Numerical study of the influence of gravity on the heat conductivity on the basis of kinetic theory

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The Boltzmann–Krook–Welander (or Bhatnagar–Gross–Krook) model of the Boltzmann equation is solved numerically for the heat transfer problem of a gas enclosed between two parallel, infinite plates kept at different temperatures, in the presence of a constant gravity field normal to the plates. At each point where the direct effect of the boundaries is negligible, a relation among the relevant local quantities (heat flux, temperature gradient, temperature, and density) holds even if the temperature varies over a length scale comparable to the mean free path. The ratio of the actual heat flux to the value predicted by the Fourier law is seen to be determined by the local Knudsen number and the local Froude number which are defined with the local mean free path, local characteristic length, and the magnitude of gravity. It is observed that the gravity produces an enhancement of the effective heat conductivity when the heat flux and the gravity field are parallel, while it produces an inhibition when both vectors are antiparallel. This deviation from the Fourier law, which vanishes in the absence of gravity, increases as the local Knudsen number increases and is more remarkable when the heat flux is parallel to the gravity field rather than otherwise. Comparison of the numerical data with an asymptotic analysis as well as with Padé approximants derived from it is also made. [S1070-6631(99)03211-0]

I. INTRODUCTION

The relation between the system of the Boltzmann equation and the one of classical fluid dynamics in describing the behavior of gas flows has been extensively studied by many physicists and mathematicians. The study has been developed by means of tools such as the Hilbert expansion,[1,2] the Chapman–Enskog expansion,[3] and the asymptotic theory (Ref. 4 for initial-value problems and Refs. 5–13 for boundary-value problems). In Refs. 7–13, the set of the fluid dynamic-type equations and its appropriate boundary conditions describing the steady behavior of slightly rarefied gas flows over smooth boundaries has been derived from the Boltzmann system (generalized slip flow theory). Recently, especially in Ref. 12, it has been clarified that phenomena which cannot be understood without the aid of the rarefaction of the gas occur in the continuum limit (ghost effect) and that the classical gas dynamics contains an essential defect.[14] On the other hand, another branch of studies has also been continued.[15–20] In these studies, where absence of boundary effects and one-dimensional situations are considered, the dependence of the heat conductivity of the gas on the local Knudsen number (the local mean free path of the gas molecules divided by a local characteristic length) is investigated when the latter is finite. When the local Knudsen number is small, the heat conductivity can be derived from the Boltzmann equation with the aid of the Hilbert expansion, etc., and the result is well-known. Asmolov *et al.*[15] and Santos *et al.*,[16,17] however, have shown that this relation also holds exactly even when the local Knudsen number is finite. Further, the study has been recently extended to situations where gravity is present and its effect on the heat conductivity has been investigated by means of perturbation expansions.[18–20] The results obtained, however, seem to indicate that the series are divergent, so that they do not seem to be applicable to situations of finite Knudsen numbers. It is worth pointing out that, from a physical point of view, a *uniform* gravity field is equivalent to a non-inertial frame of reference subject to a *constant* acceleration. The study of the frame dependence of the transport properties of a gas has been the subject of a number of papers.[21]

The goal of this paper is, by analyzing numerically a one-dimensional heat transfer problem in the presence of a gravity field on the basis of kinetic theory, to investigate the relation holding among the heat flux, the temperature gradient, the temperature, and the density of the gas, and to clarify the effect of gravity on it when the local Knudsen number is finite. To be more precise, we will analyze the steady behavior of the gas when the gas flow is absent, evaluate the ratio of the heat flux and the local temperature gradient at each point in the gas, and investigate the dependence of it on the local Knudsen number and the strength of gravity. In the

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actual numerical analysis, we will employ the Boltzmann–Krook–Welander (BKW) equation,[22] also known as the Bhatnagar–Gross–Krook (BGK) equation, in accordance with the analysis of Ref. 20. The physical problem is described in Sec. II and the numerical method used to solve it is presented in Sec. III. Section IV is devoted to a brief summary of the theoretical analysis carried out in Ref. 20. In order to improve the range of applicability of the asymptotic result, Padé approximants are considered. The numerical results and their comparison with the theoretical estimates are shown in Sec. V. The paper ends with some concluding remarks.

II. PROBLEM AND BASIC EQUATIONS

A. Setting of the problem

In order to investigate the problem stated in the previous section, let us consider a gas enclosed between two parallel plane surfaces placed at rest at $X_1=0$ and $X_1=D$ (X_i: space rectangular coordinate system), whose surface temperatures are T_0 and T_1, respectively. A uniform gravity field $(g,0,0)$ is exerted on the gas. We will analyze the steady behavior of the gas in the absence of gas flow and obtain the heat flux and the local temperature gradient of the gas. This problem contains the boundaries, which are introduced in order to produce a temperature gradient in the gas, so that we restrict ourselves to the bulk region, i.e., the region where the direct effect of the boundaries is negligible. According to Refs. 13, 23, and 24, the effect of the molecules directly coming from the boundaries is sufficiently small at points separated from the walls about 10–15 times the mean free path. At each point within this region, we evaluate the heat flux $(q,0,0)$ and the local temperature gradient $G\equiv dT/dX_1$ (T: gas temperature) and consider the ratio $-q/G$. Now, let the length scale along which the gas temperature varies appreciably be the local characteristic length L:

