Poiseuille flow driven by an external force

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The steady planar Poiseuille flow generated by a constant external force is analyzed in the context of the nonlinear Bhatnagar–Gross–Krook kinetic equation for a gas of Maxwell molecules. An exact solution is found for a particular value of the force parameter. At a hydrodynamic level, the solution is characterized by a parabolic profile of the flow velocity with respect to a space variable scaled with the local collision frequency, a parabolic profile of the temperature with respect to the same variable, and a constant pressure. The (dimensionless) ratios between the quadratic coefficients and the external force are equal to 146 for the flow velocity and 65 for the temperature, as compared with the values 1/2 and 0, respectively, in the Navier–Stokes order. The fluxes of momentum and energy are explicitly evaluated. The anisotropy of the velocity distribution is made evident by the diagonal elements of the pressure tensor: $P_{yy}/P_{xx} = 0.031$, $P_{zz}/P_{xx} = 0.081$. Finally, the velocity distribution function is obtained in terms of quadratures.

I. INTRODUCTION

Transport phenomena in dilute gases are a subject of long-lasting interest.[1] At a fundamental level, they are usually studied in the framework of the Boltzmann equation and related kinetic equations.[2] The well-known Chapman–Enskog theory[3] provides a method for solving the Boltzmann equation that is useful in many situations of practical interest. Nevertheless, the Chapman–Enskog method fails when the boundary effects are dominant (large Knudsen numbers) and/or the system is arbitrarily far from equilibrium. For the sake of simplicity, one generally separates both situations. On the one hand, a great deal of attention has been devoted to the solution of kinetic equations linearized around (local) equilibrium, but with realistic boundary conditions.[1,4] On the other hand, recent advances have been achieved in the search of solutions of fully nonlinear kinetic equations valid in the bulk, i.e., in the limit of vanishing Knudsen number.[5] In this paper, we will be concerned with the latter approach applied to planar Poiseuille flow induced by a constant external force.

Perhaps, the best known example in fluid dynamics is the Poiseuille flow[6] that was first studied by Poiseuille and Hagen about 150 years ago. It consists of the steady flow along a channel of constant cross section produced by a pressure difference at the distant ends of the channel. To fix ideas, consider a fluid enclosed between two infinite parallel plates at rest, orthogonal to the y axis and located at $y = \pm H/2$. A pressure gradient $\partial p/\partial x$ exists along the x direction. Also, a constant external force per unit mass $\mathbf{F} = F_x \hat{\mathbf{x}}$ is assumed. In practice, F_x can be the component of gravity in the direction of motion. The balance equation for momentum reads

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\rho} \nabla \cdot \mathsf{P} + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{F} = 0, \tag{1}$$

where $\mathbf{u}$ is the flow velocity, ρ is the mass density, and P is the pressure tensor. The Navier–Stokes equation is obtained by supplementing Eq. (1) with the constitutive equation

$$P_{ij} = p\delta_{ij} - \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\,\delta_{ij}\nabla\cdot\mathbf{u}\right) - \zeta\delta_{ij}\nabla\cdot\mathbf{u}, \tag{2}$$

where η is the shear viscosity and ζ is the bulk viscosity. In the Poiseuille flow, $\partial \mathbf{u}/\partial t = 0$ and $\mathbf{u}(\mathbf{r}) = u_x(y)\hat{\mathbf{x}}$. Consequently, the Navier–Stokes equation now becomes

$$\frac{\partial p}{\partial y} = 0, \tag{3a}$$

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{1}{\eta}\left(\frac{\partial p}{\partial x} - \rho F_x\right). \tag{3b}$$

Equations (3) show that $\partial p/\partial x = \text{const}$. The solution of Eq. (3b) with the boundary condition of zero flow velocity near the walls is

$$u_x(y) = -\frac{1}{2\eta}\left(\rho F_x - \frac{\partial p}{\partial x}\right)\left(y^2 - \frac{H^2}{4}\right). \tag{4}$$

This gives the parabolic profile for the flow velocity that is characteristic of the Poiseuille flow. In most textbooks, the external force is omitted ($F_x = 0$). However, it is noticeable that the same kind of profile is obtained as well in absence of pressure gradient ($p = \text{const}$) if the external force is present. Thus, at the Navier–Stokes order, one may conclude that the role of the force is to mimic a pressure gradient, and vice versa: $\rho F_x \leftrightarrow -\partial p/\partial x$.

Kadanoff *et al.*[7] have recently used a constant external force to induce a Poiseuille flow in the lattice gas automaton proposed by Frisch *et al.*[8] Their simulation results agree with Eq. (4) (with $\partial p/\partial x = 0$), from which they get the shear viscosity η. This supports the validity of a hydrodynamic description for lattice gas automata.

The main goal of this paper is to study the departure from equilibrium in a dilute gas because of the action of the external force considered above. Conditions will be such that a bulk domain can be identified far from the range of bound-

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ary layers. Some of the questions we want to address are: (i) Is the velocity profile still parabolic if the external force is so strong that a Navier–Stokes description is not expected to hold? (ii) At the Navier–Stokes order, the coefficient of the quadratic term in u_x is just proportional to F_x. Is that relationship still valid beyond the Navier–Stokes limit? (iii) How are the other hydrodynamic variables (pressure and temperature) affected by the force? (iv) How do the fluxes responsible for momentum and energy transport behave? (v) How large is the distortion from equilibrium in the velocity distribution function?

The above questions are extremely hard to answer by using the Boltzmann equation. We prefer to gain insight into the problem at the expense of describing the system by means of the Bhatnagar–Gross–Krook (BGK) model kinetic equation.[2] The BGK equation is a model of the Boltzmann equation where the detailed collision term is replaced by a single-time relaxation term toward local equilibrium:

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f+\mathbf{F}\cdot\frac{\partial}{\partial \mathbf{v}}f=-\nu(f-f_{\mathrm{LE}}). \tag{5}$$

Here, $f(\mathbf{r},\mathbf{v};t)$ is the velocity distribution function, $\nu(\mathbf{r},t)$ is the collision frequency, and $f_{\mathrm{LE}}(\mathbf{r},\mathbf{v};t)$ is the local equilibrium velocity distribution function:

$$f_{\mathrm{LE}}(\mathbf{r},\mathbf{v};t)=n(\mathbf{r},t)\left(\frac{m}{2\pi k_{\mathrm{B}}T(\mathbf{r},t)}\right)^{3/2}\times\exp\left(-\frac{m}{2k_{\mathrm{B}}T(\mathbf{r},t)}[\mathbf{v}-\mathbf{u}(\mathbf{r},t)]^2\right), \tag{6}$$

where k_{B} is the Boltzmann constant, m is the mass of a particle, $n(\mathbf{r},t)$ is the local number density, $\mathbf{u}(\mathbf{r},t)$ is the local flow velocity, and $T(\mathbf{r},t)$ is the local temperature. These fields are defined as

$$n=\int d\mathbf{v}\, f, \tag{7}$$

$$n\mathbf{u}=\int d\mathbf{v}\,\mathbf{v}f, \tag{8}$$

$$p\equiv nk_{\mathrm{B}}T=\frac{m}{3}\int d\mathbf{v}(\mathbf{v}-\mathbf{u})^2 f. \tag{9}$$

The transport of momentum and energy are described by the pressure tensor

$$P_{ij}=m\int d\mathbf{v}(v_i-u_i)(v_j-u_j)f \tag{10}$$

and the heat flux

$$\mathbf{q}=\frac{m}{2}\int d\mathbf{v}(\mathbf{v}-\mathbf{u})^2(\mathbf{v}-\mathbf{u})f, \tag{11}$$

respectively. In the case of steady planar Poiseuille flow driven by a constant external force $\mathbf{F}=F_x\hat{\mathbf{x}}$, Eq. (5) reduces to

$$v_y\frac{\partial}{\partial y}f+F_x\frac{\partial}{\partial v_x}f=-\nu(f-f_{\mathrm{LE}}). \tag{12}$$

Upon writing Eq. (12), we have assumed that the boundary conditions are such that $\nabla f\|\hat{\mathbf{y}}$. This rules out the trivial solution of Eq. (5) corresponding to the canonical equilibrium distribution $f=f_{\mathrm{LE}}$ with $\mathbf{u}=\text{const}$, $T=\text{const}$, and $n(\mathbf{r})\propto\exp(\mathbf{F}\cdot\mathbf{r}/k_{\mathrm{B}}T)$.

