Granular mixtures modeled as elastic hard spheres subject to a drag force

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Granular gaseous mixtures under rapid flow conditions are usually modeled as a multicomponent system of smooth inelastic hard disks (two dimensions) or spheres (three dimensions) with constant coefficients of normal restitution α_{ij}. In the low density regime an adequate framework is provided by the set of coupled inelastic Boltzmann equations. Due to the intricacy of the inelastic Boltzmann collision operator, in this paper we propose a simpler model of elastic hard disks or spheres subject to the action of an effective drag force, which mimics the effect of dissipation present in the original granular gas. For each collision term ij, the model has two parameters: a dimensionless factor β_{ij} modifying the collision rate of the elastic hard spheres, and the drag coefficient ζ_{ij}. Both parameters are determined by requiring that the model reproduces the collisional transfers of momentum and energy of the true inelastic Boltzmann operator, yielding $\beta_{ij}=(1+\alpha_{ij})/2$ and $\zeta_{ij} \propto 1-\alpha_{ij}^2$, where the proportionality constant is a function of the partial densities, velocities, and temperatures of species i and j. The Navier-Stokes transport coefficients for a binary mixture are obtained from the model by application of the Chapman-Enskog method. The three coefficients associated with the mass flux are the same as those obtained from the inelastic Boltzmann equation, while the remaining four transport coefficients show a general good agreement, especially in the case of the thermal conductivity. The discrepancies between both descriptions are seen to be similar to those found for monocomponent gases. Finally, the approximate decomposition of the inelastic Boltzmann collision operator is exploited to construct a model kinetic equation for granular mixtures as a direct extension of a known kinetic model for elastic collisions.

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I. INTRODUCTION

Natural and industrial granular media are generally present in polydisperse form. In some cases a certain degree of polydispersity in masses and/or sizes is unavoidable, while in other cases one is dealing with a real mixture constituted by grains belonging to species characterized by distinct mechanical parameters. In conditions of rapid flow, inelastic binary collisions are the primary mechanisms and so a kinetic theory description applied to inelastic hard spheres has proven to be adequate [1,2]. In the low density regime, all the relevant information on the state of the mixture is contained in the one-particle velocity distribution functions $f_i(\mathbf{r},\mathbf{v};t)$, which obey a set of coupled Boltzmann equations,

$$(\partial_t + \mathbf{v}\cdot\nabla) f_i = \sum_{j=1}^{N} J_{ij}^{\text{inel}}[\mathbf{v}|f_i,f_j], \tag{1.1}$$

where N is the number of species and $J_{ij}^{\text{inel}}[\mathbf{v}|f_i,f_j]$ denotes the inelastic Boltzmann operator that gives the rate of change of f_i due to collisions with particles of species j. This collision operator depends parametrically on the coefficient of normal restitution $\alpha_{ij}\leqslant 1$ (here assumed to be constant) for collisions between particles of species i and j.

Obviously, the problem posed by Eq. (1.1) is much more complicated than in the case of a single granular gas. Not only one has to deal with N coupled equations, but in addition the space of parameters is much larger: there are $N-1$ independent mole fractions, $N-1$ mass ratios, $N-1$ size ratios, and $N(N+1)/2$ coefficients of restitution. An important consequence of inelasticity is the breakdown of energy equipartition, even in homogeneous and isotropic states. This means that one can associate a different granular temperature to each species [3–5], as confirmed by computer simulations [5–9] and real experiments [9,10] of agitated mixtures. In the case of small spatial gradients, the set of Boltzmann equations (1.1) can be solved by means of the Chapman-Enskog method to Navier-Stokes (NS) order. Many attempts to determine the NS transport coefficients are restricted to the quasielastic limit ($\alpha_{ij}\approx 1$), assuming an expansion around Maxwellians at the *same* temperature [11]. A more general derivation takes into account the nonequipartition of energy and determines the transport coefficients without any *a priori* limitation on the degree of dissipation [12]. The accuracy of this latter approach has been confirmed by computer simulations in the cases of the diffusion [13,14] and shear viscosity [15] coefficients.