$$L=T/|G|. \tag{1}$$

Further, we introduce the local Knudsen number $\tilde{k}$ and the local inverse Froude number $\tilde{g}$ with the aid of this L as their characteristic length:

$$\tilde{k}=\frac{\sqrt{\pi}}{2}\frac{l}{L}, \tag{2}$$

$$\tilde{g}=\frac{gL}{2RT}, \tag{3}$$

where l is the local mean free path of the gas molecules and R is the specific gas constant. Without loss of generality, we will take the convention $T_1<T_0$, i.e., $G<0$ and $q>0$. On the other hand, g (and hence $\tilde{g}$) can be either positive or negative. In the first case, the heat flux is in the same direction as that of gravity, while $\tilde{g}<0$ means that the heat flux is in the opposite direction to that of gravity.

In the absence of gravity ($\tilde{g}=0$), according to the asymptotic theory for small Knudsen numbers, the ratio $-q/G$ is a function of the local temperature T:

$$-q/G=\lambda(T), \tag{4}$$

where λ is called the heat conductivity and its functional form is determined by the intermolecular potential. Asmolov *et al.*[15] (in the case of the Boltzmann equation for Maxwell molecules) and Santos *et al.*[16,17] (in the case of the BKW equation for an arbitrary interaction) showed, however, that the Fourier law (4) holds even for finite Knudsen number $\tilde{k}$. In the course of our present study, we will confirm this statement numerically. That is, we will examine from the numerical data that q, G, and T satisfy the relation (4) at each point in the bulk region of the gas. When the gravity is present, the relation (4) no longer holds.[18–20] We put instead

$$-q/G=\lambda(T)\tilde{\lambda}, \tag{5}$$

where the heat conductivity factor $\tilde{\lambda}$ measures the deviation of the actual heat flux from the one predicted by the Fourier law. The aims of this study are as follows: (i) We confirm that the factor $\tilde{\lambda}$ is determined by the local Knudsen number $\tilde{k}$ and the local inverse Froude number $\tilde{g}$; in other words, as far as the ratio $(-q/G)/\lambda(T)$ is concerned, it gives the same value when $\tilde{k}$ and $\tilde{g}$ are identical, even if q, G, and T are different. (ii) We clarify the functional relation $\tilde{\lambda}(\tilde{k},\tilde{g})$ for finite values of $\tilde{k}$ and $\tilde{g}$.

B. Basic equations

The BKW equation[22] for the present steady and one-dimensional problem in the presence of gravity can be written as

$$\xi_1\frac{\partial f}{\partial X_1}+g\frac{\partial f}{\partial \xi_1}=A_c\rho(f_e-f), \tag{6}$$

$$f_e=\frac{\rho}{(2\pi RT)^{3/2}}\exp\left(-\frac{(\xi_1-v)^2+\xi_2^2+\xi_3^2}{2RT}\right), \tag{7}$$

$$\rho=\int f\,d\xi_1 d\xi_2 d\xi_3, \tag{8}$$

$$v=\frac{1}{\rho}\int \xi_1 f\,d\xi_1 d\xi_2 d\xi_3, \tag{9}$$

$$T=\frac{1}{3R\rho}\int [(\xi_1-v)^2+\xi_2^2+\xi_3^2] f\,d\xi_1 d\xi_2 d\xi_3, \tag{10}$$

where ξ_i is the molecular velocity, f is the distribution function of the gas molecules, ρ is the mass density of the gas, $(v,0,0)$ is the flow velocity, A_c is a constant ($A_c\rho$ is the collision frequency of a gas molecule, assumed here to interact via the Maxwell potential), and the range of integration with respect to ξ_i is, hereafter, the whole space of ξ_i unless otherwise stated. The boundary conditions, diffuse reflection conditions, on the walls are

$$f=\frac{\rho_{w0}}{(2\pi RT_0)^{3/2}}\exp\left(-\frac{\xi_1^2+\xi_2^2+\xi_3^2}{2RT_0}\right)\quad (X_1=0,\ \xi_1>0), \tag{11}$$

$$\rho_{w0}=-\left(\frac{2\pi}{RT_0}\right)^{1/2}\int_{\xi_1<0}\xi_1 f\,d\xi_1 d\xi_2 d\xi_3\quad (X_1=0), \tag{12}$$

$$f=\frac{\rho_{w1}}{(2\pi RT_1)^{3/2}}\exp\left(-\frac{\xi_1^2+\xi_2^2+\xi_3^2}{2RT_1}\right)\quad (X_1=D,\ \ \xi_1<0), \tag{13}$$

$$\rho_{w1}=\left(\frac{2\pi}{RT_1}\right)^{1/2}\int_{\xi_1>0}\xi_1 f\, d\xi_1 d\xi_2 d\xi_3\quad (X_1=D). \tag{14}$$

Finally, we have to specify the amount of the gas or, equivalently, its mean density

$$\rho_0=\frac{1}{D}\int_0^D \rho\, dX_1 \tag{15}$$

to complete the problem.