It is illustrative to get the solution of Eq. (12) up to first order in F_x (Navier–Stokes order). As can be easily verified, it is given by

$$p=\text{const}, \tag{13}$$

$$T=\text{const}, \tag{14}$$

$$\frac{\partial^2 u_x}{\partial y^2}=-\frac{\nu}{k_{\mathrm{B}}T/m}F_x, \tag{15}$$

$$f=\left[1-\frac{1}{\nu}\left(F_x\frac{\partial}{\partial v_x}+v_y\frac{\partial}{\partial y}-\frac{v_y^2}{\nu}\frac{\partial^2}{\partial y^2}\right)\right]f_{\mathrm{LE}}$$
$$=f_{\mathrm{LE}}\left[1+\frac{F_x(v_x-u_x)}{\nu k_{\mathrm{B}}T/m}\left(1+\frac{v_y\nu y}{k_{\mathrm{B}}T/m}-\frac{v_y^2}{k_{\mathrm{B}}T/m}\right)\right], \tag{16}$$

where in the last step the origin of the y axis has been chosen at the vertex of the velocity parabola. As expected, Eq. (15) agrees with Eq. (3b) if one takes into account that the shear viscosity in the BGK model is given by $\eta=p/\nu$.[2] From Eq. (16), one can also get the fluxes to first order in F_x:

$$P_{xx}=P_{yy}=P_{zz}=p, \tag{17}$$

$$P_{xy}=nmF_x y, \tag{18a}$$

$$=-\eta\frac{\partial u_x}{\partial y}, \tag{18b}$$

$$q_x=-\eta F_x, \tag{19}$$

$$q_y=0. \tag{20}$$

Equation (18b) is consistent with the hydrodynamic equation (2). On the other hand, Eq. (19) shows that, despite the absence of a thermal gradient at this order, there exists a uniform heat flux opposite to the direction of motion. This apparent violation of the Fourier law reminds us of the fact that, in this problem, the role of a nonequilibrium perturbation parameter is played by F_x, rather than by the hydrodynamic gradients.

The organization of this paper is as follows. A simple self-consistent solution of the nonlinear equation (12) is found to exist in Sec. II for a particular value of the external force. This solution is characterized by constant pressure and parabolic velocity and temperature profiles with respect to a conveniently scaled space variable. The most relevant fluxes are then obtained in Sec. III. The explicit form for the velocity distribution function is derived in Sec. IV. Finally, Sec. V offers some comments and conclusions.

II. HYDRODYNAMIC FIELDS

In the BGK model, all the details of the interaction potential are grossly taken into account through the temperature dependence of ν/n. In the special case of Maxwell molecules (which interact via a potential inversely proportional to the fourth power of the distance), ν/n is just a constant. Henceforth, we will restrict ourselves to this interaction.

In order to solve Eq. (12), one needs to add appropriate boundary conditions. Complete accommodation of the particles on interaction with the walls is described by diffuse boundary conditions:[3,9]

$$f_\pm\left(y=\mp\frac{H}{2},\mathbf{v}\right)=n_w\left(\frac{m}{2\pi k_B T_w}\right)^{3/2}\exp\left(-\frac{mv^2}{2k_B T_w}\right). \tag{21}$$

Here, $f_\pm\equiv\Theta(\pm v_y)f$, Θ being Heaviside's step function, T_w represents the temperatures of the walls at $y=\mp H/2$, and n_w is twice the density of particles coming off the walls. The latter parameter must be determined self-consistently along with the solution of Eq. (12). Given the geometry of the problem, the following symmetry relations hold:

$$f(y,v_x,v_y,v_z)=f(y,v_x,v_y,-v_z), \tag{22}$$

$$f(y,v_x,v_y,v_z)=f(-y,v_x,-v_y,v_z). \tag{23}$$

In particular, the three hydrodynamic fields (p, u_x, and T) are even functions of y. In principle, the solution of the problem defined by Eqs. (12) and (21) implies to solve a set of three coupled singular nonlinear integral equations, which requires the use of numerical methods.[10,11] Furthermore, the solution is expected to include boundary layers within a few mean-free paths from the plates. The influence of the boundary layers is measured by the Knudsen number (ratio of the mean-free path to the system size).

One can get rid of undesired boundary effects by applying idealized boundary conditions such that the local Knudsen number vanishes near the walls. This can be accomplished if one chooses infinitely cold walls ($T_w=0$).[11] Thus, Eq. (21) becomes

$$f_\pm\left(y=\mp\frac{H}{2},\mathbf{v}\right)=0. \tag{24}$$

The formal solution of Eq. (12) is then

$$f_+(y,\mathbf{v})=\frac{1}{v_y}\int_{-H/2}^{y}dy_1\;\nu(y_1)\times\exp\left[-\frac{1}{v_y}\left(\int_{y_1}^{y}dy_2\;\nu(y_2)+(y-y_1)F_x\frac{\partial}{\partial v_x}\right)\right]f_{\rm LE}(y_1,\mathbf{v}). \tag{25}$$

Notice that the other half-distribution, f_-, can be obtained from f_+ by making use of Eq. (23). The solution (25) has a formal character because ν and $f_{\rm LE}$ still depend on f through the hydrodynamic fields. The set of coupled integral equations for p, u_x, and T can be obtained by taking moments in Eq. (25).

Rather than solving the problem numerically, we are going to use an heuristic approach in the same spirit as in previous works.[12,13] On the basis of simplicity and symmetry arguments, we "guess" the profiles and then verify their consistency. The solution in the linear case, Eqs. (13)–(15), is a good starting point. First, it is reasonable to expect that the uniformity of pressure (a quantity related to normal transfer of momentum) is a property more related to the stationary character of the flow than to the linear approximation. In fact, it is easy to check from Eq. (12) the exact property $P_{yy}=\text{const}$. We assume that the same happens with the other diagonal elements of the pressure tensor, so that

$$p=\text{const}. \tag{26}$$

This assumption is also supported by the cases of pure heat flow[12] and planar Couette flow.[13] On the other hand, there is no reason to expect Eq. (14) to hold beyond the linear regime. Since the walls are very cold, one can argue that the most energetic particles tend to concentrate far from the walls. Let T_0 be the temperature at $y=0$, which is expected to be the highest temperature in the system. We will take T_0 as a convenient unit of temperature and $v_0\equiv(k_B T_0/m)^{1/2}$ as a convenient unit of velocity. Since $T\neq\text{const}$, Eq. (26) implies that $n\neq\text{const}$. Therefore the rate at which collisions take place, as measured by ν, is nonuniform, so that it is not very convenient to measure distance with the linear space variable y. We take instead the scaled space variable s defined as

$$s(y)=\frac{1}{v_0}\int_0^y dy_1\;\nu(y_1). \tag{27}$$

The variable s measures distance in units of mean-free paths. Of course, the symmetry relation (23) is also true when y is replaced by s. The simplest nonconstant even function one can propose for the temperature is a quadratic one:

$$T=T_0(1-\omega^2 s^2), \tag{28}$$

where $\omega^{-1}\equiv s(y=H/2)$. Finally, we borrow from Eq. (15) the parabolic shape of the velocity profile and suggest

$$\frac{1}{v_0}\frac{\partial^2 u_x}{\partial s^2}=-2\epsilon, \tag{29}$$

where ϵ is a parameter to be determined.