However, most of the inhomogeneous situations are characterized by a coupling between inelasticity and spatial gradients, and so they require a description beyond the NS domain. A typical example of this coupling is represented by the simple shear flow [1,16], where non-Newtonian effects are unavoidable in the steady state. Needless to say, the analysis of this type of more general situations based on the inelastic Boltzmann equation (1.1) becomes very intricate, especially for multicomponent systems. In order to overcome these difficulties, a possible strategy consists of replacing the true Boltzmann collision operator J_{ij}^{inel} by a simpler model term that retains its physically relevant properties. Following

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an idea previously proposed for monodisperse granular gases [17], here we explore the possibility of describing the multicomponent gas of inelastic hard spheres by a model of elastic hard spheres subject to the action of an effective drag force with a different drag coefficient for each species. The parameters of the model are explicitly determined by optimizing the agreement with the collisional transfer of momentum and energy obtained from the original operator J_{ij}^{inel}. The resulting model is simpler than the original Boltzmann equation since all the dependence on the coefficients of restitution α_{ij} appears explicitly outside the collision term.

The paper is organized as follows. In Sec. II the model of elastic hard spheres subject to a drag force is formulated, some technical details being relegated to Appendix A. In order to assess the reliability of the model, in Sec. III we present its Chapman-Enskog solution, the expressions of the NS transport coefficients being given in Appendix B. In addition, the dependence of the transport coefficients on dissipation is compared with known results derived from the Boltzmann equation for inelastic hard spheres [12,18]. The paper is closed by a discussion of the results in Sec. IV, where a kinetic model equation for granular mixtures is proposed in Appendix C.

II. PROPOSAL OF THE MODEL

A. The inelastic Boltzmann equation

Consider an N-component mixture composed by smooth inelastic disks ($d=2$) or spheres ($d=3$) of masses m_i and diameters σ_i, $i=1,\ldots,N$. The inelasticity of collisions between a sphere of species i and a sphere of species j is characterized by a constant coefficient of restitution $0<\alpha_{ij}\leqslant 1$. In the low density regime, the distribution functions $f_i(\mathbf{r},\mathbf{v};t)$ are determined from the set of nonlinear Boltzmann equations (1.1), where the Boltzmann collision operator is

$$J_{ij}^{\text{inel}}[\mathbf{v}_1|f_i,f_j]=\sigma_{ij}^{d-1}\int d\mathbf{v}_2\int d\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})\times[\alpha_{ij}^{-2}f_i(\mathbf{r},\mathbf{v}_1',t)f_j(\mathbf{r},\mathbf{v}_2',t)-f_i(\mathbf{r},\mathbf{v}_1,t)f_j(\mathbf{r},\mathbf{v}_2,t)]. \tag{2.1}$$

In Eq. (2.1), d is the dimensionality of the system, $\sigma_{ij}=(\sigma_i+\sigma_j)/2$, $\hat{\boldsymbol{\sigma}}$ is a unit vector along the line of centers, Θ is the Heaviside step function, and $\mathbf{g}_{12}=\mathbf{v}_1-\mathbf{v}_2$ is the relative velocity. The primes on the velocities denote the initial values $\{\mathbf{v}_1',\mathbf{v}_2'\}$ that lead to $\{\mathbf{v}_1,\mathbf{v}_2\}$ following a binary (restituting) collision:

$$\mathbf{v}_1'=\mathbf{v}_1-\mu_{ji}(1+\alpha_{ij}^{-1})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})\hat{\boldsymbol{\sigma}}, \tag{2.2a}$$

$$\mathbf{v}_2'=\mathbf{v}_2+\mu_{ij}(1+\alpha_{ij}^{-1})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})\hat{\boldsymbol{\sigma}}, \tag{2.2b}$$

where $\mu_{ij}\equiv m_i/(m_i+m_j)$, so that $\mu_{ij}+\mu_{ji}=1$.

The relevant hydrodynamic fields are the number densities n_i, the flow velocity $\mathbf{u}$, and the temperature T. They are defined in terms of moments of the distributions f_i as

$$n_i=\int d\mathbf{v}f_i(\mathbf{v}), \tag{2.3}$$

$$\rho\mathbf{u}=\sum_{i=1}^{N}m_in_i\mathbf{u}_i=\sum_{i=1}^{N}m_i\int d\mathbf{v}\mathbf{v}f_i(\mathbf{v}), \tag{2.4}$$

$$nT=p=\sum_{i=1}^{N}n_iT_i=\sum_{i=1}^{2}\frac{m_i}{d}\int d\mathbf{v}V^2f_i(\mathbf{v}), \tag{2.5}$$

where $\mathbf{V}=\mathbf{v}-\mathbf{u}$ is the peculiar velocity, $n=\sum_{i=1}^{N}n_i$ is the total number density, $\rho=\sum_{i=1}^{N}\rho_i=\sum_{i=1}^{N}m_in_i$ is the total mass density, and p is the pressure. Furthermore, the second equality of Eq. (2.4) and the third equality of Eq. (2.5) define the flow velocity $\mathbf{u}_i$ and the kinetic temperature T_i for each species, respectively.

