ON THE RELAXATION OF THE LOW-ENERGY REGION IN THE BOLTZMANN EQUATION

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Received 11 September 1984

The relaxation of the low-energy region is numerically analyzed within the model-Boltzmann equation introduced by Tjon and Wu. For teo-delta-peak initial distributions, two overpopulation effects are observed, one of them showing a nature similar to the Tjon effect.

One of the main recent achievements in kinetic theory has been the finding of exact solutions of the Boltzmann equation [1]. For some kinds of initial conditions, the solutions present a transient overpopulation phenomenon in the high-energy region of the distribution function, usually referred to as Tjon effect [1,2]. The same effect has also been observed in dense systems by means of molecular dynamics experiments [3]. Besides, these experiments showed that a similar overpopulation effect takes place in the low-energy region.

The aim of this letter is to extend Tjon's numerical calculations [2] in order to check wether there is an overpopulation of a "proximity" effect that, although giving rise to overpopulation, has a quite different origin than the Tjon effect.

We consider the two-dimensional discretized Tjon–Wu model of the Boltzmann equation defined by [4]

$$\partial F_n(t)/\partial t + F_n(t) = \Delta x \sum_{k=n}^{\infty} \epsilon_{k-n} k^{-1} \sum_{j=0}^{k} \epsilon_j \epsilon_{k-j} F_j(t) F_{k-j}(t) , \quad (1)$$

where $F_n(t)$ is the distribution function corresponding to the energy $x = n\Delta x$ and $\epsilon_j = 1 - \frac{1}{2}\delta_{j,0}$, $\delta_{j,k}$ being the Kronecker delta. Eq. (1) has the stationary solution

$$F_n(\infty) = A r^{-n} , \quad (2)$$

with

$$r = \Delta x + [1 + (\Delta x)^2]^{1/2}, \quad A = 2(r-1)/(r+1)\Delta x .$$

We have solved (1), using the same method as in ref. [2], for initial conditions of the form

$$F_n(0) = C_\alpha \delta_{n,n_\alpha} + C_\beta \delta_{n,n\beta} , \quad (3)$$

where C_α and C_β are determined from the normalization conditions. Provided that $n_\beta > n_\alpha \neq 0$, they are given by $C_\alpha = (n_\beta - 1/\Delta x)/(n_\beta - n_\alpha)$ and $C_\beta = (1/\Delta x - n_\alpha)/(n_\beta - n_\alpha)$. Tjon [2] studied the cases $(n_\alpha, n_\beta) = (1, 5)$ and $(1, 9)$ taking $\Delta x = 40/99$. These conditions are not adequate to study the relaxation of the distribution function for $n < n_\alpha$, as this implies $n = 0$. On the other hand, according to molecular dynamics [3], the overpopulation effect we intend to observe is expected for $n < n_\alpha$. Then, we have taken $\Delta x = 4/99$ and $n_\alpha = 10$, in such a way that the energy $x_\alpha = n_\alpha \Delta x$ is

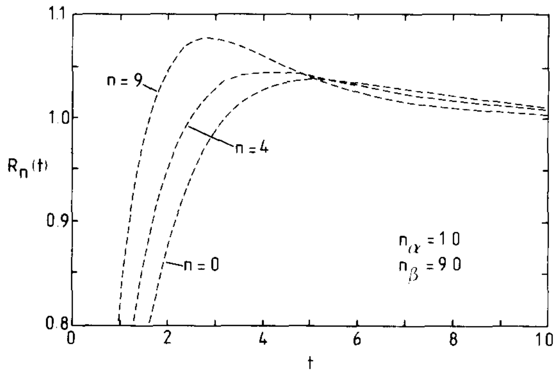

Fig. 1. Time evolution of the relative population $R_n(t) = F_n(t)/F_n(\infty)$ for several values of n smaller than n_α. The initial distribution is $F_n(0) = 0$ except for $n = n_\alpha = 10$ and $n = n_\beta = 90$.

