AN EXACTLY SOLVABLE MODEL OF THE BOLTZMANN EQUATION WITH REMOVAL

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A two-dimensional model of the Boltzmann equation including removal collisions is proposed. Its solution is found for a special value of the removal collision frequency. Unlike Maxwell molecules, the decreasing of the total number of particles is selective, the most energetic particles being more efficiently removed.

In the last few years, a great deal of attention has been devoted to the study of the spatially homogeneous Boltzmann equation (BE) incorporating removal events, interactions with an unbounded background host medium, and the presence of an external source [1,2]. Most of the studies deal with the Maxwell interaction law, for which the collision frequency is constant and the total particle density obeys an autonomous evolution equation [1]. (As a matter of fact, only the removal collision frequency needs to be constant [2].) In that context, particular exact solutions have been found [3,4], and the BE with removal when the host medium is a pure absorber can be easily transformed into the usual BE without removal [5,6]. For more general interaction models, the velocity dependence of the collision frequency complicates very much the problem and little is known about the influence of removal events on the corresponding BE [2].

The aim of this note is to explore removal effects on particles different from Maxwell molecules by using a simple model defined as follows. First, the interaction with the host medium is ignored [4,6]. Second, the two-dimensional very-hard-particle (VHP) model [7] is used to describe the collisions among the particles. The BE (without removal) for this interaction model is exactly solvable [8]. The problem is mathematically much harder when removal events are introduced, but it is shown here that there exists a particular value of the removal collision frequency affording for an exact solution for arbitrary initial conditions. This solution allows one to discuss quite generally the evolution of the distribution function when removal events are present.

Let us consider an isotropic scattering model. Then, the BE for a homogeneous and isotropic velocity distribution function becomes a closed kinetic equation for the energy distribution function [7]. The VHP interaction model is defined by assuming that the collision rate is proportional to the energy [7,8], giving rise to a collision frequency linear in the energy. Although the VHP model does not correspond to any physical interaction potential (it would represent an interaction "harder" than that of hard spheres), it has the mathematical advantage that its general solution can be obtained in closed form for two dimensions [8]. Here, we take the BE for this interaction model and introduce removal events in it. A fraction μ of the binary encounters between particles is assumed to give rise to removal, the remainder corresponding to elastic collisions. Therefore, the nonnegative constant μ represents the (relative) removal collision frequency. The BE for the above model reads, in dimensionless variables,

$$\frac{\partial}{\partial t}F(x,t)=\int_x^\infty du\int_0^u dy\,[F(y,t)F(u-y,t)$$

$$-(1+\mu)F(x,t)F(u-x,t)]$$

$$=\int_x^\infty du\int_0^u dy\,F(y,t)F(u-y,t)$$

$$-(1+\mu)[M_1(t)+xM_0(t)]F(x,t)\,, \qquad (1)$$

where $F(x, t)$ is the energy distribution function, and

$$M_n(t)=\int_0^\infty dx\,x^n F(x,t)$$

are the moments. In absence of removal ($\mu=0$), M_0, M_1=const, and eq. (1) becomes the BE already studied by Ernst [7,8]. From eq. (1) one gets the evolution equation for M_0:

$$dM_0/dt+2\mu M_0 M_1=0\,. \qquad (2)$$

In order to study eq. (1) it is advantageous to introduce the variables [5,6]

$$\tilde{F}(x,\tau)=F(x,t)/M_0(t),\quad \tau=\int_0^t dt'\,M_0(t')\,. \qquad (3)$$