A collision ij conserves the particle number of each species and the total momentum:

$$\int d\mathbf{v}J_{ij}^{\text{inel}}[\mathbf{v}|f_i,f_j]=0, \tag{2.6}$$

$$m_i\int d\mathbf{v}\mathbf{v}J_{ij}^{\text{inel}}[\mathbf{v}|f_i,f_j]+m_j\int d\mathbf{v}\mathbf{v}J_{ji}^{\text{inel}}[\mathbf{v}|f_j,f_i]=\mathbf{0}. \tag{2.7}$$

However, unless $\alpha_{ij}=1$, the collision ij does not conserve the kinetic energy, so that

$$m_i\int d\mathbf{v}v^2J_{ij}^{\text{inel}}[\mathbf{v}|f_i,f_j]+m_j\int d\mathbf{v}v^2J_{ji}^{\text{inel}}[\mathbf{v}|f_j,f_i]=-\Omega_{ij}, \tag{2.8}$$

where $\Omega_{ij}\geqslant 0$. The total "cooling rate" due to inelastic collisions among all species is given by

$$\zeta=\frac{1}{2dnT}\sum_{i,j=1}^{N}\Omega_{ij}, \tag{2.9}$$

so that the rate of change of the granular temperature T due to all the collisions is

$$\left.\frac{\partial T}{\partial t}\right|_{\text{coll}}\equiv\frac{1}{nd}\sum_{i,j}m_i\int d\mathbf{v}V^2J_{ij}^{\text{inel}}[\mathbf{v}|f_i,f_j]=-\zeta T. \tag{2.10}$$

B. Model of elastic hard spheres with a drag force

The dependence of the collision operator J_{ij}^{inel} on α_{ij} is rather involved since it appears as the factor α_{ij}^{-2} in the gain term and also through the scattering rules (2.2). This represents an additional difficulty with respect to the elastic operator J_{ij}^{el}. In order to simplify the α_{ij} dependence of the inelastic collision operator, we propose the following model of elastic particles subject to a drag force [17]:

$$J_{ij}^{\text{inel}}[\mathbf{v}|f_i,f_j]\to\beta_{ij}J_{ij}^{\text{el}}[\mathbf{v}|f_i,f_j]+\frac{\zeta_{ij}}{2}\frac{\partial}{\partial\mathbf{v}}\cdot[(\mathbf{v}-\mathbf{u}_i)f_i(\mathbf{v})]\equiv K_{ij}[\mathbf{v}|f_i,f_j], \tag{2.11}$$

where β_{ij} and ζ_{ij} are determined by optimizing the agreement between the model and the true operator. The dimensionless factor β_{ij} modifies the collision rate of the elastic spheres to mimic that of the inelastic spheres. The quantity $\zeta_{ij}\geqslant 0$ is the coefficient of the drag force $\mathbf{F}_{ij}=-(m_i\zeta_{ij}/2)(\mathbf{v}-\mathbf{u}_i)$ felt by the elastic spheres of species i. This nonconservative force intends to mimic the loss of energy that the true inelastic spheres of species i suffer when colliding with spheres of species j. For simplicity, the drag force $\mathbf{F}_{ij}$ has been chosen proportional to the velocity relative to the mean flow velocity of species i. According to the model (2.11), the Boltzmann equation (1.1) becomes

$$\partial_t f_i + \mathbf{v}\cdot\nabla f_i + \frac{1}{m_i}\sum_{j=1}^{N}\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{F}_{ij}f_i) = \sum_{j=1}^{N}\beta_{ij}J_{ij}^{\mathrm{el}}[\mathbf{v}|f_i,f_j]. \tag{2.12}$$

In this way, the problem of a mixture of *inelastic* hard spheres is mapped, via a renormalization of the collision rate and the introduction of a drag force, onto the problem of a mixture of *elastic* hard spheres.