The heat flux of the gas $(q,0,0)$ is defined by the moment of the distribution function as

$$q=\frac{1}{2}\int (\xi_1-v)[(\xi_1-v)^2+\xi_2^2+\xi_3^2]f\, d\xi_1 d\xi_2 d\xi_3. \tag{16}$$

Multiplying by 1 or $\xi_1^2+\xi_2^2+\xi_3^2$ both sides of Eq. (6), integrating with respect to ξ_i, and considering the boundary conditions (11) and (13), we get

$$v=0\quad \text{and}\quad q=\text{const.} \tag{17}$$

These relations will be used to test the accuracy of the numerical results. The local mean free path[25] l of the gas molecules and the heat conductivity λ of the gas without gravity for the BKW equation are

$$l=\frac{2}{\sqrt{\pi}}\frac{(2RT)^{1/2}}{A_c\rho}, \tag{18}$$

$$\lambda=\frac{5}{2}\frac{R^2T}{A_c}. \tag{19}$$

If we make some appropriate nondimensionalization, we find that the boundary-value problem posed by Eqs. (6)–(15) is characterized by the following three dimensionless parameters:

$$k_0\equiv\frac{(2RT_0)^{1/2}}{A_c\rho_0 D},\quad g_0\equiv\frac{gD}{2RT_0},\quad \frac{T_1}{T_0}. \tag{20}$$

The first parameter represents the reference Knudsen number, the second parameter represents the reference inverse Froude number, and the third parameter is simply the temperature ratio. We will give them a set of actual values and solve the problem. As stated before, we will evaluate $\tilde{k}$, $\tilde{g}$, and $\tilde{\lambda}\equiv(-q/G)/\lambda(T)$ at each point X_1 in the bulk region of the gas. From a lot of solutions for various sets of values of Eq. (20), we construct the relation among $\tilde{k}$, $\tilde{g}$, and $\tilde{\lambda}$ for a wide range of $\tilde{k}$ and $\tilde{g}$.

III. OUTLINE OF THE NUMERICAL ANALYSIS

In this section, we briefly summarize the method of the numerical analysis employed. According to Chu,[26] the boundary-value problem (6)–(15) can be reduced to one with two independent variables, X_1 and ξ_1, by eliminating the molecular velocity components ξ_2 and ξ_3 parallel to the walls. We solve this reduced problem numerically. First, the range of ξ_1, which is from $-\infty$ to $+\infty$, is restricted to some finite interval, the width of which is chosen appropriately depending on the parameters (20). The validity of this restriction is confirmed *a posteriori* from the numerical result, i.e., it is confirmed that the distribution function practically vanishes sufficiently near the edge of this interval. Next, we introduce (nonuniform) lattice points in the $X_1-\xi_1$ plane and define the distribution function only on these points. The derivatives $\partial/\partial X_1$ and $\partial/\partial\xi_1$ are approximated with a standard second-order finite difference formula. The finite difference equation thus obtained is solved numerically using an iterative method. [In the actual computation, the time derivative term is added to Eq. (6) and the initial-value problem with a uniform equilibrium initial state is solved to obtain the steady solution as the limiting one for very long time.] The convergence is rather fast unless k_0 is very small. The finite difference scheme for a one-dimensional problem without gravity is explained in detail in Ref. 27. In the present problem, although a derivative term with respect to ξ_1 is added, the difference scheme is obtained as a straightforward extension. Thus we will omit the explicit form of our scheme. Incidentally, the scheme used here is essentially the same as the one used to obtain one-dimensional solutions of the Bénard problem in Refs. 28 and 29.

In our problem, although the boundaries are plane in shape, an external force is exerted on the gas molecules and thus the characteristics of Eq. (6) are curved to be convex to the opposite direction to that of gravity. Because of this, a discontinuity of the distribution function develops from the upper wall into the gas[29] (here, ''upper wall'' means the boundary at $X_1=D$ when $g<0$ and the one at $X_1=0$ when $g>0$). Such discontinuity cannot be described correctly with a standard finite difference scheme and special techniques are required. Our aim is, however, to investigate the gas behavior in the region located a long distance, as compared with the mean free path, apart from the boundaries, where the discontinuity is estimated to vanish owing to intermolecular collisions. In addition, an inaccurate treatment of this discontinuity does not seem to disturb the behavior of the gas in that region. For these reasons, in our present study, we do not analyze this discontinuity accurately and we employ a conventional finite difference scheme. Incidentally, on the lower boundary the discontinuity does not exist and the distribution function is continuous even at the boundary (see footnote 29 of Ref. 29).

IV. SUMMARY OF THE THEORETICAL ANALYSIS

For the sake of completeness, in this section we give a brief summary of the results of the asymptotic analysis of Ref. 20. First, we take an *arbitrary* point $X_1=a$ in the bulk region and define the following dimensionless quantities:

$$\rho^*=\rho/\rho_a,\quad T^*=T/T_a,\quad \xi_i^*=\xi_i/(2RT_a)^{1/2},\quad f^*=\rho_a^{-1}(2RT_a)^{3/2}f, \tag{21}$$

$$x=(2RT_a)^{-1/2}A_c\int_a^{X_1}\rho\, dX_1', \tag{22}$$

where the subscript a denotes quantities evaluated at $X_1 = a$. Thus, Eq. (6) becomes

$$\xi_1^* \frac{\partial f^*}{\partial x} + \frac{\tilde{g}_a \tilde{k}_a}{\rho^*} \frac{\partial f^*}{\partial \xi_1^*} = f_e^* - f^*, \tag{23}$$

where $f_e^* = \rho_a^{-1}(2RT_a)^{3/2} f_e$. In the absence of gravity ($\tilde{g}_a = 0$), an exact solution of Eq. (23) yields[16]

$$T^* = \frac{1}{\rho^*} = 1 - \tilde{k}_a x, \quad \tilde{\lambda}_a = 1. \tag{24}$$

