress differences appear although the flow is incompressible. To account for these g^2-order effects, which are relevant when the hydrodynamic quantities change over a characteristic length of the order of the mean free path, it is shown that the Chapman–Enskog expansion should be carried out three steps beyond the Navier–Stokes level. © *1998 American Institute of Physics.* [S1070-6631(98)01404-4]

I. INTRODUCTION

One of the most well-known textbook examples in fluid dynamics is the Poiseuille flow,[1] first studied by Poiseuille and Hagen in the past century. It consists of the steady flow along a channel of constant cross section produced by a pressure difference at the distant ends of the channel. The same effect is obtained if the pressure is constant along the channel but a longitudinal force (e.g., gravitation) exists.[2–4] To fix ideas, let us consider a fluid enclosed between two infinite parallel plates normal to the y axis and located at $y = \pm H$, which are kept at rest. A constant external force per unit mass $\mathbf{g} = g\hat{\mathbf{x}}$ is applied along a direction $\hat{\mathbf{x}}$ parallel to the plates. After a certain transient stage, a steady state with gradients along the y direction is reached. Under these conditions, the conservation equations for momentum and energy reduce to

$$\frac{\partial P_{xy}}{\partial y} = \rho g, \tag{1.1a}$$

$$\frac{\partial P_{yy}}{\partial y} = 0, \tag{1.1b}$$

$$P_{xy} \frac{\partial u_x}{\partial y} + \frac{\partial q_y}{\partial y} = 0, \tag{1.2}$$

where ρ is the mass density, **u** is the flow velocity, P is the pressure tensor, and **q** is the heat flux. Equations (1.1) and (1.2) become a closed set if one assumes the validity of the Navier–Stokes (NS) constitutive equations, namely

$$P_{xx} = P_{yy} = p, \tag{1.3a}$$

$$P_{xy} = -\eta \frac{\partial u_x}{\partial y}, \tag{1.3b}$$

$$q_x = 0, \tag{1.4a}$$

$$q_y = -\kappa \frac{\partial}{\partial y} T, \tag{1.4b}$$

where $p = \frac{1}{3} \text{Tr } \mathsf{P}$ is the hydrostatic pressure, T is the temperature, and η and κ are the shear viscosity and the thermal conductivity, respectively. Combination of Eqs. (1.1)–(1.4) yield

$$\frac{\partial p}{\partial y} = 0, \tag{1.5}$$

$$\frac{\partial}{\partial y} \eta \frac{\partial u_x}{\partial y} = -\rho g, \tag{1.6}$$

$$\frac{\partial}{\partial y} \kappa \frac{\partial T}{\partial y} = -\eta \left(\frac{\partial u_x}{\partial y} \right)^2. \tag{1.7}$$

Equation (1.6) gives a paraboliclike velocity profile, that is characteristic of the Poiseuille flow. The temperature profile has, according to Eq. (1.7), a quarticlike shape. Strictly speaking, these NS profiles are more complicated than just polynomials due to the temperature dependence of the transport coefficients. Since the hydrodynamic profiles must be symmetric with respect to the plane $y = 0$, their odd derivatives must vanish at $y = 0$. Thus from Eqs. (1.6) and (1.7) we have

$$\left. \frac{\partial^2 u_x}{\partial y^2} \right|_{y=0} = -\frac{\rho_0 g}{\eta_0}, \tag{1.8}$$

$$\left. \frac{\partial^2 T}{\partial y^2} \right|_{y=0} = 0 \tag{1.9}$$

$$\left. \frac{\partial^4 T}{\partial y^4} \right|_{y=0} = -2 \frac{\rho_0^2 g^2}{\eta_0 \kappa_0}, \tag{1.10}$$

where henceforth the subscript 0 will denote quantities evaluated at $y=0$. Therefore, Eqs. (1.8)–(1.10) imply that *both* the flow velocity and the temperature have a maximum at the middle layer $y=0$.

It must be kept in mind that, while Eqs. (1.1) and (1.2) are exact, Eqs. (1.3)–(1.10) are consequences of the NS approximation. The NS constitutive relations are expected to hold outside the boundary layers (i.e., for distances from the boundaries much larger than a relevant microscopic scale ℓ, such as the mean free path in the case of gases) and for small gradients (i.e., small relative variations of the hydrodynamic quantities over a distance equal to ℓ). The first limitation can be dealt with if $H \gg \ell$ and appropriate slip boundary conditions are applied. On the other hand, the second constraint implies the assumption of a sufficiently weak field g. In view of Eqs. (1.6) and (1.7), one could expect that the NS profiles are valid at least through order g^2. Nevertheless, this expectation fails in the simple case of a dilute gas, according to a recent perturbation solution[5] of the Bhatnagar–Gross–Krook (BGK) model kinetic equation[6] through order g^5. While the corrections to the velocity profile, Eq. (1.6), are indeed of order g^3 (due to symmetry reasons), deviations from Eq. (1.7) are already of order g^2. In particular, Eq. (1.9) is replaced by $(\partial^2 T/\partial y^2)|_{y=0} = (38/25)g^2\rho_0^2 T_0/p_0^2 > 0$. The main qualitative change is that $T(y)$ has now a local minimum at $y=0$ rather than a maximum. Analogously, the pressure is not constant, but $(\partial^2 p/\partial y^2)|_{y=0} = (12/5)g^2\rho_0^2/p_0$. Recent simulations of the Boltzmann equation for hard spheres[7] have shown a qualitative and semi-quantitative agreement with these theoretical predictions.