The model (2.11) trivially satisfies the mass conservation equation (2.6). In addition, if we assume the symmetry relation $\beta_{ij}=\beta_{ji}$ (to be confirmed later), the momentum conservation equation (2.7) is also verified. Finally, Eq. (2.8) yields

$$n_i\zeta_{ij}\widetilde{T}_i + n_j\zeta_{ji}\widetilde{T}_j = \frac{1}{d}\Omega_{ij}, \tag{2.13}$$

where we have introduced the quantity

$$\widetilde{T}_i = \frac{m_i}{dn_i}\int d\mathbf{v}(\mathbf{v}-\mathbf{u}_i)^2 f_i = T_i - \frac{m_i}{d}(\mathbf{u}_i-\mathbf{u})^2. \tag{2.14}$$

Note that, according to Eqs. (2.9) and (2.13), the cooling rate ζ of the mixture can be expressed in terms of the drag coefficients ζ_{ij} as

$$\zeta = \frac{1}{T}\sum_{i,j=1}^{N} x_i\zeta_{ij}\widetilde{T}_i, \tag{2.15}$$

where $x_i=n_i/n$ is the mole fraction of species i.

Thus far, the parameters β_{ij} and ζ_{ij} of the model remain unknown, except for the constraint (2.13). In order to determine them, we impose that the collisional transfer of momentum and energy of species i due to collisions with particles of species j must be the same as those given by the true Boltzmann equation,

$$\int d\mathbf{v}\mathbf{v}J_{ij}^{\mathrm{inel}}[\mathbf{v}|f_i,f_j] = \int d\mathbf{v}\mathbf{v}K_{ij}[\mathbf{v}|f_i,f_j], \tag{2.16}$$

$$\int d\mathbf{v}v^2 J_{ij}^{\mathrm{inel}}[\mathbf{v}|f_i,f_j] = \int d\mathbf{v}v^2 K_{ij}[\mathbf{v}|f_i,f_j]. \tag{2.17}$$

This gives

$$\beta_{ij} = \frac{\int d\mathbf{v}\mathbf{v}J_{ij}^{\mathrm{inel}}[\mathbf{v}|f_i,f_j]}{\int d\mathbf{v}\mathbf{v}J_{ij}^{\mathrm{el}}[\mathbf{v}|f_i,f_j]}, \tag{2.18}$$

$$\zeta_{ij} = \frac{m_i}{dn_i\widetilde{T}_i}\left\{\beta_{ij}\int d\mathbf{v}v^2 J_{ij}^{\mathrm{el}}[\mathbf{v}|f_i,f_j] - \int d\mathbf{v}v^2 J_{ij}^{\mathrm{inel}}[\mathbf{v}|f_i,f_j]\right\}. \tag{2.19}$$

The collision integrals can be simplified by using the property [2]

$$\int d\mathbf{v}_1 h(\mathbf{v}_1)J_{ij}^{\mathrm{inel}}[\mathbf{v}_1|f_i,f_j] = \sigma_{ij}^{d-1}\int d\mathbf{v}_1\int d\mathbf{v}_2 f_i(\mathbf{v}_1)f_j(\mathbf{v}_2) \times\int d\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12}) \times[h(\mathbf{v}_1'')-h(\mathbf{v}_1)], \tag{2.20}$$

where $h(\mathbf{v})$ is an arbitrary test function and

$$\mathbf{v}_1'' = \mathbf{v}_1 - \mu_{ji}(1+\alpha_{ij})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})\hat{\boldsymbol{\sigma}} \tag{2.21}$$

is the postcollisional velocity of a particle of species i. Taking $h(\mathbf{v})=\mathbf{v}$ in Eq. (2.20), Eq. (2.18) simply becomes

$$\beta_{ij} = \frac{1+\alpha_{ij}}{2}, \tag{2.22}$$

in agreement with the symmetry relation $\beta_{ij}=\beta_{ji}$. Next, taking $h(\mathbf{v})=v^2$ and making use of

$$v_1''^2 - v_1^2 = -\mu_{ji}(1+\alpha_{ij})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})[2(\hat{\boldsymbol{\sigma}}\cdot\mathbf{G}_{12}) + \mu_{ji}(1-\alpha_{ij}) \times(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})], \tag{2.23}$$

where $\mathbf{G}_{12}=\mu_{ij}\mathbf{v}_1+\mu_{ji}\mathbf{v}_2$ is the center of mass velocity, one obtains from Eq. (2.19)

$$\zeta_{ij} = \frac{\pi^{(d-1)/2}}{d\Gamma[(d+3)/2]}(1-\alpha_{ij}^2)\frac{n_j m_i\mu_{ji}^2\sigma_{ij}^{d-1}}{\widetilde{T}_i}\langle g_{12}^3\rangle_{ij}. \tag{2.24}$$