Equations (26), (28), and (29) constitute our guess of hydrodynamic profiles for an exact solution of Eq. (12). Notice that the simplicity of the profiles (28) and (29) is not so apparent if one uses the variable y. For Maxwell molecules and constant pressure, $\nu(y)=\nu_0 T_0/T(y)$, where ν_0 is the collision frequency at $y=0$. Thus Eqs. (27) and (28) give the following nonlinear relationship between y and s:

$$y=(v_0/\nu_0)s(1-\tfrac{1}{3}\omega^2 s^2). \tag{30}$$

Hence $H=(v_0/\nu_0)(4/3\omega)$. Equations (28) and (29) can be combined to show that T is a linear function of u_x with a slope equal to $(T_0/v_0)\omega^2/\epsilon$.

Once the hydrodynamic profiles (26), (28), and (29) are inserted into the right side of Eq. (25), we obtain an explicit expression for the velocity distribution function. This gives the solution of the problem, however, only if the following self-consistency conditions are verified:

$$\int d\mathbf{v}\,f=\int d\mathbf{v}\,f_{\rm LE}, \tag{31a}$$

$$\int d\mathbf{v}\;\mathbf{v}f=\int d\mathbf{v}\;\mathbf{v}f_{\rm LE}, \tag{31b}$$

$$\int d\mathbf{v}\;v^2 f=\int d\mathbf{v}\;v^2 f_{\rm LE}. \tag{31c}$$

In order to check these conditions, a series representation for f is far more convenient than the integral representation (25). First, we rewrite Eq. (12) as

$$f=f_{\rm LE}-v_y^*\frac{\partial}{\partial s}f-\mu T^*\frac{\partial}{\partial v_x^*}f, \tag{32}$$

where $\mu \equiv F_x/(v_0 \nu_0)$ is a dimensionless parameter measuring the strength of the external force, $T^* \equiv T/T_0$, and $\mathbf{v}^* \equiv \mathbf{v}/v_0$. Assumption (26) has already been taken into account in Eq. (32). Repeated iteration of Eq. (32) yields

$$f = f_{\rm LE} + \sum_{k=1}^{\infty} (-1)^k (v_y^* \partial_s + \mu T^* D_x)^k f_{\rm LE}, \quad (33)$$

where $\partial_s \equiv \partial/\partial s$ and $D_x \equiv \partial/\partial v_x^*$. The expansion of the operator acting on $f_{\rm LE}$ in Eq. (33) has the form

$$\begin{aligned}(v_y^* \partial_s + \mu T^* D_x)^k &= v_y^{*k} \partial_s^k + v_y^{*k-1} \mu D_x \sum_{l=0}^{k-1} \partial_s^l T^* \partial_s^{k-1-l} \\ &+ v_y^{*k-2} \mu^2 D_x^2 \sum_{l=0}^{k-2} \sum_{r=0}^{k-2-l} \partial_s^l T^* \\ &\times \partial_s^r T^* \partial_s^{k-2-l-r} + \cdots, \end{aligned} \quad (34)$$

where the ellipsis denotes terms involving at least D_x^3. The consistency conditions (31) are then equivalent to

$$\sum_{k=1}^{\infty} (-1)^k \int d\mathbf{v}^* (v_y^* \partial_s + \mu T^* D_x)^k f_{\rm LE}^* = 0, \quad (35a)$$

$$\sum_{k=1}^{\infty} (-1)^k \int d\mathbf{v}^* \, \mathbf{v}^* (v_y^* \partial_s + \mu T^* D_x)^k f_{\rm LE}^* = 0, \quad (35b)$$

$$\sum_{k=1}^{\infty} (-1)^k \int d\mathbf{v}^* \, v^{*2} (v_y^* \partial_s + \mu T^* D_x)^k f_{\rm LE}^* = 0, \quad (35c)$$

where

$$\begin{aligned} f_{\rm LE}^* &\equiv (k_B T_0/p) v_0^3 f_{\rm LE} \\ &= (2\pi)^{-3/2} T^{*-5/2} \exp[-(\mathbf{v}^* - \mathbf{u}^*)^2/2T^*], \\ \mathbf{u}^* &\equiv \mathbf{u}/v_0. \end{aligned}$$

Let us start with Eq. (35a). In that case, only the first term in the right side of Eq. (34) matters:

$$\int d\mathbf{v}^* (v_y^* \partial_s + \mu T^* D_x)^k f_{\rm LE}^* = C_k \, \partial_s^k T^{*(k-2)/2}, \quad (36)$$

where $C_k \equiv (k-1)!!$ if k is even and zero otherwise. So far, only Eq. (26) has been used. In addition, Eq. (29) shows that $\partial_s^k T^{*(k-2)/2} = 0$ $(k \geqslant 2)$, so that Eq. (35a) is automatically satisfied. The same happens in Eq. (35b) with the y and z components of the velocity. Nevertheless, the consistency condition for the x component imposes the following relationship between ω, ϵ, and μ (see Appendix A):

$$\frac{\epsilon}{\mu} = \Phi(\omega) \equiv \frac{1 - \frac{2}{3}\omega^2 [F_1(\omega) + 2F_2(\omega)]}{2F_1(\omega)}, \quad (37)$$

where the functions $F_r(\omega)$ are defined by Eqs. (A5) and (A6). The first few terms in the (asymptotic) expansion of $\Phi(\omega)$ around the origin are

$$\Phi(\omega) = \tfrac{1}{2} + 17\omega^2 - 2028\omega^4 + \cdots . \quad (38)$$

Figure 1 shows that $\Phi(\omega)$ is a rapidly increasing function.

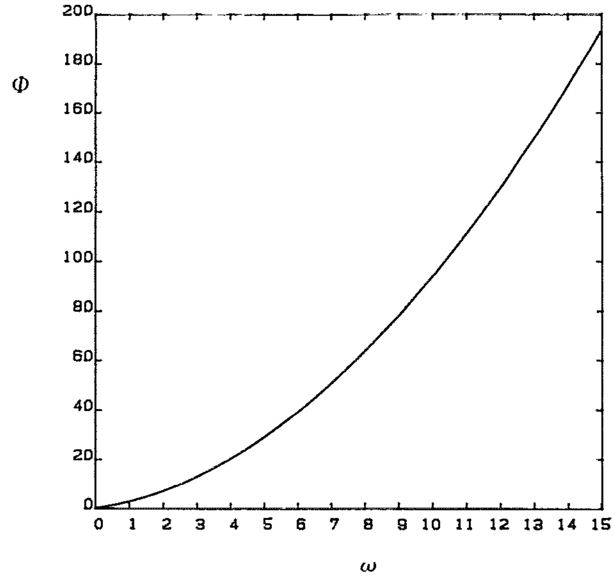

FIG. 1. Plot of the function $\Phi(\omega)$. This function gives the ratio ϵ/μ.