The aim of this paper is to use a more fundamental description to investigate the limitations of the NS equations in the planar Poiseuille flow.[8] To that end, we solve the Boltzmann equation[6,9,10] for Maxwell molecules [i.e., particles interacting via a potential $\phi(r)=K/r^4$] through second order in the external field. The results agree with those obtained from the BGK model,[5] except for changes in the numerical coefficients. We have chosen Maxwell molecules because they lend themselves to a more detailed analysis in the context of the Boltzmann equation.[10,11] Nevertheless, the simulation results of Ref. 7 show that the effects we want to analyze are not an artifact of the Maxwell interaction.

II. BOLTZMANN EQUATION

In a dilute gas, all the relevant physical information is contained in the one-particle velocity distribution function $f(\mathbf{r},\mathbf{v},t)$. The hydrodynamic quantities and their fluxes can be expressed in terms of moments of f:

$$\rho \equiv mn = m \int d\mathbf{v}f, \tag{2.1}$$

$$n\mathbf{u} = \int d\mathbf{v}\mathbf{v}f, \tag{2.2}$$

$$\frac{3}{2}nk_B T = \frac{m}{2}\int d\mathbf{v}V^2 f, \tag{2.3}$$

$$\mathsf{P} = m\int d\mathbf{v}\mathbf{V}\mathbf{V}f, \tag{2.4}$$

$$\mathbf{q} = \frac{m}{2}\int d\mathbf{v}V^2\mathbf{V}f. \tag{2.5}$$

In these equations, m is the mass of a particle, n is the number density, k_B is the Boltzmann constant, and $\mathbf{V} \equiv \mathbf{v} - \mathbf{u}$ is the peculiar velocity. The equation of state is that of an ideal gas, namely $p=nk_B T$. The time evolution of f is governed by the nonlinear Boltzmann equation,[6,9,10] which in standard notation reads

$$\frac{\partial}{\partial t}f + \mathbf{v}\cdot\nabla f + \frac{\mathbf{F}}{m}\cdot\frac{\partial}{\partial\mathbf{v}}f = \int d\mathbf{v}_1 \int d\Omega|\mathbf{v}-\mathbf{v}_1|$$
$$\times \sigma(|\mathbf{v}-\mathbf{v}_1|, \cos\theta)[f'f'_1 - ff_1]$$
$$\equiv J[f,f], \tag{2.6}$$

where $\mathbf{F}$ is an external force. The influence of the interaction potential $\phi(r)$ appears through the dependence of the cross section σ on the relative velocity $|\mathbf{v}-\mathbf{v}_1|$ and the scattering angle θ.

In the steady planar Poiseuille flow the distribution function is expected to depend on space only through the coordinate y, i.e., $f=f(y,\mathbf{v})$. In that case, the Boltzmann equation, Eq. (2.6), becomes

$$v_y\frac{\partial}{\partial y}f + g\frac{\partial}{\partial v_x}f = J[f,f], \tag{2.7}$$

that must be complemented by the adequate boundary conditions at $y = \pm H$. The balance equations (1.1) and (1.2) can be easily obtained from Eq. (2.7). They are just the first few members of the infinite hierarchy of equations for the velocity moments of f. Let us introduce the moments

$$M_{k_1,k_2,k_3}(y) = \int d\mathbf{v}V_x^{k_1}V_y^{k_2}V_z^{k_3}f(y,\mathbf{v}). \tag{2.8}$$

From Eq. (2.7) one gets

$$\frac{\partial}{\partial y}M_{k_1,k_2+1,k_3} + \frac{\partial u_x}{\partial y}k_1 M_{k_1-1,k_2+1,k_3} - gk_1 M_{k_1-1,k_2,k_3}$$
$$= J_{k_1,k_2,k_3}, \tag{2.9}$$

where

$$J_{k_1,k_2,k_3}(y) = \int d\mathbf{v}V_x^{k_1}V_y^{k_2}V_z^{k_3}J[f,f]. \tag{2.10}$$

In the sequel, we will use the roman boldface $\mathbf{k}$ to denote the triad $\{k_1,k_2,k_3\}$ and the italic lightface k to denote the sum $k_1+k_2+k_3$. Thus $M_k \equiv M_{k_1,k_2,k_3}$ is a moment of order $k \equiv k_1+k_2+k_3$. Because of the symmetry properties of the problem, M_{k_1,k_2,k_3} is an even (odd) function of y if k_2 = even (odd). Seen as a function of g, M_{k_1,k_2,k_3} is even (odd) if k_1 = even (odd). Finally, $M_{k_1,k_2,k_3} = 0$ if k_3 = odd.