Here,

$$\langle g_{12}^3\rangle_{ij} = \frac{1}{n_in_j}\int d\mathbf{v}_1\int d\mathbf{v}_2 f_i(\mathbf{v}_1)f_j(\mathbf{v}_2)g_{12}^3 \tag{2.25}$$

and use has been made of the result [19]

$$\int d\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g}_{12})^3 = \frac{\pi^{(d-1)/2}}{\Gamma[(d+3)/2]}g_{12}^3. \tag{2.26}$$

Inserting Eq. (2.24) into Eq. (2.13) we obtain

$$\Omega_{ij} = \frac{\pi^{(d-1)/2}}{\Gamma[(d+3)/2]}(1-\alpha_{ij}^2)n_in_j\frac{m_im_j}{m_i+m_j}\sigma_{ij}^{d-1}\langle g_{12}^3\rangle_{ij}. \tag{2.27}$$

According to Eq. (2.24), the drag coefficient ζ_{ij} is a positive definite quantity which only vanishes if the ij collisions are elastic ($\alpha_{ij}=1$). It can be interpreted as the cooling rate of species i due to the inelasticity of collisions with particles of species j. To make this more explicit, let us consider the contribution of collisions with particles of species j to the rate of change of the partial temperature $\widetilde{T}_i$, i.e.,

$$\left.\frac{\partial\widetilde{T}_i}{\partial t}\right|_{\text{coll},j} \equiv \frac{m_i}{dn_i}\int d\mathbf{v}(\mathbf{v}-\mathbf{u}_i)^2 J_{ij}^{\text{inel}}[\mathbf{v}|f_i,f_j]$$
$$= \beta_{ij}\frac{m_i}{dn_i}\int d\mathbf{v}(\mathbf{v}-\mathbf{u}_i)^2 J_{ij}^{\text{el}}[\mathbf{v}|f_i,f_j] - \zeta_{ij}\widetilde{T}_i, \tag{2.28}$$

where in the last step use has been made of Eqs. (2.16) and (2.17). The first term on the right-hand side represents the contribution to the rate of change not directly associated with inelasticity (except for the presence of the factor β_{ij}). This term can be either positive or negative, depending on f_i and f_j. In particular, if $\mathbf{u}_i=\mathbf{u}_j$ and f_i and f_j are approximated by Maxwellians, the sign of this term is the same as that of the temperature difference $\widetilde{T}_j-\widetilde{T}_i$. Therefore, the second term on the right-hand side of Eq. (2.28) is the genuine contribution associated with inelasticity. It must be emphasized that Eq. (2.28), with β_{ij} and ζ_{ij} given by Eqs. (2.22) and (2.24), respectively, is exact and so it is not restricted to the model (2.11).

While Eq. (2.24) is formally exact, it involves the average $\langle g_{12}^3\rangle_{ij}$, which is a functional of f_i and f_j. An estimate of this average can be obtained by assuming Gaussian forms for f_i and f_j given by

$$f_i(\mathbf{v}) = n_i\left(\frac{m_i}{2\pi\widetilde{T}_i}\right)^{d/2}\exp\left[-\frac{m_i(\mathbf{v}-\mathbf{u}_i)^2}{2\widetilde{T}_i}\right] \tag{2.29}$$

with an analogous form for $f_j(\mathbf{v})$. The distribution (2.29) is the one that, sharing with the exact distribution the first $d+2$ velocity moments, maximizes the missing information defined as $-\int d\mathbf{v} f_i(\mathbf{v})\ln f_i(\mathbf{v})$. The corresponding average $\langle g_{12}^3\rangle_{ij}$ is obtained in Appendix A by neglecting terms of order fourth and higher in the difference $\mathbf{u}_i-\mathbf{u}_j$. Inserting Eq. (A7) into Eq. (2.24), one obtains

$$\zeta_{ij} = \frac{1}{2}\xi_{ij}\mu_{ji}^2\left[1+\frac{m_i\widetilde{T}_j}{m_j\widetilde{T}_i}+\frac{3}{2d}\frac{m_i}{\widetilde{T}_i}(\mathbf{u}_i-\mathbf{u}_j)^2\right](1-\alpha_{ij}^2), \tag{2.30}$$

where

$$\xi_{ij} = \frac{4\pi^{(d-1)/2}}{d\Gamma(d/2)}n_j\sigma_{ij}^{d-1}\left(\frac{2\widetilde{T}_i}{m_i}+\frac{2\widetilde{T}_j}{m_j}\right)^{1/2} \tag{2.31}$$

is an effective collision frequency of species i due to collisions with particles of species j. Note that $n_i\xi_{ij}=n_j\xi_{ji}$, while $m_in_i\widetilde{T}_i\zeta_{ij}=m_jn_j\widetilde{T}_j\zeta_{ji}$. Equation (2.30) reduces to the one derived in Ref. [17] when $\mathbf{u}_i=\mathbf{u}_j$.