This implies that the temperature has a linear profile with respect to the nonlinear space variable x, the pressure is constant, and the heat conductivity is independent of the value of the local Knudsen number $\tilde{k}_a$, so that Eq. (4) is exactly verified. These predictions have been confirmed by comparison with numerical solutions.[30,31]

On the other hand, the extension of the above exact solution to the case $\tilde{g}_a \neq 0$ for finite $\tilde{k}_a$ and $\tilde{g}_a$ does not seem to be feasible. In Ref. 20, the solution of Eq. (23) was obtained in the form of a perturbation series expansion in powers of $\tilde{g}_a$ through order $\tilde{g}_a^6$. Here we quote the main results through third order:

$$\begin{aligned} T^* = {} & 1 - \tilde{k}_a x + \tilde{k}_a^2 x^2 \tilde{g}_a - \tfrac{4}{15}\tilde{k}_a^3 x^2 (5x + 99\tilde{k}_a)\tilde{g}_a^2 \\ & - \tfrac{2}{75}\tilde{k}_a^4 x^2 [317\,952\tilde{k}_a^2 - 3460\tilde{k}_a x - 15(5x^2 - 32)]\tilde{g}_a^3 \\ & + \mathcal{O}(\tilde{g}_a^4), \end{aligned} \tag{25}$$

$$\rho^* T^* = 1 + 2\tilde{k}_a x \tilde{g}_a + \tfrac{552}{5}\tilde{k}_a^5 x \tilde{g}_a^3 + \mathcal{O}(\tilde{g}_a^4), \tag{26}$$

$$\begin{aligned} \tilde{\lambda}_a = {} & 1 + \tfrac{58}{5}\tilde{k}_a^2 \tilde{g}_a + \tfrac{32}{25}\tilde{k}_a^2 (1499\tilde{k}_a^2 + 5)\tilde{g}_a^2 \\ & + \tfrac{8}{125}\tilde{k}_a^4 (21\,427\,694\tilde{k}_a^2 + 45\,985)\tilde{g}_a^3 + \mathcal{O}(\tilde{g}_a^4). \end{aligned} \tag{27}$$

Since $X_1 = a$ represents an arbitrary point, we can drop the subscript a in Eq. (27), so that it then refers to local quantities evaluated at any given point.

These results, and the more extensive ones of Ref. 20, indicate that the expansion in powers of $\tilde{g}$ seems to be asymptotic (i.e., it is not a convergent one), so that it will be only useful for small values of $|\tilde{g}|$. Here we will retain only the terms through order $\tilde{g}^2$. The corresponding truncated series is

$$\tilde{\lambda}_{[2,0]} = 1 + \tfrac{58}{5}\tilde{k}^2 \tilde{g} + \tfrac{32}{25}\tilde{k}^2 (1499\tilde{k}^2 + 5)\tilde{g}^2. \tag{28}$$

The validity of the above approximation is limited to those values of $\tilde{g}$ and $\tilde{k}$ for which the first neglected term (i.e., the third-order term) is much smaller than the last term in Eq. (28). This yields $\tilde{k}^2|\tilde{g}| \ll 20(1499\tilde{k}^2 + 5)/(21\,427\,694\tilde{k}^2 + 45\,985) \sim 10^{-3}$.

On the other hand, in order to enlarge the range of values of $\tilde{g}$ for which one can expect to get reliable results, it is useful to construct Padé approximants[32] based on Eq. (28). More specifically, we consider the Padé approximants [1,1] and [0,2]:

$$\tilde{\lambda}_{[1,1]}(\tilde{k}, \tilde{g}) = \frac{1 - \frac{16}{29}(1 + \frac{11\,151}{40}\tilde{k}^2)\tilde{g}}{1 - \frac{16}{29}(1 + \frac{1499}{5}\tilde{k}^2)\tilde{g}}, \tag{29}$$

$$\tilde{\lambda}_{[0,2]}(\tilde{k}, \tilde{g}) = [1 - \tfrac{58}{5}\tilde{k}^2 \tilde{g} - \tfrac{32}{5}(1 + \tfrac{11\,151}{40}\tilde{k}^2)\tilde{k}^2 \tilde{g}^2]^{-1}. \tag{30}$$