In the special case of Maxwell molecules, $\phi(r)=K/r^4$, the collision rate is independent of the velocity, i.e.,

$|\mathbf{v}-\mathbf{v}_1|\sigma(|\mathbf{v}-\mathbf{v}_1|,\cos\theta)=\sigma_0(\cos\theta)$, and $J_{\mathbf{k}}$ can be expressed as a bilinear combination of moments of order equal to or smaller than k:[10]

$$J_{\mathbf{k}}=\sum_{\mathbf{k}',\mathbf{k}''}{}^{\dagger}C^{\mathbf{k}}_{\mathbf{k}',\mathbf{k}''}M_{\mathbf{k}'}M_{\mathbf{k}''}, \qquad (2.11)$$

where the dagger denotes the constraint $k'+k''=k$. The coefficients $C^{\mathbf{k}}_{\mathbf{k}',\mathbf{k}''}$ are linear combinations of the eigenvalues[10,12]

$$\lambda_{r\ell}=\int d\Omega\,\sigma_0(\cos\theta)\left[1+\delta_{r0}\delta_{\ell0}-\cos^{2r+\ell}\frac{\theta}{2}\right.$$
$$\left.\times P_\ell\left(\cos\frac{\theta}{2}\right)-\sin^{2r+\ell}\frac{\theta}{2}\,P_l\left(\sin\frac{\theta}{2}\right)\right] \qquad (2.12)$$

of the linearized collision operator. The explicit expressions of $J_{\mathbf{k}}$ through order $k=5$ are given in Ref. 11. The thermal conductivity and shear viscosity for Maxwell molecules are[9]

$$\kappa(T)=\frac{5k_B}{2m}\frac{p}{n\lambda_{11}}, \qquad (2.13a)$$

$$\eta(T)=\frac{p}{n\lambda_{02}}, \qquad (2.13b)$$

where $\lambda_{02}=\frac{3}{2}\lambda_{11}=0.436\times3\,\pi\sqrt{2K/m}$.

Let us define an effective collision frequency as

$$\nu=n\lambda_{02} \qquad (2.14)$$

and introduce the scaled space variable

$$s(y)=(k_BT_0/m)^{-1/2}\int_0^y dy'\,\nu(y'), \qquad (2.15)$$

where the subscript 0 refers to quantities at $y=0$. The variable s essentially measures distance in units of a local mean free path. Thus the value $s(H)$ represents an inverse Knudsen number. It is convenient to introduce other dimensionless quantities:

$$g^*=(k_BT_0/m)^{-1/2}\nu_0^{-1}g, \qquad (2.16)$$

$$u_x^*(s)=(k_BT_0/m)^{-1/2}u_x(y), \qquad (2.17)$$

$$T^*(s)=T_0^{-1}T(y), \qquad (2.18)$$

$$\mathsf{P}^*(s)=p_0^{-1}\mathsf{P}(y), \qquad (2.19)$$

$$M_{\mathbf{k}}^*(s)=n_0^{-1}(k_BT_0/m)^{-k/2}M_{\mathbf{k}}(y), \qquad (2.20)$$

$$J_{\mathbf{k}}^*(s)=n_0^{-1}(k_BT_0/m)^{-k/2}\frac{1}{\nu(y)}J_{\mathbf{k}}(y). \qquad (2.21)$$

A compelling argument in favor of using s rather than y as space variable is that the profiles can have a simpler form in terms of the former, as happens, for instance, in the case of the Couette flow.[13] In the case of the Poiseuille flow, Eq. (1.1a) can be integrated to give the simple result $P_{xy}^*=g^*s$ exactly. Of course, once the density profile is known, one can go back to the actual variable y by means of Eq. (2.15). In terms of dimensionless quantities, Eq. (2.9) can be rewritten as

$$\frac{\partial}{\partial s}M^*_{k_1,k_2+1,k_3}+\frac{\partial u_x^*}{\partial s}k_1M^*_{k_1-1,k_2+1,k_3}$$
$$-g^*\frac{T^*}{p^*}k_1M^*_{k_1-1,k_2k_3}=J^*_{k_1,k_2,k_3}, \qquad (2.22)$$

where $M^*_{000}=p^*/T^*$, $M^*_{100}=M^*_{010}=0$, and $M^*_{200}+M^*_{020}+M^*_{002}=3p^*$.

The scaled field g^* can be expressed as the ratio $g^*=\ell_0/\sqrt{2}h_0$, where $\ell_0=(2k_BT_0/m)^{1/2}\eta_0/p_0$ is an effective mean free path and $h_0=(k_BT_0/m)/g$ represents the characteristic length over which the gravitational field would produce a velocity increment of the order of the thermal velocity on a free particle. The quantity g^* and the Knudsen number $\mathrm{Kn}=\ell_0/2H$ are the only parameters characterizing the nonequilibrium state. At a hydrodynamic level, it is common in gravity related problems to introduce the Froude number[1,8]

$$\mathrm{Fr}=\left(\frac{5k_BT_0}{4mgH}\right)^{1/2}. \qquad (2.23)$$

While $\mathrm{Fr}\sim(h_0/H)^{1/2}$ is a measure of the field strength on the scale of the system size, g^* measures the strength on the scale of the mean free path. The three parameters are related by $g^*=\frac{5}{2}\mathrm{Kn}/\mathrm{Fr}^2$.

The task of solving the infinite hierarchy (2.22) (plus the corresponding boundary conditions) for arbitrary values of Kn and g^* (or, equivalently, Fr) is an unsurmountable one. In order to get explicit results, we make here two assumptions. First, we assume that the system is sufficiently large, so that we can focus on the bulk region of the system, i.e., $-H+\delta<y<H-\delta$, where $\delta\sim\ell_0$ is the width of the boundary layers. The existence of such a region implies that the Knudsen number must be small enough, say $\mathrm{Kn}<0.1$. Our second assumption is that the external field is weak enough as to make $\mathrm{Fr}\gtrsim1$. Taking together both assumptions, we can restrict ourselves to small values of g^* and perform a perturbation expansion in powers of g^*, neglecting terms of third and higher order. Since this series expansion is possibly only asymptotic,[5] it is not useful from a practical point of view to consider those higher order terms.