The last consistency condition, Eq. (35c), is much more difficult to deal with. The detailed calculations are done in Appendix B and here we only quote the result:

$$s^2\mu^2\Psi(\omega) + \mu^2\Theta(\omega) - \Omega(\omega) = 0, \quad (39)$$

where $\Psi(\omega)$, $\Theta(\omega)$, and $\Omega(\omega)$ are expressed in terms of $F_0(\omega)$, $F_1(\omega)$, and $F_2(\omega)$ in Appendix B. The asymptotic series of these functions are

$$\Psi(\omega) = 2 - 82\omega^2 + 38\,628\omega^4 - \cdots, \quad (40)$$

$$\Theta(\omega) = 14 - 3916\omega^2 + 2\,599\,128\omega^4 - \cdots, \quad (41)$$

$$\Omega(\omega) = 10\omega^2 - 504\omega^4 + \cdots . \quad (42)$$

Complete consistency of the assumed profiles, Eqs. (26), (28), and (29), requires Eq. (39) to be verified at any point. Thus

$$\Psi(\omega) = 0, \quad (43)$$

$$\mu^2 = \Lambda(\omega) \equiv \Omega(\omega)/\Theta(\omega), \quad (44a)$$

$$= \tfrac{5}{7}\omega^2 + \tfrac{8026}{49}\omega^4 - \cdots . \quad (44b)$$

Equation (44b) shows that $\omega^2 = O(\mu^2)$. So, in the linear limit (i.e., all terms of order μ^2 and higher are neglected), Eq. (39) is identically satisfied. In that order, Eq. (37) gives $\epsilon/\mu = 1/2$. Consequently, Eqs. (26), (28), and (29) become Eqs. (13)–(15), respectively. On the other hand, Eqs. (26), (28), and (29) also hold in the fully nonlinear case, provided that the parameters μ, ϵ, and ω take values given by the solution of Eqs. (37), (43), and (44a). The functions $\Psi(\omega)$ and $\Lambda(\omega)$ are plotted in Figs. 2 and 3, respectively. The solution of Eq. (43) is

$$\omega = 12.797. \quad (45)$$

For this value, one finds $F_0 = 1.0048 \times 10^{-2}$, $F_1 = 2.3821 \times 10^{-3}$, and $F_2 = 2.0947 \times 10^{-4}$. Thus Eqs. (44a) and (37) give

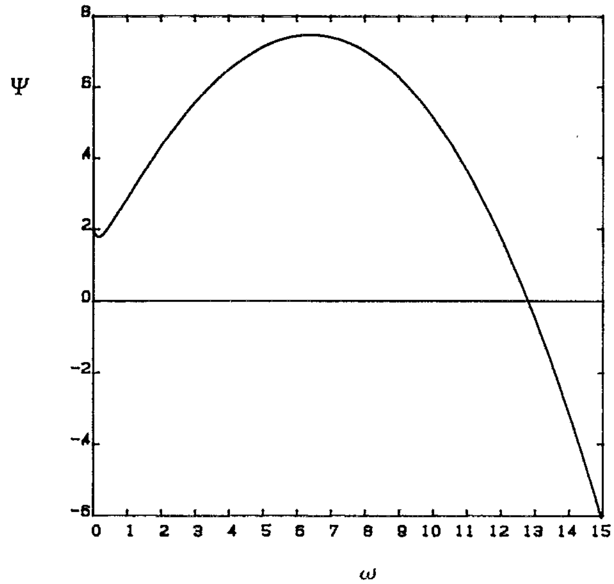

FIG. 2. Plot of the function $\Psi(\omega)$. The zero of this function gives the value of the parameter ω in the self-consistent solution.

$$\mu = 2.5240, \tag{46}$$

$$\epsilon = 367.78. \tag{47}$$

To sum up, we have proved that there exists an exact solution of the nonlinear equation (12) that is consistent with the hydrodynamic profiles given by Eqs. (26), (28), and (29). This solution does not apply to arbitrary values of the external force, but only to the (dimensionless) value given by Eq. (46). In that case, the parameters ω and ϵ appearing in Eqs. (28) and (29) take the values (45) and (47), respectively. The temperature is a linear function of the flow velocity with a (dimensionless) slope $\omega^2/\epsilon = 0.445$. We remind that Eqs. (26), (28), and (29) also hold if μ is asymptotically small (linear regime), in which case $\omega^2 = 0$, $\epsilon = \mu/2$, and $s \propto y$. The importance of nonlinear effects is made evident by the contrast with Eqs. (45) and (47).

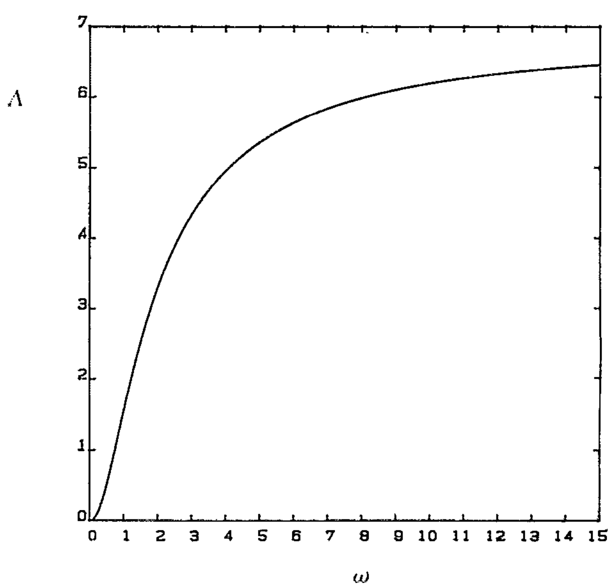

FIG. 3. Plot of the function $\Lambda(\omega)$. This function gives the parameter μ^2.

III. FLUXES

In the previous section, a particular exact solution of the Poiseuille problem in the presence of an external force has been identified by the hydrodynamic variables. We proceed now to the calculation of some of the fluxes. Let us define the moments

$$M_{klr} = \int d\mathbf{v}^* (v_x^* - u_x^*)^k v_y^{*l} v_z^{*r} f^*, \tag{48}$$

where $f^* \equiv (k_B T_0/p) v_0^3 f$. Equation (32) then yields the following hierarchy:

$$\frac{\partial}{\partial s} M_{k,l+1,r} - 2\epsilon k s M_{k-1,l+1,r} - \mu k(1-\omega^2 s^2) M_{k-1,l,r} = -M_{klr} + M_{klr}^{\rm LE}. \tag{49}$$

Here, $M_{klr}^{\rm LE} = C_k C_l C_r T^{*(k+l+r-2)/2}$, where C_k is defined below Eq. (36) and $C_0 = 1$. Equation (49) allows one to obtain $\partial M_{klr}/\partial s$ in terms of moments $M_{k'l'r}$ with $k' \leqslant k$, $k'+l' < k+l$. Starting from $M_{000} = T^{*-1}$, $M_{100} = M_{010} = M_{001} = 0$, it is straightforward to see that M_{klr}, with $2k+l+r \geqslant 2$, is a polynomial in s of degree $2k+l+r-2$ and the same parity as l. The latter property is a direct consequence of the symmetry relation (23). Also, as can be seen from Eq. (22), $M_{klr} = 0$ if r is odd. From Eq. (49), one can get the moments of odd degree from the knowledge of lower degree moments. On the other hand, Eq. (49) does not provide the values of even moments at $s = 0$. To get those values, one needs to perform calculations similar to the ones in Appendix B.