It must be kept in mind that these approximants are intended to yield reasonable estimates of $\tilde{\lambda}$ for small values of $|\tilde{g}|$ only, as they differ from the exact asymptotic series [Eq. (27)] to order $\tilde{g}^3$ and beyond. One could also construct higher-order approximants, but their expressions become progressively more complicated without any significant increase in their range of applicability, due to the asymptotic character of the series (27). Thus, as a compromise between simplicity and accuracy, we restrict ourselves to the above two approximants. There is no definite *a priori* criterion to choose between $\tilde{\lambda}_{[1,1]}$ and $\tilde{\lambda}_{[0,2]}$. From a practical point of view, comparison with the numerical results discussed in Sec. V shows that $\tilde{\lambda}_{[1,1]}$ is clearly superior to $\tilde{\lambda}_{[0,2]}$ for $\tilde{g} < 0$, while $\tilde{\lambda}_{[0,2]}$ is slightly better than $\tilde{\lambda}_{[1,1]}$ for $\tilde{g} > 0$. In view of this, we take

$$\tilde{\lambda}_{\text{est}}(\tilde{k}, \tilde{g}) = \begin{cases} \tilde{\lambda}_{[0,2]}(\tilde{k}, \tilde{g}), & (\tilde{g} > 0), \\ \tilde{\lambda}_{[1,1]}(\tilde{k}, \tilde{g}), & (\tilde{g} < 0) \end{cases} \tag{31}$$

as our best theoretical estimate, at least for the range $-0.6 \leqslant \tilde{g} \leqslant 0.4$, $\tilde{k} \lesssim 0.1$ considered in the numerical solutions.[33]

V. RESULTS OF THE NUMERICAL ANALYSIS

In this section, we show the results of the numerical analysis. As stated at the end of Sec. II, once a solution of Eqs. (6)–(15) is obtained, we evaluate the local Knudsen number $\tilde{k}$ [Eq. (2)], the local inverse Froude number $\tilde{g}$ [Eq. (3)], and the local ratio $\tilde{\lambda} \equiv (-q/G)/\lambda(T)$ [Eq. (5)] at some points in the gas where the direct effects from the boundaries are negligible. To be more precise, we employ the following criterion. The number of collisions that a molecule emitted from the boundary $X_1 = 0$ experiences while it proceeds to some distance $X_1 = \delta$ is estimated in the average sense as $\mathcal{N}_0(\delta) = \int_0^\delta l^{-1}\, dX_1$. If we consider a point $X_1 = \delta$ where this integral is sufficiently large, the direct effect of the boundary is considered to vanish owing to intermolecular collisions. In the present study, we consider that this direct effect vanishes where the value of $\mathcal{N}_0(\delta)$ is greater than 15 (for the cases of $g > 0$) or greater than 10 (for $g < 0$). This threshold value is chosen judging from the results. A similar criterion is adopted also on the opposite boundary ($X_1 = D$). For the convenience of the following discussions, we evaluate $\tilde{k}$ and $\tilde{\lambda}$ at which $\tilde{g}$ assumes a specified value with the aid of interpolation. The values of $\tilde{\lambda}$ thus obtained, which are taken from a lot of solutions of Eqs. (6)–(15), are shown as a function of $\tilde{k}$ for some values of $\tilde{g}$ in Table I and Fig. 1. We note that $\tilde{g}$ is taken as positive when the heat flux is in the same direction as that of gravity and negative when the heat flux is in the opposite direction to that of gravity. The theoretical results corresponding to the truncated series $\tilde{\lambda}_{[2,0]}$, Eq. (28), and to the estimate $\tilde{\lambda}_{\text{est}}$, Eqs. (29)–(31), are also shown in Table I and Fig. 1 for comparison.

As seen from Fig. 1, the data for any given value of $\tilde{g}$ lie on a smooth curve, although they are taken from various solutions of Eqs. (6)–(15). This suggests that, as predicted

TABLE I. The heat conductivity factor $\tilde{\lambda}$ as a function of $\tilde{k}$ and $\tilde{g}$. The theoretical approximations $\tilde{\lambda}_{[2,0]}$ and $\tilde{\lambda}_{\mathrm{est}}$ are given by Eq. (28) and Eqs. (29)–(31), respectively. The columns are common in this table.