Therefore, $\tilde{F}$ is a probability density function. While M_0 is proportional to the total number of particles present in the system, $\tilde{F}$ measures how the energy is distributed on those particles. In terms of the new variables, eq. (1) yields

$$\frac{\partial}{\partial\tau}\tilde{F}(x,\tau)=\int_x^\infty du\int_0^u dy\,\tilde{F}(y,\tau)\tilde{F}(u-y,\tau)$$

$$-[(1-\mu)\tilde{M}_1(\tau)+(1+\mu)x]\tilde{F}(x,\tau)\,, \qquad (4)$$

where

$$\tilde{M}_1(\tau)=\int_0^\infty dx\,x\tilde{F}(x,\tau)$$

is the mean energy. Once this quantity is known, application of eqs. (2) and (3) gives $M_0(t)$ and the relationship between t and τ:

$$M_0(t)\equiv N(\tau)=\exp\left(-2\mu\int_0^\tau d\tau'\,\tilde{M}_1(\tau')\right)\,, \qquad (5)$$

$$t=\int_0^\tau \frac{d\tau'}{N(\tau')}\,. \qquad (6)$$

The structure of eq. (4) suggests the introduction of the Laplace transform (or generating function of moments)

$$\tilde{G}(z,\tau)=\int_0^\infty dx\,\exp(-zx)\tilde{F}(x,\tau)\,.$$

Then, eq. (4) yields

$$\frac{\partial}{\partial\tau}\tilde{G}(z,\tau)+(1-\mu)\tilde{M}_1(\tau)-(1+\mu)\frac{\partial}{\partial z}\tilde{G}(z,\tau)$$

$$=z^{-1}[1-\tilde{G}^2(z,\tau)]\,. \qquad (7)$$

Eq. (7) is still very involved due to the presence of $\tilde{M}_1=-\partial\tilde{G}/\partial z|_{z=0}$. However, in the particular case $\mu=1$ (i.e., 50% of the collisions giving rise to particle removal), eq. (7) becomes a conventional partial differential equation whose general solution is

$$\tilde{G}(z,\tau)=\frac{\Phi(z+2\tau)-z}{\Phi(z+2\tau)+z}\quad(\mu=1)\,, \qquad (8)$$

where the function $\Phi(z)$ can be expressed in terms of the initial distribution just by making $\tau=0$ in eq. (8). This function also gives the evolution of the mean energy as $\tilde{M}_1(\tau)=2/\Phi(2\tau)$. Since $\tilde{G}(z,0)$ must vanish when $z\to\infty$ [#1], one has $\Phi(z)\to z$ in that limit. Consequently, $\tilde{M}_1(\infty)=0$ and $\tilde{G}(z,\infty)=1$, which implies $\tilde{F}(x,\infty)=\delta(x)$. Although this asymptotic form has been obtained from the exact solution in the case $\mu=1$, it is also true for any positive value of μ, as can be easily checked by substitution into eq. (7).

Thus, we can generally distinguish two removal effects on the distribution function. First, the total number of particles decreases in time. Second, due to the velocity dependence of the collision frequency for interactions repulsive enough, the most energetic

[#1] Since $\tilde{F}(x,0)$ must have a finite norm, $\tilde{F}(x,0)\sim x^{-\epsilon}$, with $\epsilon<1$, for small x. Then $\tilde{G}(z,0)\sim z^{-(1-\epsilon)}$ for large z.

particles are removed more frequently. Then, only particles with vanishing energy remain in the long time limit. On the other hand, the second effect is not present in the special case of Maxwell molecules, so that $\tilde{F}$ satisfies the usual BE and asymptotically evolves towards a maxwellian distribution [5,6]. The advantage of our VHP model for $\mu=1$ is that it allows one to work out in detail some of the general features of removal phenomena in the BE.

As an example of application of the solution (8), let us consider a maxwellian initial distribution, i.e. $F(x,0)=e^{-x}$. Then, $\tilde{G}(z,0)=(z+1)^{-1}$ and $\Phi(z)=z+2$. From eqs. (8), (5), and (6), one easily gets

$$\tilde{F}(x,\tau)=(\tau+1)e^{-(\tau+1)x} ,$$

$$\tilde{M}_1(\tau)=1/(\tau+1) , \qquad (9)$$

$$\tau+1=(3t+1)^{1/3} , \qquad (10)$$

$$M_0(t)=(3t+1)^{-2/3} . \qquad (11)$$

In this particular example, the total population decreases as $t^{-2/3}$, while the mean energy behaves as $t^{-1/3}$.

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