The first nontrivial odd moment is $M_{110} \equiv P_{xy}/p$. Equation (49) yields

$$P_{xy} = p\mu s, \tag{50a}$$

$$= -\frac{\mu}{2\epsilon} \eta \frac{\partial u_x}{\partial y}. \tag{50b}$$

Equation (50a) shows that P_{xy} is a linear function of s with a coefficient that is just proportional to the external force. Thus this result formally coincides with that obtained in the linear approach, Eq. (18a), if s is replaced by $(m/k_B T) v_0 \nu_0 y$. On the other hand, Eq. (50b) differs from Eq. (18b), except in the limit $\mu \to 0$, in which case $\mu/2\epsilon = 1$. According to Eqs. (46) and (47), $\mu/2\epsilon = 0.003\,43$, which is a value clearly associated to non-Newtonian effects.

The remaining nonzero elements of the pressure tensor are even moments, so that Eq. (49) is not helpful. According to Eqs. (33) and (B1),

$$P_{yy} = p(1 + I_y), = p\{1 - 2\omega^2[F_1(\omega) + 2F_2(\omega)]\}, \tag{51}$$

where use has been made of Eq. (B2) in the last step. Similarly,

$$P_{zz} = p(1 + I_z) = p[1 - 2\omega^2 F_1(\omega)]. \tag{52}$$

Finally, $P_{xx} = 3p - P_{yy} - P_{zz}$, which implies

$$P_{xx} = p\{1 + 4\omega^2[F_1(\omega) + F_2(\omega)]\}. \tag{53}$$

In the linear limit, we recover Eq. (17). On the other hand, nonlinear effects give rise to a large anisotropy, as shown up by the particular solution reported in this paper. For the value given by Eq. (45), we have $P_{xx}/p = 2.6976$, $P_{yy}/p = 0.0826$, and $P_{zz}/p = 0.2198$.

Let us consider now the next odd moments. Making $(k,l,r) = (2,0,0)$, $(0,2,0)$, and $(0,0,2)$ in Eq. (49), one obtains, respectively,

$$M_{210} = -4\omega^2[F_1(\omega) + F_2(\omega)]s + \tfrac{4}{3}\epsilon\mu s^3, \tag{54}$$

$$M_{030} = 2\omega^2[F_1(\omega) + 2F_2(\omega)]s, \tag{55}$$

$$M_{012} = 2\omega^2 F_1(\omega)s. \tag{56}$$

The y component of the heat flux is simply

$$q_y = \tfrac{1}{2}pv_0(M_{210} + M_{030} + M_{012}),$$
$$= pv_0\tfrac{2}{3}\epsilon\mu s^3. \tag{57}$$

Notice that q_y is opposite to the thermal gradient. However, it is proportional to s^3, rather than to s, as could be expected from the Fourier law. The failure of the Fourier law is related to nonlinear effects and also to the fact that the relevant nonequilibrium parameter is the external force instead of the hydrodynamic variables. In this sense, Eq. (19) shows that the Fourier law fails even in the linear regime.

IV. VELOCITY DISTRIBUTION

Although the velocity moments provide a good deal of knowledge about the nonequilibrium state of the system, the most complete information is contained in the velocity distribution function. The formal solution to Eq. (12) with the boundary conditions (24) is given in Eq. (25). After having found in Sec. II a self-consistent solution in terms of the hydrodynamic profiles, Eq. (25) is no longer a formal expression. Inserting Eqs. (26)–(29), we have

$$f^*_+(s,\mathbf{v}^*) = \frac{1}{v_y^*}\int_{-1/\omega}^{s} ds_1 \exp\left\{-\frac{1}{v_y^*}\left[s - s_1 + \mu D_x\left(s - s_1 - \frac{\omega^2}{3}(s^3 - s_1^3)\right)\right]\right\} f^*_{\mathrm{LE}}(s_1,\mathbf{v}^*). \tag{58}$$

In this equation, the velocity is measured in the laboratory frame and is reduced with respect to the thermal velocity at the middle layer. Physically, it is more interesting to adopt the Lagrangian reference frame and refer the velocity to the *local* thermal velocity. Thus we define

$$\boldsymbol{\xi} \equiv (\mathbf{v}^* - \mathbf{u}^*)/\sqrt{(2T^*)} \tag{59}$$

and

$$\phi(s,\boldsymbol{\xi}) \equiv f^*(s,\mathbf{v}^*)/f^*_{\mathrm{LE}}(s,\mathbf{v}^*). \tag{60}$$

After carrying out the change of variable $s_1 \to t = (1 + \omega s_1)/(1 + \omega s)$, Eq. (58) becomes

$$\phi_+(s,\boldsymbol{\xi}) = \mathscr{A}(s)\frac{e^{\xi^2}}{\xi_y}\int_0^1 dt\,[\mathscr{B}(s,t)]^{-5/2} \times e^{-(1-t)\mathscr{A}(s)/\xi_y}\exp\left\{-\frac{1}{\mathscr{B}(s,t)} \times\left[\left(\xi_x - \epsilon\mathscr{C}(s,t) - \frac{\mu}{\xi_y}\mathscr{D}(s,t)\right)^2 + \xi_y^2 + \xi_z^2\right]\right\}, \tag{61}$$

where

$$\mathscr{A}(s) \equiv \frac{1 + \omega s}{\omega[2T^*(s)]^{1/2}} = \frac{1}{\omega}\left(\frac{1 + \omega s}{2(1 - \omega s)}\right)^{1/2}, \tag{62}$$

$$\mathscr{B}(s,t) \equiv \frac{T^*(s_1)}{T^*(s)} = \frac{2 - (1 + \omega s)t}{1 - \omega s}t, \tag{63}$$

$$\mathscr{C}(s,t) \equiv \frac{s^2 - s_1^2}{[2T^*(s)]^{1/2}} = \frac{1}{\omega^2}\left(\frac{1 + \omega s}{2(1 - \omega s)}\right)^{1/2}[\omega s - 1 + 2t - t^2(1 + \omega s)], \tag{64}$$

$$\mathscr{D}(s,t) \equiv \frac{s - s_1 - (\omega^2/3)(s^3 - s_1^3)}{2T^*(s)} = \frac{1 + \omega s}{6\omega(1 - \omega s)}\{2 - \omega s + t^2[(1 + \omega s)t - 3]\}. \tag{65}$$