$\tilde{k}$	$\tilde{\lambda}$	$\tilde{\lambda}_{[2,0]}$	$\tilde{\lambda}_{\mathrm{est}}$	$\tilde{k}$	$\tilde{\lambda}$	$\tilde{\lambda}_{[2,0]}$	$\tilde{\lambda}_{\mathrm{est}}$
			$\tilde{g}=0$				
0.02	1.0000	1.0000	1.0000	0.08	0.9998	1.0000	1.0000
0.04	0.9999	1.0000	1.0000	0.10	0.9998	1.0000	1.0000
0.06	0.9998	1.0000	1.0000	0.13	0.9998	1.0000	1.0000
			$\tilde{g}=0.05$				$\tilde{g}=-0.05$
0.029 64	1.0004	1.0005	1.0005	0.021 36	0.9997	0.9997	0.9997
0.047 13	1.0011	1.0013	1.0013	0.042 71	0.9990	0.9990	0.9990
0.059 87	1.0020	1.0022	1.0022	0.056 31	0.9981	0.9983	0.9983
0.067 14	1.0025	1.0028	1.0028	0.066 44	0.9975	0.9976	0.9976
0.074 38	1.0031	1.0034	1.0034	0.111 61	0.9934	0.9937	0.9936
			$\tilde{g}=0.1$				$\tilde{g}=-0.1$
0.029 50	1.0009	1.0011	1.0011	0.024 34	0.9993	0.9994	0.9994
0.041 04	1.0019	1.0021	1.0021	0.042 10	0.9981	0.9981	0.9981
0.060 03	1.0044	1.0047	1.0047	0.056 18	0.9965	0.9967	0.9967
0.072 58	1.0066	1.0070	1.0070	0.074 87	0.9941	0.9945	0.9943
0.078 74	1.0078	1.0083	1.0083	0.100 11	0.9900	0.9909	0.9905
			$\tilde{g}=0.2$				$\tilde{g}=-0.2$
0.029 79	1.0022	1.0023	1.0023	0.021 99	0.9990	0.9990	0.9990
0.039 91	1.0041	1.0043	1.0043	0.043 33	0.9963	0.9964	0.9963
0.055 11	1.0085	1.0085	1.0086	0.065 69	0.9920	0.9925	0.9920
0.069 06	1.0141	1.0140	1.0141	0.081 72	0.9878	0.9896	0.9884
0.077 61	1.0183	1.0183	1.0184	0.096 62	0.9836	0.9874	0.9847
			$\tilde{g}=0.4$				$\tilde{g}=-0.4$
0.022 54	1.0027	1.0030	1.0030	0.022 56	0.9983	0.9982	0.9981
0.032 41	1.0063	1.0063	1.0063	0.038 75	0.9949	0.9953	0.9947
0.041 18	1.0110	1.0105	1.0105	0.046 57	0.9927	0.9936	0.9926
0.048 64	1.0163	1.0151	1.0152	0.069 11	0.9850	0.9897	0.9856
				0.080 84	0.9801	0.9895	0.9817
			$\tilde{g}=-0.6$				
0.021 24	0.9981	0.9980	0.9977	0.054 59	0.9875	0.9923	0.9873
0.036 59	0.9942	0.9950	0.9936	0.067 71	0.9815	0.9932	0.9821
0.044 48	0.9914	0.9935	0.9910				

by the theoretical analysis of Sec. IV, $\tilde{\lambda}$ is determined only by the local values of $\tilde{k}$ and $\tilde{g}$ irrespective of the actual ones of q and G. To confirm this, we show in Table II a few examples of the explicit values of q, G, and T at which $\tilde{k}$ and $\tilde{\lambda}$ are evaluated from different solutions of Eqs. (6)–(15). For each pair of these cases, the values of $\tilde{\lambda}$ are obtained at those points in the gas at which the values of $\tilde{g}$ are identical and those of $\tilde{k}$ are as close as possible. Although the values of q, G, and T differ considerably, those of $\tilde{\lambda}$ are nearly identical if $\tilde{k}$ are very close. Thus it may be concluded that the factor $\tilde{\lambda}$ is a function of local values of $\tilde{k}$ and $\tilde{g}$.

Now we examine the features of the relation among $\tilde{k}$, $\tilde{g}$, and $\tilde{\lambda}$. When the gravity is absent ($\tilde{g}=0$), $\tilde{\lambda}$ is very close to unity irrespective of $\tilde{k}$. As we have mentioned in Sec. II, an exact solution for this situation was obtained by Santos *et al.*,[16,17] according to which $\tilde{\lambda}=1$ regardless the value of $\tilde{k}$

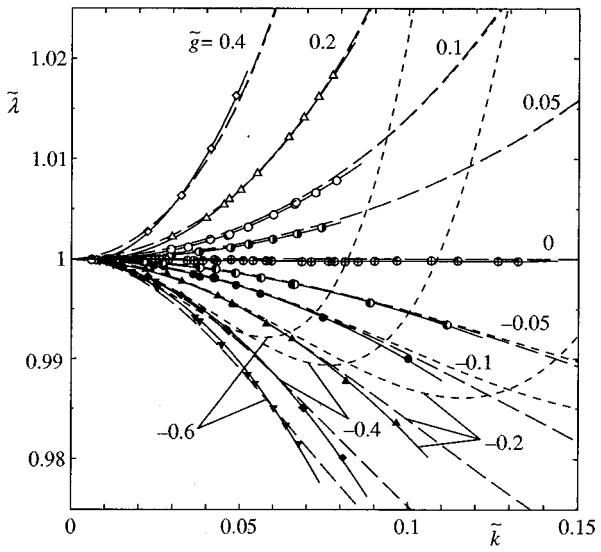

FIG. 1. Plot of the heat conductivity factor $\tilde{\lambda}$ vs $\tilde{k}$ for several values of $\tilde{g}$. The symbols are the numerical results, the solid lines represent the curves which are smoothly fitted to the data, the dotted lines represent the truncated series $\tilde{\lambda}_{[2,0]}$, Eq. (28), and the dashed lines represent the theoretical estimate $\tilde{\lambda}_{\mathrm{est}}$, Eqs. (29)–(31).