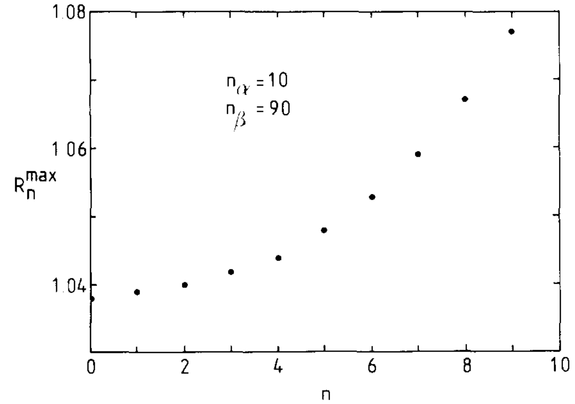

Fig. 2. Maximum value R_n^{max} reached by the relative population for $n < n_\alpha$. The initial condition corresponds to (n_α, n_β) = (10, 90).

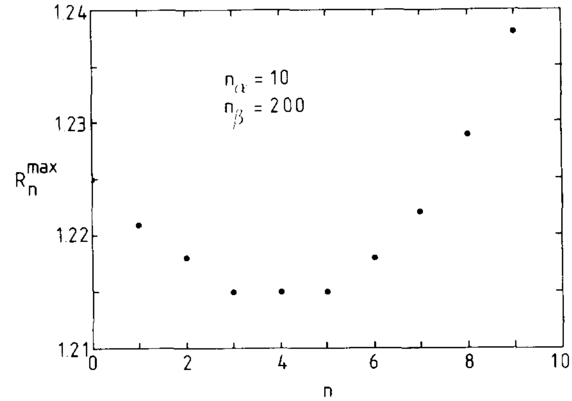

Fig. 4. The same as in fig. 2, but for the initial condition (n_α, n_β) = (10, 200).

the same as above. We have considered values of n_β for which the Tjon effect is expected according to Hauge's criterion [5], namely, n_β = 90 and 200.

In fig. 1 we show the time evolution of the ratio $R_n(t) = F_n(t)/F_n(\infty)$ for n = 0, 4, and 9, corresponding to the initial conditions (n_α, n_β) = (10, 90). It is seen that the relaxation is not monotone, but there is a transient overpopulation. Then, one could conclude that something analogous to the Tjon effect happens at low energies. To analize more clearly the nature of this overpopulation, we have plotted in fig. 2 the maximum value R_n^{max} for all $n < n_\alpha$. We notice that R_n^{max} decreases as the distance $n_\alpha - n$ increases. One of the main features of the Tjon effect is that it increases as $n - n_\beta$ does. For this reason, the overpopulation observed in fig. 1, at least its dominant part, cannot be

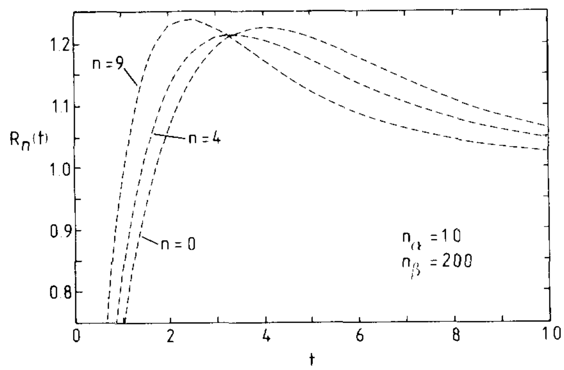

Fig. 3. The same as in fig. 1, but for the initial condition (n_α, n_β) = (10, 200).

associated with an effect similar to the one noticed by Tjon. Instead, it is related to the broadening of the initial peak at $n = n_\alpha$. This "proximity effect" has also been observed in molecular dynamics [3] and it is always present at both sides of n_α and n_β, even when there is no Tjon effect.

According to Hauge's criterion, the Tjon effect for a given n is expected to increase with n_β when keeping n_α constant. On the other hand, the physical picture of the "proximity effect" shows that, near n_α, it should be qualitatively insensitive to the value of n_β. So, in order to make the Tjon-like effect dominant, we have increased the value of n_β. In figs. 3 and 4 we have represented the same as in figs. 1 and 2, respectively, but for the initial condition (n_α, n_β) = (10, 200). Now, in addition to the "proximity effect", we can identify an overpopulation that increases with the distance from n_α, i.e. a Tjon-like effect. It is likely also present in the case (10, 90), but it is hidden by the "proximity effect". Let us remark that C_α is smaller in the case (10, 90) than in the case (10, 200), and this explains why the values of R_n^{max} for n near n_α are greater in the latter case.

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