The other half-distribution can be obtained from Eq. (61) by changing s to $-s$ and ξ_y to $-\xi_y$:

$$\phi_-(s,\boldsymbol{\xi}) = \mathscr{A}(s)\frac{e^{\xi^2}}{|\xi_y|}\int_1^{2/(1+\omega s)} dt\,[\mathscr{B}(s,t)]^{-5/2} \times e^{-(t-1)\mathscr{A}(s)/|\xi_y|}\exp\left\{-\frac{1}{\mathscr{B}(s,t)} \times\left[\left(\xi_x - \epsilon\mathscr{C}(s,t) + \frac{\mu}{|\xi_y|}\mathscr{D}(s,t)\right)^2 + \xi_y^2 + \xi_z^2\right]\right\}, \tag{66}$$

where we have performed the change of variable $t \to t' = [2 - (1 - \omega s)t]/(1 + \omega s)$. Equations (61) and (66) give the explicit expression of the velocity distribution function corresponding to the solution found in Sec. II. A more closed expression can be obtained for the distribution of particles moving with a velocity orthogonal to the gradient. Performing the change of variable $t \to \tau = (1 - t)/\xi_y$ in Eqs. (61) or (66) and then taking the limit $\xi_y \to 0$, we obtain

$$\phi(s,\xi_x,\xi_y = 0,\xi_z) = \mathscr{A}(s)\int_0^\infty d\tau\, e^{-\tau\mathscr{A}(s)}\exp\left[\xi_x^2 - \left(\xi_x - \mu\frac{1 + \omega s}{2\omega}\tau\right)^2\right]$$
$$= \frac{\sqrt{\pi}}{\mu[2(1 - \omega^2 s^2)]^{1/2}}\exp\left[\left(\frac{1}{\mu[2(1 - \omega^2 s^2)]^{1/2}} - \xi_x\right)^2\right] \times\mathrm{erfc}\left(\frac{1}{\mu[2(1 - \omega^2 s^2)]^{1/2}} - \xi_x\right). \tag{67}$$

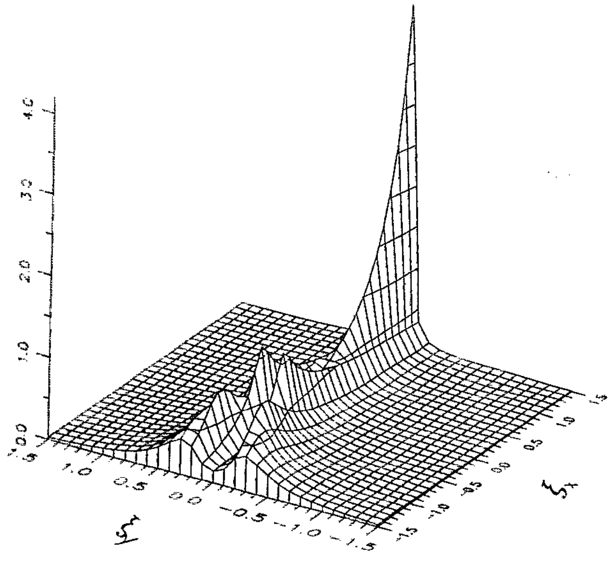

FIG. 4. Surface plot of the reduced velocity distribution function $\bar{\phi}(s,\xi_x,\xi_y)$ at the point $s=0$.

For plotting purposes, it is convenient to integrate over the z component of the velocity. Thus we define the distribution

$$\bar{\phi}(s,\xi_x,\xi_y)=\frac{\int_{-\infty}^{\infty}d\xi_z\,\phi(s,\xi)e^{-\xi^2}}{\int_{-\infty}^{\infty}d\xi_z\,e^{-\xi^2}}. \tag{68}$$

Figure 4 is a surface plot of $\bar{\phi}$ at $s=0$, where both the thermal and the velocity gradients are zero. The distortion with respect to local equilibrium ($\bar{\phi}=1$) is quite apparent. In particular, we can observe the strong concentration of the particle population along the axis ξ_x. This gives rise to a value of P_{xx} about 33 times larger than that of P_{yy}, as seen in Sec. III. This anisotropy feature appears again in Fig. 5, where $\bar{\phi}$ is plotted at $s=-0.01$. (The maximum absolute value of s is $1/\omega=0.078\,143$.) However, the symmetry $\xi_y\leftrightarrow-\xi_y$ has now disappeared, so that $P_{xy}<0$, as indicated by Eq. (50a).

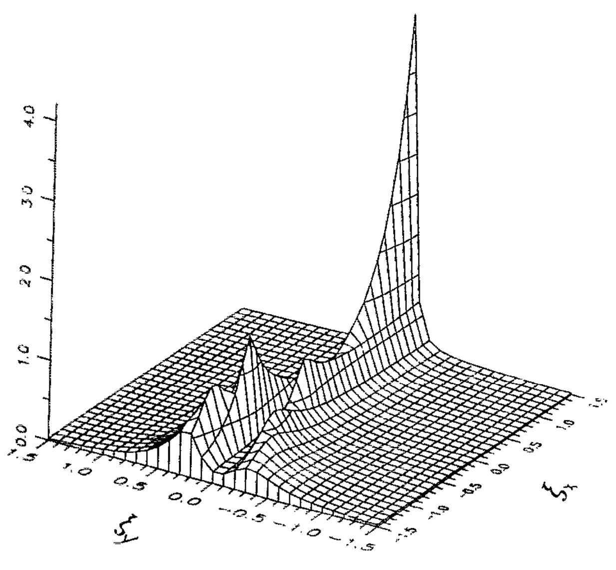

FIG. 5. Surface plot of the reduced velocity distribution function $\bar{\phi}(s,\xi_x,\xi_y)$ at the point $s=-0.01$.

V. CONCLUSIONS

In this paper, we have been concerned with the planar Poiseuille flow induced by a constant external force in the absence of gradients along the flow direction. An exact solution of the steady nonlinear Bhatnagar–Gross–Krook (BGK) equation for Maxwell molecules has been found. The solution holds for a particular value of the external force and is characterized by a constant pressure and parabolic profiles for the temperature and the flow velocity. As a matter of fact, the parabolic dependence does not take place with respect to the actual distance (y), but with respect to a space variable (s) conveniently scaled with the local collision frequency. Equation (30) gives the relationship between y and s. The hydrodynamic profiles are given by Eqs. (26), (28), and (29), where the parameters ω and ϵ take the values (45) and (47), respectively; the dimensionless value of the external force (μ) is given by Eq. (46). Idealized boundary conditions of zero wall temperatures get rid of boundary layers, so that the solution applies to the bulk region of the system.

It is remarkable that the nonlinear BGK equation admits such a simple solution. Similar examples are provided by the planar Fourier[12] and Couette[13] flows. In contrast to the latter cases, however, the solution reported here is restricted to a particular value of the parameter measuring the distance from equilibrium. If the force parameter μ does not take exactly the value (46), then the mathematical expressions for the hydrodynamic fields are not as simple as found here, even with idealized boundary conditions of infinitely cold walls. Numerical solution of a set of three coupled nonlinear integral equations seems to be unavoidable. Nonetheless, one may argue that the difference is not so large at a qualitative level. We expect the velocity profile to be paraboliclike, the pressure to be nearly uniform, and the temperature to have a strong space dependence. The parameters ϵ and ω could then be interpreted as measures of the degree of curvature of the velocity and temperature profiles, respectively, at the middle layer $s=0$. In fact, Eq. (43) is not necessary if we take the license of making $s=0$ in Eq. (39). Although this line of reasoning lacks mathematical rigor, we speculate that decent estimates of ϵ and ω for any value of μ would be provided by Eqs. (37) and (44a).