III. PERTURBATION EXPANSION

Let us expand the moments in powers of g^*:

$$M_{\mathbf{k}}^*(s)=M_{\mathbf{k}}^{(0)}+M_{\mathbf{k}}^{(1)}(s)g^*+M_{\mathbf{k}}^{(2)}(s)g^{*2}+\mathcal{O}(g^{*3}), \qquad (3.1)$$

$$u_x^*(s)=u_x^{(1)}(s)g^*+\mathcal{O}(g^{*3}), \qquad (3.2)$$

$$p^*(s)=1+p^{(2)}(s)g^{*2}+\mathcal{O}(g^{*4}), \qquad (3.3)$$

$$T^*(s)=1+T^{(2)}(s)g^{*2}+\mathcal{O}(g^{*4}). \qquad (3.4)$$

In Eq. (3.1), $M_{\mathbf{k}}^{(0)}$ are the reduced moments at equilibrium, i.e., $M^{(0)}_{k_1,k_2,k_3}=(k_1-1)!!(k_2-1)!!(k_3-1)!!$ if $k_1,k_2,k_3=$ even, being zero otherwise. Because of our choice of units, $p^{(2)}(0)=T^{(2)}(0)=0$. We also choose a reference frame stationary with the fluid at $s=0$, so that $u_x^{(1)}(0)=0$. The corresponding expansion of the collisional moments is

$$J_{\mathbf{k}}^*(s)=J_{\mathbf{k}}^{(1)}(s)g^*+J_{\mathbf{k}}^{(2)}(s)g^{*2}+\mathcal{O}(g^{*3}) \qquad (3.5)$$

TABLE I. Coefficients $\mu_{\mathbf{k}}^{(1,\alpha)}$ as obtained from the Boltzmann equation (B) and the Bhatnagar–Gross–Krook model (BGK).

	$\mu_{\mathbf{k}}^{(1,1)}$		$\mu_{\mathbf{k}}^{(1,0)}$	
$\mathbf{k}$	B	BGK	B	BGK
100	–	–	0	0
110	1	1	–	–
102	–	–	$-\frac{1}{3}$	0
120	–	–	$-\frac{5}{3}$	-2
300	–	–	-1	0
112	1	1	–	–
130	3	3	–	–
310	3	3	–	–

where

$$J_{\mathbf{k}}^{(1)} = \sum_{\mathbf{k}',\mathbf{k}''} {}^{\dagger} \frac{C_{\mathbf{k}',\mathbf{k}''}^{\mathbf{k}}}{\lambda_{02}} (M_{\mathbf{k}'}^{(0)} M_{\mathbf{k}''}^{(1)} + M_{\mathbf{k}'}^{(1)} M_{\mathbf{k}''}^{(0)}), \qquad (3.6)$$

$$J_{\mathbf{k}}^{(2)} = \sum_{\mathbf{k}',\mathbf{k}''} {}^{\dagger} \frac{C_{\mathbf{k}',\mathbf{k}''}^{\mathbf{k}}}{\lambda_{02}} (M_{\mathbf{k}'}^{(0)} M_{\mathbf{k}''}^{(2)} + M_{\mathbf{k}'}^{(1)} M_{\mathbf{k}''}^{(1)} + M_{\mathbf{k}'}^{(2)} M_{\mathbf{k}''}^{(0)}). \qquad (3.7)$$

Insertion of the above expansions into the hierarchy (2.22) gives

$$\frac{\partial}{\partial s} M_{k_1,k_2+1,k_3}^{(1)} + \frac{\partial u_x^{(1)}}{\partial s} k_1 M_{k_1-1,k_2+1,k_3}^{(0)} - k_1 M_{k_1-1,k_2,k_3}^{(0)}$$

$$= J_{k_1,k_2,k_3}^{(1)}, \qquad (k_1 = \text{odd}), \qquad (3.8)$$

$$\frac{\partial}{\partial s} M_{k_1,k_2+1,k_3}^{(2)} + \frac{\partial u_x^{(1)}}{\partial s} k_1 M_{k_1-1,k_2+1,k_3}^{(1)} - k_1 M_{k_1-1,k_2,k_3}^{(1)}$$

$$= J_{k_1,k_2,k_3}^{(2)}, \qquad (k_1 = \text{even}). \qquad (3.9)$$

Let us first consider Eq. (3.8). Its structure and the solution of the BGK model[5] suggest that $M_{k_1,k_2,k_3}^{(1)}$ is a linear function of s:

$$M_{k_1,k_2,k_3}^{(1)} = \begin{cases} \mu_{k_1,k_2,k_3}^{(1,0)}, & k_2 = \text{even}, \\ \mu_{k_1,k_2,k_3}^{(1,1)} s, & k_2 = \text{odd}. \end{cases} \qquad (3.10)$$