[see Eq. (24)]. The slight discrepancy (deviation from unity) is due to the unavoidable influence of the boundaries and to the error of the present numerical analysis. Thus we can estimate that our numerical results for $\tilde{\lambda}$ have an error not greater than 0.02%. When the gravity is present ($\tilde{g}\neq 0$), $\tilde{\lambda}$ varies depending both on $\tilde{k}$ and $\tilde{g}$. The heat conductivity factor $\tilde{\lambda}$ is greater than unity when $\tilde{g}>0$ (heat flux is in the same direction as that of gravity) and less than unity when $\tilde{g}<0$ (heat flux is in the opposite direction to that of gravity). In both cases, the deviation of $\tilde{\lambda}$ from unity increases monotonically as $\tilde{k}$ increases, this effect being more remarkable for $\tilde{g}>0$ than for $\tilde{g}<0$. In addition, for a given value of $\tilde{k}$ the deviation is larger if $|\tilde{g}|$ is larger. In obtaining numerically the relation among $\tilde{k}$, $\tilde{g}$, and $\tilde{\lambda}$, we have been forced to restrict ourselves to the ranges $\tilde{k}\lesssim 0.1$ and $-0.6\leqslant\tilde{g}\leqslant 0.4$. This is because for the two-surface heat transfer problem stated in Sec. II A, it is difficult for any proper set of values of Eq. (20) to make a point in the gas where the local Knud-

TABLE II. Examples of pairs of values of $\tilde{\lambda}$ for which $(\tilde{k},\tilde{g})$ are close. Here, $q_0=(\rho_0/2)(2RT_0)^{3/2}$ and $G_0=T_0/D$. All the cases correspond to $T_1/T_0=0.05$.

q/q_0	$-G/G_0$	T/T_0	$\tilde{g}$	$\tilde{k}$	$\tilde{\lambda}$
0.023 20	1.3333	0.3508	−0.15	0.073 92	0.9920
0.007 38	3.3333	0.1489	−0.15	0.073 44	0.9920
0.023 20	0.8000	0.5832	−0.25	0.046 87	0.9946
0.007 38	2.0000	0.2474	−0.25	0.046 56	0.9949
0.016 52	3.5000	0.1204	0.1	0.046 28	1.0023
0.020 02	1.0000	0.4466	0.1	0.046 77	1.0024
0.023 13	2.5000	0.1646	0.1	0.066 77	1.0056
0.023 61	2.0000	0.2100	0.1	0.066 34	1.0055

sen and inverse Froude numbers are large. This is especially so if $\tilde{g}>0$ and this is why we could not include the case $\tilde{g}=0.6$ in Table I and Fig. 1.

Next we compare the truncated series $\tilde{\lambda}_{[2,0]}$ [Eq. (28)] as well as the Padé approximant $\tilde{\lambda}_{\mathrm{est}}$ [Eqs. (29)–(31)] with the numerical result. First of all, it should be remembered that the truncated series $\tilde{\lambda}_{[2,0]}$, where the terms of order $\tilde{g}^3$ and higher are neglected, has been introduced as a formula to be valid for $\tilde{k}^2|\tilde{g}|\ll 10^{-3}$, as stated in Sec. IV. In addition, the Padé approximant $\tilde{\lambda}_{\mathrm{est}}$ is also valid in the same region because it is constructed to agree with the asymptotic series, Eq. (27), up to the order of $\tilde{g}^2$. It is seen in Fig. 1 and Table I that these two formulas really agree with the numerical result within this range. Besides this, it is also seen that whenever these two formulas coincide each other, they also agree with the numerical result. When $\tilde{g}>0$, $\tilde{\lambda}_{[2,0]}$ and $\tilde{\lambda}_{\mathrm{est}}$ coincide indistinguishably with each other and agree with the numerical result fairly well up to the order of $\tilde{k}^2|\tilde{g}|\sim 10^{-3}$, beyond the restriction stated above. At $\tilde{g}=0.05$ and $\tilde{g}=0.1$ they overestimate slightly the numerical values, while at $\tilde{g}=0.4$ they underestimate them; thus at the intermediate value $\tilde{g}=0.2$ the agreement is especially good. When $\tilde{g}<0$, on the other hand, the truncated series $\tilde{\lambda}_{[2,0]}$ deviates from the numerical result rapidly as the Knudsen number $\tilde{k}$ increases and increases beyond unity, so that the agreement is considerably poor. This is because the last term on the right-hand side of Eq. (28) increases rapidly with $\tilde{k}$ due to its large numerical factor. For this reason, it is clear that merely employing terms in the asymptotic series [Eq. (27)] will not give any improvement. The Padé approximant $\tilde{\lambda}_{\mathrm{est}}$, however, suppresses this increasing character and gives a better agreement with the numerical result for a wider range of $\tilde{k}$. In addition, the fact that the effect of gravity on $\tilde{\lambda}$ is stronger when $\tilde{g}>0$ rather than when $\tilde{g}<0$ is well expressed qualitatively. From these reasons, it might be said that the Padé approximants [Eqs. (29)–(31)] provide a better formula in describing the behavior of the numerical result than the truncared asymptotic result [Eq. (28)], at least in a finite range $\tilde{k}\lesssim 0.1$.