In the limit of a force so weak that terms of second and higher order in μ can be neglected (Navier–Stokes order), the profiles are given by Eqs. (13)–(15) and the fluxes by Eqs. (17)–(20). The main effects due to nonlinear terms can be inferred from the particular solution discussed here. Some key points are (i) the parabolic velocity profile, characteristic of the Poiseuille flow, no longer exists in real space. Of course, any profile can be mapped onto a parabola with an appropriate change of space variable. (ii) The coefficient measuring the curvature of the velocity is not just proportional to the strength of the force. The ratio $2\epsilon/\mu$ is equal to 1 in the Navier–Stokes regime, while it takes the value 291.43 in our particular solution. (iii) Concerning the remaining hydrodynamic variables, the hydrostatic pressure does not change over the scale of variation of the velocity. In fact, conservation of momentum implies that the element P_{yy} of the pressure tensor is strictly constant. The temperature is also constant in the linear limit. However, our results indicate that the space dependence of the temperature is coupled to that of the velocity. The former is a linear function of the latter with a (dimensionless) slope equal to $\omega^2/\epsilon = 0.445$. (iv) The fluxes exhibit a rich nonlinear behavior. In contrast to Eq. (17), one has $P_{xx} > P_{zz} > P_{yy}$. In particular, we have found $P_{yy}/P_{xx} = 0.031$ and $P_{zz}/P_{xx} = 0.081$ if $\mu = 2.5240$. The Newton law, Eq. (18b), is generalized by Eq. (50b), where $\mu/2\epsilon = 0.003\,43$ (shear thinning effect). The heat flux parallel to the gradient direction, Eq. (57), is a nonlinear function of the distance, while it vanishes in the limit of small force strength. (v) Finally, the explicit expression for the velocity distribution function allows one to analyze with more detail the anisotropy and nonlinear features already learned from its first few moments.

The solution derived in this paper can be useful to gaining insight into some of the peculiarities of the nonlinear behavior of even simple systems (such as a dilute gas of Maxwell molecules) in simple nonequilibrium states (such as the planar Poiseuille flow). It must be emphasized that the system has been assumed to be described by the BGK model, rather than by the Boltzmann equation. The latter would require the use of numerical techniques, such as the direct simulation Monte Carlo method.[14]

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APPENDIX A: CONSISTENCY CONDITION FOR THE FLOW VELOCITY

The consistency condition (35b) for the x component of the velocity is analyzed in this appendix. Only the first two terms in the right side of Eq. (34) contribute. Therefore

$$\int d\mathbf{v}^*\, v_x^*(v_y^*\,\partial_s + \mu T^* D_x)^k f_{\rm LE}^* = C_k\,\partial_s^k u_x^*\, T^{*(k-2)/2} - \mu C_{k-1} \sum_{l=0}^{k-1} \partial_s^l T^*\,\partial_s^{k-1-l} T^{*(k-3)/2}$$
$$= -\epsilon C_k k!(-\omega^2)^{(k-2)/2} - \mu C_{k-1}(-\omega^2)^{(k-1)/2} \sum_{l=0}^{k-1} \partial_s^l s^2\,\partial_s^{k-1-l} s^{k-3}, \tag{A1}$$

for $k \geqslant 2$. The summation in the last term is

$$\sum_{l=0}^{k-1} \partial_s^l s^2\,\partial_s^{k-1-l} s^{k-3} = (k-3)! \sum_{l=0}^{k-1} l(l-1) = \frac{k!}{3}. \tag{A2}$$

On the other hand, for $k = 1$ we have

$$\int d\mathbf{v}^*\, v_x^*(v_y^*\,\partial_s + \mu T^* D_x) f_{\rm LE}^* = -\mu. \tag{A3}$$

Putting together all of this in Eq. (35b), we obtain

$$\frac{\epsilon}{\mu} = \Phi(\omega) = \frac{1 - (\omega^2/3)\,\Sigma_{k=0}^{\infty}\,(2k+3)!(2k+1)!!(-\omega^2)^k}{\Sigma_{k=0}^{\infty}\,(2k+2)!(2k+1)!!(-\omega^2)^k}. \tag{A4}$$

The function $\Phi(\omega)$ is expressed in terms of asymptotic series. A more useful representation can be obtained by means of Borel summations.[13,15] Let us introduce the auxiliary function

$$F_0(\omega) = \frac{2}{\omega^2} \int_0^\infty dt\, t e^{-t^2/2} K_0(2\sqrt{t/\omega}), \tag{A5}$$

K_0 being the zeroth-order modified Bessel function. For computational purposes, the function F_0 can also be represented by a generalized Frobenius series around the point at infinity ($\omega^{-1} = 0$). Its explicit expression has been obtained elsewhere[16] and will not be repeated here. We also define

$$F_r(\omega) = \left(\frac{1}{2\omega}\frac{d}{d\omega}\,\omega^2\right)^r F_0(\omega). \tag{A6}$$

The expansion of F_0 around $\omega = 0$ is asymptotic.[16] From it, one can obtain that of F_r:

$$F_r(\omega) = \sum_{k=0}^{\infty} (k+1)^r (2k+1)!(2k+1)!!(-\omega^2)^k. \tag{A7}$$

Comparison with Eq. (A4) shows that

$$\Phi(\omega) = \frac{1 - \frac{2}{3}\omega^2[F_1(\omega) + 2F_2(\omega)]}{2F_1(\omega)}. \tag{A8}$$

APPENDIX B: CONSISTENCY CONDITION FOR THE TEMPERATURE

The implications of Eq. (35c) are worked out in this appendix. First, we define

$$I_{x,y,z} \equiv \sum_{k=1}^{\infty} (-1)^k \int d\mathbf{v}^*\, v_{x,y,z}^{*2} \times (v_y^*\partial_s + \mu T^* D_x)^k f_{\rm LE}^*. \tag{B1}$$

Equation (35c) is then equivalent to $I_x + I_y + I_z = 0$. Here, I_y and I_z are easy to compute:

$$I_y = \sum_{k=1}^{\infty} (-1)^k \int d\mathbf{v}^*\, v_y^{*k+2}\,\partial_s^k f_{\rm LE}^* = \sum_{k=1}^{\infty} C_{k+2}\,\partial_s^k T^{*k/2}$$

$$= -\omega^2 \sum_{k=0}^{\infty} (2k+3)!(2k+1)!!(-\omega^2)^k$$

$$= -2\omega^2[F_1(\omega) + 2F_2(\omega)], \quad \text{(B2)}$$

$$I_z = \sum_{k=1}^{\infty} (-1)^k \int d\mathbf{v}^* \, v_z^{*2} v_y^{*k} \partial_s^k f_{\rm LE}^*$$

$$= \sum_{k=1}^{\infty} C_k \, \partial_s^k T^{*k/2}$$

$$= -\omega^2 \sum_{k=0}^{\infty} (2k+2)!(2k+1)!!(-\omega^2)^k$$

$$= -2\omega^2 F_1(\omega), \quad \text{(B3)}$$

where in the last steps we have made use of Eq. (A7).