In summary, the model is defined by the replacement (2.11) with the parameters β_{ij} and ζ_{ij} given by Eqs. (2.22) and (2.30), respectively. While β_{ij} depends only on the coefficient of restitution α_{ij}, the cooling rate ζ_{ij} is also a function of the masses and sizes of particles of species i and j, as well as of the first few velocity moments (density, flow velocity, and partial temperature) of both distributions f_i and f_j. The expressions of β_{ij} and ζ_{ij} preserve the collisional transfer of momentum and energy of the true inelastic Boltzmann equation, although in the latter case we have considered the Gaussian forms (2.29) for f_i and f_j parametrized by their first $d+2$ moments in order to obtain explicit results.

It is worth mentioning that, while the model (2.11) is mathematically simpler than the original Boltzmann operator, its functional dependence on f_i and f_j through the corresponding flow velocities and partial temperatures is highly nonlinear. Therefore, the bilinear property

$$J_{ij}^{\text{inel}}[\mathbf{v}|\lambda_i f_i,\lambda_j f_j] = \lambda_i\lambda_j J_{ij}^{\text{inel}}[\mathbf{v}|f_i,f_j] \tag{2.32}$$

is satisfied, but not the other bilinear property

$$J_{ij}^{\text{inel}}[\mathbf{v}|f_{i1}+f_{i2},f_{j1}+f_{j2}] = J_{ij}^{\text{inel}}[\mathbf{v}|f_{i1},f_{j1}] + J_{ij}^{\text{inel}}[\mathbf{v}|f_{i1},f_{j2}] + J_{ij}^{\text{inel}}[\mathbf{v}|f_{i2},f_{j1}] + J_{ij}^{\text{inel}}[\mathbf{v}|f_{i2},f_{j2}]. \tag{2.33}$$

A consequence of the failure of K_{ij} to satisfy the property (2.33) is that, in general, one does not obtain a closed equation for the total distribution function $f=\Sigma_i f_i$ in the case of mechanically equivalent particles, unless $\mathbf{u}_i=\mathbf{u}$ and $T_i=T$. However, this drawback is not relevant in most of the situations of physical interest, such as nonequilibrium steady states, since in those cases the existence of different velocities and/or temperatures is a consequence of the particles being mechanically different.

C. Homogeneous cooling state

The simplest application of the model corresponds to the so-called homogeneous cooling state (HCS) [4]. This state is characterized by the absence of gradients, so that $\mathbf{u}_i=\mathbf{u}$. In that case, the model (2.11) yields

$$\partial_t T_i = -\zeta_i T_i, \tag{2.34}$$

where

$$\zeta_i = \sum_{j=1}^N\left(-\frac{m_i}{dn_iT_i}\beta_{ij}\int d\mathbf{v}V^2 J_{ij}^{\text{el}}[\mathbf{v}|f_i,f_j]+\zeta_{ij}\right). \tag{2.35}$$

As said above in connection with Eq. (2.28), Eq. (2.35) is also valid for the inelastic Boltzmann equation, provided that ζ_{ij} is given by Eq. (2.24).

To evaluate the collision integral in Eq. (2.35), we can take again Eq. (2.29), which (since $\mathbf{u}_i=\mathbf{u}$) becomes

$$f_i \to f_{i,M} = n_i\left(\frac{m_i}{2\pi T_i}\right)^{d/2}\exp(-m_iV^2/2T_i). \tag{2.36}$$

This yields

$$-\frac{m_i}{dn_iT_i}\int d\mathbf{v}V^2 J_{ij}^{\text{el}}[\mathbf{v}|f_i,f_j] = 2\xi_{ij}\frac{m_im_j}{(m_i+m_j)^2}\frac{T_i-T_j}{T_i}. \tag{2.37}$$

Therefore,

$$\zeta_i=\sum_{j=1}^{N}\xi_{ij}\frac{m_im_j}{(m_i+m_j)^2}(1+\alpha_{ij})\left[\frac{T_i-T_j}{T_i}+\frac{1-\alpha_{ij}}{2}\left(\frac{m_j}{m_i}+\frac{T_j}{T_i}\right)\right]. \tag{2.38}$$

This expression coincides with the one obtained from the original Boltzmann equation when the approximation (2.36) is used.