By making $(k_1,k_2,k_3)=(1,0,0)$ in Eq. (3.8), we get $\mu_{110}^{(1,1)} = 1$, which is equivalent to the known result $P_{xy}^* = g^* s$. Next, we make $(k_1,k_2,k_3)=(1,1,0)$ to obtain $\partial u_x^{(1)}/\partial s = -M_{110}^{(1)}$, which gives $u_x^{(1)}(s) = -\frac{1}{2}s^2$. Thus Eq. (3.8) decouples into the following two hierarchies:

$$-k_1 M_{k_1-1,k_2+1,k_3}^{(0)} s = J_{k_1,k_2,k_3}^{(1)}, \qquad (k_1 = \text{odd}, \ k_2 = \text{odd}), \qquad (3.11)$$

$$\mu_{k_1,k_2+1,k_3}^{(1,1)} - k_1 M_{k_1-1,k_2,k_3}^{(0)} = J_{k_1,k_2,k_3}^{(1)},$$

$$(k_1 = \text{odd}, \ k_2 = \text{even}). \qquad (3.12)$$

Equation (3.11) can be recursively solved to obtain $\mu_{k_1,k_2,k_3}^{(1,1)}$. Once those coefficients are known, Eq. (3.12) al-

lows one to get $\mu_{k_1,k_2,k_3}^{(1,0)}$. The first few coefficients are given in Table I. For the sake of comparison, the values obtained from the BGK model[5] are also listed.

The second-order hierarchy, Eq. (3.9), is more difficult to handle. According to the analysis of Sec. I and the BGK solution,[5] $T^{(2)}$ must be a quartic function of s. In general, we assume

$$M_{k_1,k_2,k_3}^{(2)}$$

$$= \begin{cases} \mu_{k_1,k_2,k_3}^{(2,0)} + \mu_{k_1,k_2,k_3}^{(2,2)} s^2 + \mu_{k_1,k_2,k_3}^{(2,4)} s^4, & k_2 = \text{even}, \\ \mu_{k_1,k_2,k_3}^{(2,1)} s + \mu_{k_1,k_2,k_3}^{(2,3)} s^3, & k_2 = \text{odd}. \end{cases} \qquad (3.13)$$

Consequently, Eq. (3.9) decouples into

$$3\mu_{k_1,k_2+1,k_3}^{(2,3)} s^2 + \mu_{k_1,k_2+1,k_3}^{(2,1)} - k_1 \mu_{k_1-1,k_2+1,k_3}^{(1,1)} s^2$$

$$- k_1 \mu_{k_1-1,k_2,k_3}^{(1,0)} = J_{k_1,k_2,k_3}^{(2)}, \qquad (k_1 = \text{even}, \ k_2 = \text{even}), \qquad (3.14)$$

$$4\mu_{k_1,k_2+1,k_3}^{(2,4)} s^3 + 2\mu_{k_1,k_2+1,k_3}^{(2,2)} s - k_1 \mu_{k_1-1,k_2+1,k_3}^{(1,0)} s$$

$$- k_1 \mu_{k_1-1,k_2,k_3}^{(1,1)} s = J_{k_1,k_2,k_3}^{(2)}, \qquad (k_1 = \text{even}, \ k_2 = \text{odd}). \qquad (3.15)$$

The left-hand side of Eq. (3.14) is a polynomial of second degree, while the right-hand side is of fourth degree. Thus, the coefficient of the fourth-degree term on the right-hand side must vanish and this allows us to obtain $\mu_{k_1,k_2,k_3}^{(2,4)}$ recursively. Once those coefficients are known, Eq. (3.15) can be used to get $\mu_{k_1,k_2,k_3}^{(2,3)}$. In general, the solution scheme proceeds as follows:

$$\{\mu_k^{(2,4)}\} \xrightarrow{(3.15)} \{\mu_{k-1}^{(2,3)}\} \xrightarrow{(3.14)} \{\mu_{k-2}^{(2,2)}\}$$

$$\xrightarrow{(3.15)} \{\mu_{k-3}^{(2,1)}\} \xrightarrow{(3.14)} \{\mu_{k-4}^{(2,0)}\}, \qquad (3.16)$$

where $\{\mu_k^{(2,\alpha)}\}$ denotes the set $\{\mu_{k_1,k_2,k_3}^{(2,\alpha)}; \ k_1+k_2+k_3=k\}$. In order to determine $\{\mu_2^{(2,0)}\}$ we need to start from Eq. (3.14) with $k_1+k_2+k_3=6$. Since, to the best of our knowledge, the collisional moments of sixth order are not available in the literature, here we have used a recent evaluation of those moments.[14] The coefficients obtained by following the scheme (3.16) are listed in Table II. The blank entries correspond to coefficients not evaluated, since they are not needed for the complete determination of the hydrodynamic quantities and their fluxes. Comparison with the BGK model shows that the latter gives values for $\mu_{\mathbf{k}}^{(2,4)}$ that are $\frac{3}{2}$ times larger than the correct ones; this is a consequence of the incorrect Prandtl number given by the BGK model.[6] On the other hand, the BGK values of $\mu_{\mathbf{k}}^{(2,3)}$ are correct.