We proceed now to the evaluation of I_x. Only the three terms explicitly written in Eq. (34) give nonvanishing contributions. Let us analyze each one separately: $I_x = I_x^{(1)} + I_x^{(2)} + I_x^{(3)}$, where

$$I_x^{(1)} = \sum_{k=1}^{\infty} (-1)^k \int d\mathbf{v}^* \, v_x^{*2} v_y^{*k} \partial_s^k f_{\rm LE}, \quad \text{(B4)}$$

$$I_x^{(2)} = -2\mu \sum_{k=1}^{\infty} (-1)^k \sum_{l=0}^{k-1} \partial_s^l T^* \partial_s^{k-1-l} \times \int d\mathbf{v}^* \, v_x^* v_y^{*k-1} f_{\rm LE}, \quad \text{(B5)}$$

$$I_x^{(3)} = 2\mu^2 \sum_{k=2}^{\infty} (-1)^k \sum_{l=0}^{k-2} \sum_{r=0}^{k-2-l} \partial_s^l T^* \partial_s^r T^* \times \partial_s^{k-2-l-r} \int d\mathbf{v}^* \, v_y^{*k-2} f_{\rm LE}. \quad \text{(B6)}$$

The first contribution is

$$I_x^{(1)} = I_z + \sum_{k=1}^{\infty} C_k \, \partial_s^k u_x^{*2} \, T^{*(k-2)/2}$$

$$= I_z + \epsilon^2[\Theta^{(1)}(\omega) + s^2 \Psi^{(1)}(\omega)], \quad \text{(B7)}$$

where

$$\Theta^{(1)}(\omega) = \sum_{k=0}^{\infty} (2k+4)!(2k+3)!!(k+1)(-\omega^2)^k$$

$$= 2\omega^{-2}[F_1(\omega) - F_2(\omega)], \quad \text{(B8)}$$

$$\Psi^{(1)}(\omega) = \frac{1}{2} \sum_{k=0}^{\infty} (2k+4)!(2k+1)!!(-\omega^2)^k$$

$$= F_1(\omega) + 2F_2(\omega) + \frac{\omega^{-2}}{2}[1 - F_0(\omega)]. \quad \text{(B9)}$$

Next, we consider $I_x^{(2)}$:

$$I_x^{(2)} = 2\mu\left(u_x^* + \sum_{k=3}^{\infty} C_{k-1} \sum_{l=0}^{k-1} \partial_s^l T^* \times \partial_s^{k-1-l} u_x^* \, T^{*(k-3)/2}\right). \quad \text{(B10)}$$

The l summation yields

$$\sum_{l=0}^{k-1} \partial_s^l T^* \partial_s^{k-1-l} u_x^* \, T^{*(k-3)/2} = -\epsilon\Bigg[s^2(-\omega^2)^{(k-1)/2} \frac{(k-1)!}{2} \sum_{l=0}^{k-1} (l+1)(l+2) + (-\omega^2)^{(k-3)/2}(k-3)! \sum_{l=0}^{k-1}\left((k-1)(k-2) + \frac{k-3}{2} l(l-1)\right)\Bigg]$$

$$= -\epsilon\left(s^2(-\omega^2)^{(k-1)/2} \frac{(k+2)!}{6} + (-\omega^2)^{(k-3)/2} \frac{k+3}{6} k!\right). \quad \text{(B11)}$$

Thus Eq. (B10) becomes

$$I_x^{(2)} = -2\mu\epsilon[\Theta^{(2)}(\omega) + s^2 \Psi^{(2)}(\omega)], \quad \text{(B12)}$$

where

$$\Theta^{(2)}(\omega) = \frac{1}{3} \sum_{k=0}^{\infty} (2k+3)!(2k+1)!!(k+3)(-\omega^2)^k$$

$$= F_1(\omega) + 2F_2(\omega) + \frac{\omega^{-2}}{6}[1 - F_0(\omega)], \quad \text{(B13)}$$

$$\Psi^{(2)}(\omega) = 1 + \frac{1}{6} \sum_{k=1}^{\infty} (2k+3)!(2k-1)!!(-\omega^2)^k$$

$$= \frac{1}{3}[1 + F_0(\omega) + F_1(\omega)] - \frac{2}{3}\omega^2[F_1(\omega) + 2F_2(\omega)]. \quad \text{(B14)}$$

Finally, we evaluate $I_x^{(3)}$. After performing the velocity integration, Eq. (B6) becomes

$$I_x^{(3)} = 2\mu^2\left(T^* + \sum_{k=4}^{\infty} C_{k-2} \sum_{l=0}^{k-2} \sum_{r=0}^{k-2-l} \partial_s^l T^* \times \partial_s^r T^* \partial_s^{k-2-l-r} T^{*(k-4)/2}\right). \quad \text{(B15)}$$

After a tedious calculation, the r and l summations yield

$$\sum_{l=0}^{k-2} \sum_{r=0}^{k-2-l} \partial_s^l T^* \partial_s^r T^* \partial_s^{k-2-l-r} T^{*(k-4)/2}$$

$$= s^2(-\omega^2)^{k/2} \frac{(k-4)!}{2} \sum_{l=0}^{k-2} \sum_{r=0}^{k-2-l} (l+2)(l+1)(l+r)(l+r-1) + (-\omega^2)^{(k-2)/2}(k-6)!(k-4)$$

$$\times \sum_{l=0}^{k-2} \sum_{r=0}^{k-2-l} \left[(k-5)(l+r)(l+r-1) + l(l-1)\left((k-5) + \frac{l+r-2}{2}(l+r-3) \right) \right]$$

$$= s^2(-\omega^2)^{k/2} \frac{(k+2)!}{36} + (-\omega^2)^{(k-2)/2} \frac{k+8}{36} k!. \tag{B16}$$

Insertion of Eq. (B16) into Eq. (B15) gives

$$I_x^{(3)} = \mu^2[\Theta^{(3)}(\omega) + s^2\Psi^{(3)}(\omega)], \tag{B17}$$

where

$$\Theta^{(3)}(\omega) = 2 + \frac{1}{9}\sum_{k=1}^{\infty} (2k+2)!(2k-1)!!(k+5)(-\omega^2)^k = \frac{1}{18}[25 + 9F_0(\omega) + 2F_1(\omega)] - \omega^2[F_1(\omega) + 2F_2(\omega)], \tag{B18}$$

$$\Psi^{(3)}(\omega) = -2\omega^2 + \frac{1}{18}\sum_{k=2}^{\infty}(2k+2)!(2k-3)!!(-\omega^2)^k = -\frac{\omega^2}{9}\{9 + 3F_0(\omega) + 4F_1(\omega) + 2F_2(\omega) - 6\omega^2[F_1(\omega) + 2F_2(\omega)]\}. \tag{B19}$$

Consequently, the condition (35) implies that

$$s^2[\epsilon^2\Psi^{(1)}(\omega) - 2\epsilon\mu\Psi^{(2)}(\omega) + \mu^2\Psi^{(3)}(\omega)] + [\epsilon^2\Theta^{(1)}(\omega) - 2\epsilon\mu\Theta^{(2)}(\omega) + \mu^2\Theta^{(3)}(\omega)] - \Omega(\omega) = 0, \tag{B20}$$

with

$$\Omega(\omega) = -(I_y + 2I_z) = 2\omega^2[3F_1(\omega) + 2F_2(\omega)]. \tag{B21}$$

Taking into account that ϵ, μ, and ω are already related by Eq. (37), Eq. (B20) becomes

$$s^2\mu^2\Psi(\omega) + \mu^2\Theta(\omega) - \Omega(\omega) = 0, \tag{B22}$$

where

$$\psi(\omega) \equiv \psi^{(1)}(\omega)[\Phi(\omega)]^2 - 2\psi^{(2)}(\omega)\Phi(\omega) + \psi^{(3)}(\omega), \tag{B23}$$

$$\Theta(\omega) \equiv \Theta^{(1)}(\omega)[\Phi(\omega)]^2 - 2\Theta^{(2)}(\omega)\Phi(\omega) + \Theta^{(3)}(\omega). \tag{B24}$$

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[16] See Appendix B of Ref. 13. Notice a misprint in Eq. (B3): e^{-u^2} should read $e^{-u^2/2}$.