Finally, we briefly summarize the computational conditions and the results of accuracy tests. We introduced 799 lattice points within the interval $0\leqslant X_1/D\leqslant 1$. The lattice size is uniform $(0.002D)$ within $0.1\leqslant X_1/D\leqslant 0.9$ and nonuniform otherwise; approaching the boundaries it becomes finer and takes a minimum value $(3.35\times 10^{-6}D)$ on the boundaries $X_1=0$ and D. The computational range with respect to ξ_1 is limited as $-\xi_D\leqslant\xi_1\leqslant\xi_D$ with $\xi_D=5.2(2RT_0)^{1/2}$. Within this interval, we introduced 1601 lattice points, the size of which is a monotonically increasing function of $|\xi_1|$: $1.25\times 10^{-4}(2RT_0)^{1/2}$ at $\xi_1=0$ and $0.02421(2RT_0)^{1/2}$ at $\xi_1=\pm\xi_D$. The convergence criterion of the iteration process is that the variation of the dimensionless macroscopic variables ρ/ρ_0 and T/T_0 in 100 iterations is less than 10^{-10} on each lattice point. The results of the accuracy tests are as follows. It is confirmed that the dimensionless marginal distribution function $\rho_0^{-1}(2RT_0)^{1/2}\times\int\int d\xi_2 d\xi_3$ decays rapidly with $|\xi_1|$ and is less than 8.0×10^{-11} on the lattice points $\xi_1=\pm\xi_D$. As mentioned in Sec. II, the heat flux q should be uniform theoretically [Eq. (17)], but it varies slightly due to numerical error. Conversely this variation can be an estimate of the numerical error. The present results show that the relative variation of the heat flux, $|(q_{\max}-q_{\min})/q_{\min}|$, is less than 1.2×10^{-4}, where $q_{\max}$ and $q_{\min}$ are, respectively, the maximum and minimum value of q within the bulk region (see the first paragraph of this section). In addition, the flow velocity v, which should vanish everywhere, is evaluated as $|v|/(2RT_0)^{1/2}<3.6\times 10^{-6}$ within the same region. We have also made recomputations for eight typical cases of the parameters in Eq. (20) using finer lattice systems (one is double in X_1 and the other double in ξ_1). The values of $\tilde{k}$ and $\tilde{\lambda}$ for given $\tilde{g}$ obtained with these finer systems differ less than 0.031% from the ones obtained with the standard system.

The computation was carried out on VT-Alpha 433 computers (Visual Technology Inc., CPU: Alpha 21164 433 MHz) of the first author's laboratory.

VI. CONCLUDING REMARKS

In this paper we have been concerned with a one-dimensional heat transfer problem in the presence of a gravity field and have analyzed it on the basis of kinetic theory. To that end, we have numerically solved the BKW (BGK) model of the Boltzmann equation for Maxwell molecules. Our aim has been to investigate the influence of gravity on the heat conductivity of the gas for those points sufficiently separated from the walls so that the boundary effects on them are negligible. At those points we have measured the heat flux q, the temperature T, the density ρ, and the temperature gradient G. Three independent characteristic distances can be defined, the mean free path l [Eq. (18)], the characteristic length L associated with the temperature gradient [Eq. (1)], and the characteristic length $h\equiv 2RT/g$ associated with the gravity, the ratios of which define the local Knudsen number $\tilde{k}=(\sqrt{\pi}/2)l/L$ and the local inverse Froude number $\tilde{g}=L/h$. In addition, the ratio of the actual heat flux to the value predicted by the Fourier law, $\tilde{\lambda}=(-q/G)/\lambda(T)$, where $\lambda(T)$ is the heat conductivity in the absence of gravity, is evaluated.

The first conclusion drawn from our results is that the factor $\tilde{\lambda}$ is a unique function of $\tilde{k}$ and $\tilde{g}$. This means that if different points corresponding to different solutions have the same $(\tilde{k},\tilde{g})$, then they also have the same $\tilde{\lambda}$. We have also verified that, in accordance with the exact solution of Ref. 16, $\tilde{\lambda}=1$ in the absence of gravity $(\tilde{g}=0)$, irrespective of the value of $\tilde{k}$. On the other hand, $\tilde{\lambda}$ depends both on $\tilde{k}$ and $\tilde{g}$ when gravity is present. The results show that the effective heat conductivity is enhanced $(\tilde{\lambda}>1)$ when the heat flux points in the same direction as that of gravity $(\tilde{g}>0)$, while it is inhibited $(\tilde{\lambda}<1)$ when pointing in the opposite direction to that of gravity $(\tilde{g}<0)$, both effects becoming more remarkable as the Knudsen number $\tilde{k}$ increases. At a given value of $\tilde{k}$ and a given sign of $\tilde{g}$, the larger the value of $|\tilde{g}|$

the larger the deviation $|\tilde{\lambda}-1|$. Further, the dependence of $\tilde{\lambda}$ on $\tilde{g}$ is not symmetric, i.e., $\tilde{\lambda}(\tilde{k},\tilde{g})-1>1-\tilde{\lambda}(\tilde{k},-\tilde{g})$.

We have also compared our numerical results with the asymptotic analysis of Ref. 20. The asymptotic result, taking the first few terms in the expansion as a series in powers of $\tilde{g}$, shows a fair agreement with the numerical one when $\tilde{g}>0$, while the agreement is considerably poor when $\tilde{g}<0$. To overcome this, we have also constructed a theoretical estimate with the aid of Padé approximants. This estimate, although being a considerably simple formula, shows reasonable agreement with the numerical results, so that it provides a practical formula valid within the range of the local Knudsen number and the local inverse Froude number considered here.

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