IV. SUMMARY AND DISCUSSION

From a physical point of view, the most important output of our analysis is the knowledge of the hydrodynamic quantities and their fluxes, to second order in the external field. Let us express them in real units. First, we notice that the relationship between s and y can be obtained from Eq. (2.15) and the first line of Table II with the result

TABLE II. Coefficients $\mu_{\mathbf{k}}^{(2,\alpha)}$ as obtained from the Boltzmann equation (B) and the Bhatnagar–Gross–Krook model (BGK).

k	$\mu_{\mathbf{k}}^{(2,4)}$ B	BGK	$\mu_{\mathbf{k}}^{(2,3)}$ B	BGK	$\mu_{\mathbf{k}}^{(2,2)}$ B	BGK	$\mu_{\mathbf{k}}^{(2,1)}$ B	BGK	$\mu_{\mathbf{k}}^{(2,0)}$ B	BGK
000	$\frac{1}{45}$	$\frac{1}{30}$	–	–	0.1847	$\frac{11}{25}$	–	–	0	0
010	–	–	0	0	–	–	0	0	–	–
002	0	0	–	–	$\frac{4}{5}$	$\frac{4}{5}$	–	–	-0.2175	$-\frac{22}{25}$
020	0	0	–	–	0	0	–	–	-6.2602	$-\frac{306}{25}$
200	0	0	–	–	$\frac{14}{5}$	$\frac{14}{5}$	–	–	6.4777	$\frac{328}{25}$
012	–	–	$\frac{2}{15}$	$\frac{2}{15}$	–	–	0.2175	$\frac{22}{25}$	–	–
030	–	–	$\frac{2}{5}$	$\frac{2}{5}$	–	–	6.2602	$\frac{306}{25}$	–	–
210	–	–	$\frac{2}{15}$	$\frac{2}{15}$	–	–	-6.4777	$-\frac{328}{25}$	–	–
004	$-\frac{1}{15}$	$-\frac{1}{10}$	–	–	3.7166	$\frac{87}{25}$	–	–		$-\frac{132}{25}$
022	$-\frac{1}{45}$	$-\frac{1}{30}$	–	–	-0.1631	$-\frac{11}{25}$	–	–		$-\frac{492}{25}$
040	$-\frac{1}{15}$	$-\frac{1}{10}$	–	–	-4.6952	$-\frac{153}{25}$	–	–		$-\frac{852}{5}$
202	$-\frac{1}{45}$	$-\frac{1}{30}$	–	–	3.3226	$\frac{79}{25}$	–	–		$\frac{306}{25}$
220	$-\frac{1}{45}$	$-\frac{1}{30}$	–	–	4.1916	$\frac{139}{25}$	–	–		$\frac{1358}{25}$
400	$-\frac{1}{15}$	$-\frac{1}{10}$	–	–	16.2190	$\frac{387}{25}$	–	–		$\frac{1968}{25}$
014	–	–	$\frac{4}{5}$	$\frac{4}{5}$	–	–		$\frac{132}{25}$	–	–
032	–	–	$\frac{4}{5}$	$\frac{4}{5}$	–	–		$\frac{492}{25}$	–	–
050	–	–	4	4	–	–		$\frac{852}{5}$	–	–
212	–	–	$\frac{4}{15}$	$\frac{4}{15}$	–	–		$-\frac{306}{25}$	–	–
230	–	–	$\frac{4}{5}$	$\frac{4}{5}$	–	–		$-\frac{1458}{25}$	–	–
410	–	–	$\frac{4}{5}$	$\frac{4}{5}$	–	–		$-\frac{1968}{25}$	–	–
006	$-\frac{2}{3}$	-1	–	–		$\frac{114}{5}$	–	–		$-\frac{198}{5}$
024	$-\frac{2}{15}$	$-\frac{1}{5}$	–	–		$-\frac{66}{25}$	–	–		$-\frac{2034}{25}$
042	$-\frac{2}{15}$	$-\frac{1}{5}$	–	–		$-\frac{246}{25}$	–	–		$-\frac{1158}{5}$
060	$-\frac{2}{3}$	-1	–	–		$-\frac{426}{5}$	–	–		$-\frac{11466}{5}$
204	$-\frac{2}{15}$	$-\frac{1}{5}$	–	–		$\frac{264}{25}$	–	–		$\frac{852}{25}$
222	$-\frac{2}{45}$	$-\frac{1}{15}$	–	–		$\frac{128}{25}$	–	–		$\frac{1172}{25}$
240	$-\frac{2}{15}$	$-\frac{1}{5}$	–	–		$\frac{504}{25}$	–	–		$\frac{1572}{5}$
402	$-\frac{2}{15}$	$-\frac{1}{5}$	–	–		$\frac{414}{25}$	–	–		$\frac{1902}{25}$
420	$-\frac{2}{15}$	$-\frac{1}{5}$	–	–		$\frac{834}{25}$	–	–		$\frac{9066}{25}$
600	$-\frac{2}{3}$	-1	–	–		$\frac{564}{5}$	–	–		$\frac{2952}{5}$

$$s(y)=y^*+\left(\frac{1}{225}\,y^{*5}+\frac{0.1847}{3}\,y^{*3}\right)g^{*2}+\mathcal{O}(g^{*4}),$$
(4.1)

where $y^*\equiv y\nu_0/(k_BT_0/m)^{1/2}=\sqrt{2}\,y/l_0$. The profiles of the hydrodynamic quantities are

$$u_x(y)=-\frac{\rho_0 g y^2}{2\eta_0}+\mathcal{O}(g^3),$$
(4.2a)

$$p(y)=p_0\left[1+\frac{6}{5}\left(\frac{mgy}{k_BT_0}\right)^2\right]+\mathcal{O}(g^4),$$
(4.2b)

$$T(y)=T_0\left[1-\frac{\rho_0^2 g^2 y^4}{12\eta_0\kappa_0 T_0}+1.0153\left(\frac{mgy}{k_BT_0}\right)^2\right]+\mathcal{O}(g^4).$$
(4.2c)