The HCS condition is $\partial_t T_i/T=0$, which implies $\zeta_1=\zeta_2=\cdots=\zeta_N$. This gives the $N-1$ temperature ratios $\gamma_i\equiv T_i/T$ as functions of the mole fractions, the mass ratios, the size ratios, and the coefficients of restitution of the mixture. Comparison with computer simulations [5–7] shows an excellent agreement with the results obtained from Eq. (2.38), even for strong dissipation.

III. NAVIER-STOKES TRANSPORT COEFFICIENTS OF THE MODEL

To assess the reliability of the model described by Eqs. (2.11), (2.22), and (2.30) in inhomogeneous situations, in this section we will consider the corresponding expressions for the NS transport coefficients of a binary mixture ($N=2$) and will compare them with the results derived from the original Boltzmann equation [12,18].

The NS coefficients are defined through the constitutive equations [12]

$$\mathbf{j}_1=-\frac{m_1m_2n}{\rho}D\,\nabla x_1-\frac{\rho}{T}D'\,\nabla T-\frac{\rho}{p}D_p\,\nabla p,\quad \mathbf{j}_2=-\mathbf{j}_1, \tag{3.1}$$

$$P_{k\ell}=p\delta_{k\ell}-\eta\left(\nabla_k u_\ell+\nabla_\ell u_k-\frac{2}{d}\delta_{k\ell}\nabla\cdot\mathbf{u}\right), \tag{3.2}$$

$$\mathbf{q}=-T^2D''\,\nabla x_1-\lambda\,\nabla T-L\,\nabla p, \tag{3.3}$$

where $\mathbf{j}_1=m_1n_1(\mathbf{u}_1-\mathbf{u})$ is the mass flux of species 1, $P_{k\ell}$ is the pressure tensor, and $\mathbf{q}$ is the heat flux. The transport coefficients in the constitutive equations are

$$\begin{pmatrix} D\\ D'\\ D_p\\ \eta\\ D''\\ \lambda\\ L\end{pmatrix}=\begin{pmatrix}\text{diffusion coefficient}\\ \text{thermal diffusion coefficient}\\ \text{pressure diffusion coefficient}\\ \text{shear viscosity}\\ \text{Dufour coefficient}\\ \text{thermal conductivity}\\ \text{pressure energy coefficient}\end{pmatrix}. \tag{3.4}$$

Explicit expressions for the above coefficients are obtained by solving the inelastic Boltzmann equation by means of the Chapman-Enskog method. These coefficients are formally given in terms of the solutions of coupled linear integral equations involving the linearized Boltzmann collision operators

$$\mathcal{L}_1f_1^{(1)}=-J_{11}^{\text{inel}}[f_1^{(0)},f_1^{(1)}]-J_{11}^{\text{inel}}[f_1^{(1)},f_1^{(0)}]-J_{12}^{\text{inel}}[f_1^{(1)},f_2^{(0)}], \tag{3.5}$$

$$\mathcal{M}_1f_2^{(1)}=-J_{12}^{\text{inel}}[f_1^{(0)},f_2^{(1)}]. \tag{3.6}$$

The corresponding expressions for the operators $\mathcal{L}_2$ and $\mathcal{M}_2$ can be easily obtained from Eqs. (3.5) and (3.6) by just making the changes $1\leftrightarrow 2$. In the above equations, $f_i^{(0)}$ is the local version of the distribution function of species i in the HCS [4].

In the case of the model (2.11), one can follow the same formal steps as in the case of the true Boltzmann equation, except that the operators $\mathcal{L}_1$ and $\mathcal{M}_1$ are now

$$\begin{aligned}\mathcal{L}_1f_1^{(1)}=&-\frac{1+\alpha_{11}}{2}(J_{11}^{\text{el}}[f_1^{(0)},f_1^{(1)}]+J_{11}^{\text{el}}[f_1^{(1)},f_1^{(0)}])\\&-\frac{1+\alpha_{12}}{2}J_{12}^{\text{el}}[f_1^{(1)},f_2^{(0)}]-\frac{\zeta_{11}+\zeta_{12}}{2}\\&\times\left(\frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{V}f_1^{(1)}-\frac{\mathbf{j}_1}{\rho_1}\cdot\frac{\partial}{\partial\mathbf{v}}f_1^{(0)}\right),\end{aligned} \tag{3.7}$$

$$\mathcal{M}_1f_2^{(1)}=-\frac{1+\alpha_{12}}{2}J_{12}^{\text{el}}[f_1^{(0)},f_2^{(1)}]. \tag{3.8}$$