The nonzero elements of the pressure tensor are

$$P_{xx}(y)=p_0\left[1+\frac{14}{5}\left(\frac{mgy}{k_BT_0}\right)^2+6.4777\,\frac{m\eta_0^2 g^2}{k_BT_0 p_0^2}\right]+\mathcal{O}(g^4),$$
(4.3a)

$$P_{yy}(y)=p_0\left(1-6.2602\,\frac{m\eta_0^2 g^2}{k_BT_0 p_0^2}\right)+\mathcal{O}(g^4),$$
(4.3b)

$$P_{zz}(y)=p_0\left[1+\frac{4}{5}\left(\frac{mgy}{k_BT_0}\right)^2-0.2175\,\frac{m\eta_0^2 g^2}{k_BT_0 p_0^2}\right]+\mathcal{O}(g^4),$$
(4.3c)

$$P_{xy}(y) = \rho_0 g y \left[1 + \frac{\rho_0^2 g^2 y^4}{60 \eta_0 \kappa_0 T_0} + \frac{0.1847}{3} \left(\frac{mgy}{k_B T_0} \right)^2 \right]$$
$$+ \mathcal{O}(g^5). \tag{4.3d}$$

The fact that $P_{xy}^* = g^* s$ exactly has allowed us to determine P_{xy} through fourth order in the field. Finally, the nonzero components of the heat flux vector are

$$q_x = -\frac{3}{2} \eta_0 g + \mathcal{O}(g^3), \tag{4.4a}$$

$$q_y = \frac{\rho_0^2 g^2 y^3}{3 \eta_0} + \mathcal{O}(g^4). \tag{4.4b}$$

Comparison with the NS constitutive relations, Eqs. (1.3) and (1.4), shows that the latter are not self-consistent to second order in the field, with the exception of Eq. (1.3b). The most important qualitative difference from the NS profiles arises in the case of the temperature. Although Eq. (1.10) is correct, Eq. (1.9) must be replaced by $(\partial^2 T / \partial y^2)|_{y=0} = 2.0306 g^2 \rho_0^2 T_0 / p_0^2$. Consequently, $T(y)$ has a local minimum at $y = 0$ surrounded by two maxima (T_{max}) at $y = \pm \sqrt{6.092 \eta_0 \kappa_0 T_0} / p_0 \simeq \pm 3.4 l_0$. The relative difference between the maxima and the minimum is of the same order as the temperature variations over a few mean free paths, namely $(T_{max} - T_0)/T_0 \simeq 11.60 g^{*2}$. The ratio third to second term in Eq. (4.2c) is of order $(l_0/y)^2$. This means that the correction to the NS temperature profile is only noticeable for distances from the middle layer of the order of or smaller than the mean free path. In the continuum limit (Kn→0), this represents a vanishing portion of the bulk region. On the other hand, if H comprises a decade or so of mean free paths the deviations from the NS profiles are relevant over most of the bulk domain.

It is worthwhile to point out that Eq. (1.4a) fails even to first order. The presence of a longitudinal component of the heat flux is indeed a Burnett (B) order effect:[9,10]

$$q_x^B = \theta_3 \frac{\eta^2}{2\rho p} \frac{\partial u_x}{\partial y} \frac{\partial p}{\partial y} + \theta_5 \frac{3 \eta^2}{2\rho T} \frac{\partial u_x}{\partial y} \frac{\partial T}{\partial y} + \theta_4 \frac{\eta^2}{2\rho} \frac{\partial^2 u_x}{\partial y^2}, \tag{4.5}$$

where, for Maxwell molecules, $\theta_3 = -3$, $\theta_4 = 3$, and $\theta_5 = 39/4$. While the first two terms are of order g^3, the last term is of first order and agrees with Eq. (4.4a). Nevertheless, the discrepancies between Eqs. (1.3), (1.4) and Eqs. (4.3), (4.4) go well beyond the Burnett order. On the one hand, P_{ii}^B contains terms of the form $\partial^2 T / \partial y^2$ (thermal transpiration), $\partial^2 p / \partial y^2$, and $(\partial u_x / \partial y)^2$, which are of order g^2, plus terms of the form $(\partial T / \partial y)^2$, $(\partial p / \partial y)^2$, and $(\partial T / \partial y) \times (\partial p / \partial y)$, which are of order g^4. On the other hand, contributions of order g^2 to q_y and P_{ii} also come from the super-Burnett term $\partial^3 T / \partial y^3$ and the super-super-Burnett term $\partial^4 T / \partial y^4$, respectively. In general, in order to obtain the fluxes through order g^{2n} one would need to consider the Chapman–Enskog expansion[9] through order $2(n+1)$ in the gradients; but this also would provide many extra terms of order higher than g^{2n}, that should be discarded.

The Chapman–Enskog expansion, of which the NS equations are the first approximation, is formally different from the perturbation expansion performed in this paper. The former is an expansion in the gradients, while the latter is an expansion in the field strength. Although it is the field what induces the gradients, both expansions are not equivalent and have different domains of applicability. The Chapman–Enskog expansion holds if the hydrodynamic quantities are practically constant over distances of the order of the mean free path l_0, what implies small values of a uniformity parameter ϵ and, in addition,[4] $g^* \sim \epsilon^2$. On the other hand, the series expansion in powers of the field is useful even for significant changes over distances close to l_0, i.e., $\epsilon \sim 1$, provided that the scaled field g^* is small. Therefore, it is important to emphasize that the results obtained in this paper extend but do not invalidate the NS description of the Poiseuille flow in its expected range of applicability. This contrasts with recent studies[15] demonstrating the invalidity of the NS system in the continuum limit when the boundaries have a nonuniform temperature distribution.

As said in Sec. I, recent Monte Carlo simulations[7] prove the correctness of the pressure and temperature profiles predicted by our perturbation analysis. In their simulations of the Boltzmann equation for hard spheres, Malek Mansour et al. considered the cases (a) Kn≃0.1, Fr≃2.2 ($g^* \simeq 0.05$), (b) Kn≃0.05, Fr≃3.1 ($g^* \simeq 0.01$), and (c) Kn ≃0.025, Fr≃4.3 ($g^* \simeq 0.003$). They compared the simulation results with the BGK predictions,[5] which coincide with Eqs. (4.2) except that the coefficient 1.0153 in Eq. (4.2c) is replaced by 19/25. In case (c) the external field was so weak that the temperature profile was sufficiently well described by the NS equations. This contrasts with what happened in cases (b) and (c), where a ''dimple'' in the temperature profile was clearly seen, in agreement with the kinetic theory prediction. In case (b) the quantitative agreement was hindered by the fact that the series expansion in powers of g^* is only asymptotic[5] and $g^* \simeq 0.05$ is not small enough to neglect terms of order $\mathcal{O}(g^{*4})$. On the other hand, the agreement in case (b) was very good, even at a quantitative level.

In summary, the planar Poiseuille flow induced by an external force provides a nice and simple example to illustrate the limitations of a purely hydrodynamic (Navier–Stokes) description and the need of kinetic theory methods whenever the hydrodynamic quantities change on the mean-free-path scale. Last but not least, the problem addressed here confirms the usefulness of the BGK kinetic model to pinpoint the correct qualitative behavior exhibited by the much more complicated Boltzmann equation.

ACKNOWLEDGMENTS

[1]D. J. Tritton, *Physical Fluid Dynamics* (Oxford University Press, Oxford, 1988); R. B. Bird, W. E. Stewart, and E. W. Lightfoot, *Transport Phenomena* (Wiley, New York, 1960); H. Lamb, *Hydrodynamics* (Dover, New York, 1945).
[2]L. P. Kadanoff, G. R. McNamara, and G. Zanetti, ''A Poiseuille viscometer for lattice gas automata,'' Complex Syst. **1**, 791 (1987); ''From au-

Phys. Fluids, Vol. 10, No. 4, April 1998

Tij, Sabbane, and Santos   1027

tomata to fluid flow: Comparison of simulation and theory,'' Phys. Rev. A **40**, 4527 (1989).

[3]M. Alaoui and A. Santos, ''Poiseuille flow driven by an external force,'' Phys. Fluids A **4**, 1273 (1992).

[4]R. Esposito, J. L. Lebowitz, and R. Marra, ''A hydrodynamic limit of the stationary Boltzmann equation in a slab,'' Commun. Math. Phys. **160**, 49 (1994).

[5]M. Tij and A. Santos, ''Perturbation analysis of a stationary nonequilibrium flow generated by an external force,'' J. Stat. Phys. **76**, 1399 (1994).

[6]C. Cercignani, *The Boltzmann Equation and Its Applications* (Springer-Verlag, New York, 1988).

[7]M. Malek Mansour, F. Baras, and A. L. Garcia, ''On the validity of hydrodynamics in plane Poiseuille flows,'' Physica A **240**, 255 (1997).

[8]For a similar analysis but for the Fourier flow, see M. Tij, V. Garzó, and A. Santos, ''Nonlinear heat transport in a dilute gas in the presence of gravitation,'' Phys. Rev. E **56**, 6729 (1997).

[9]S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, 1970).

[10]J. A. McLennan, *Introduction to Nonequilibrium Statistical Mechanics* (Prentice Hall, Englewood Cliffs, 1989).

[11]C. Truesdell and R. G. Muncaster, *Fundamentals of Maxwell's Kinetic Theory of a Simple Monatomic Gas* (Academic, New York, 1980).

[12]Z. Alterman, K. Frankowski, and C. L. Pekeris, ''Eigenvalues and eigenfunctions of the linearized Boltzmann collision operator for a Maxwell gas and for a gas of rigid spheres,'' Astrophys. J. **7**, 291 (1962).

[13]M. Tij and A. Santos, ''Combined heat and momentum transport in a dilute gas,'' Phys. Fluids **7**, 2858 (1995).

[14]M. Tij and M. Sabbane (unpublished).

[15]Y. Sone, K. Aoki, S. Takata, H. Sugimoto, and A. V. Bobylev, ''Inappropriateness of the heat-conduction equation for description of a temperature field of a stationary gas in the continuum limit: Examination by asymptotic analysis and numerical computation of the Boltzmann equation,'' Phys. Fluids **8**, 628 (1996); Y. Sone, S. Takata, and H. Sugimoto, ''The behavior of a gas in the continuum limit in the light of kinetic theory: The case of cylindrical Couette flows with evaporation and condensation,'' Phys. Fluids **8**, 3403 (1996).