In Eq. (3.7), the drag coefficients (2.30) are those of the HCS, namely

$$\zeta_{ij}=\frac{1}{2}\xi_{ij}\mu_{ji}^2\left(1+\frac{m_i\gamma_j}{m_j\gamma_i}\right)(1-\alpha_{ij}^2). \tag{3.9}$$

The integral equations defining the transport coefficients are usually solved by expanding in Sonine polynomials. For practical purposes, only the leading terms are retained. In addition, we will use the Maxwellian distribution (2.36) as a trial function for $f_i^{(0)}$. The final expressions for the seven transport coefficients of the mixture are given in Appendix B. A symbolic code providing the transport properties under arbitrary values of composition, masses, sizes, and coefficients of restitution can be downloaded from the website given in Ref. [20].

Let us compare the predictions of the model (2.11) for the NS transport coefficients with the results derived from the original inelastic Boltzmann equation [12,18,21]. The three coefficients D, D', and D_p associated with the mass flux are identical in both descriptions. This is a consequence of the requirement (2.16). For this reason, these three coefficients will not be shown here. Their behaviors for several representative cases have been analyzed elsewhere [18]. The other four coefficients differ in both descriptions, except trivially in the elastic case ($\alpha_{ij}=1$). Since the parameter space of the problem is six dimensional, namely $\{x_1,m_1/m_2,\sigma_1/\sigma_2,\alpha_{11},\alpha_{12},\alpha_{22}\}$, it is convenient to choose some specific cases. First, we consider hard spheres ($d=3$) with a common coefficient of restitution, i.e., $\alpha_{11}=\alpha_{12}=\alpha_{22}=\alpha$, (case A) and also with the choice of the coefficients of restitution $\alpha_{11}=\alpha$, $\alpha_{12}=(1+\alpha)/2$, $\alpha_{22}=(3+\alpha)/4$ (case B). Note that in case B one has $\alpha_{11}<\alpha_{12}<\alpha_{22}$. Consequently, for given values of x_1, m_1/m_2, σ_1/σ_2, and $\alpha_{11}=\alpha$, the system A is more inelastic than the system B. Next, we restrict ourselves to equimolar mixtures $(x_1=\frac{1}{2})$. This reduces the parameter space to three quantities, namely $\{m_1/m_2,\sigma_1/\sigma_2,\alpha\}$.

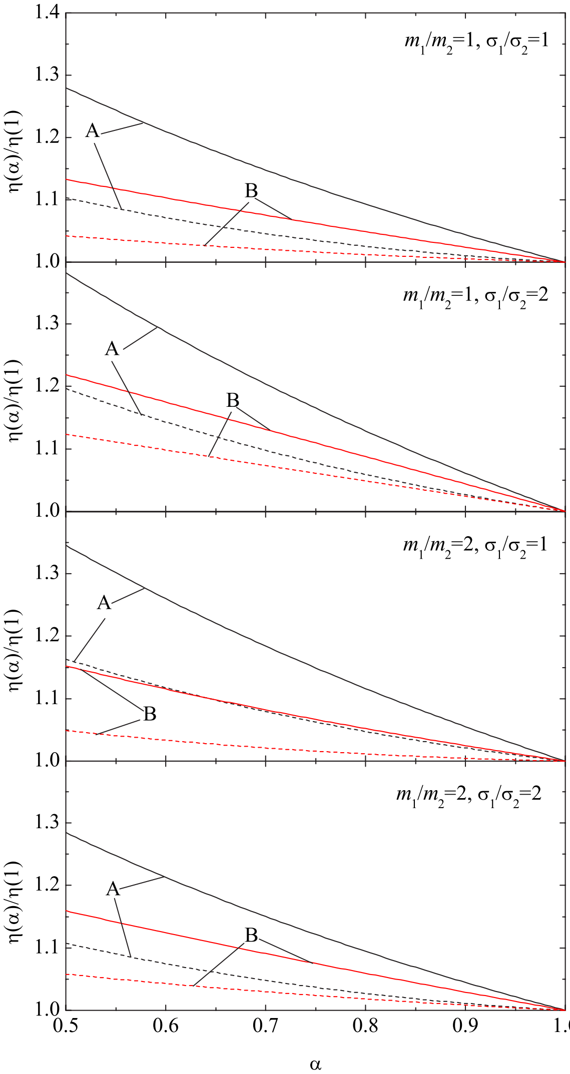

FIG. 1. (Color online) Plot of the reduced shear viscosity coefficient $\eta(\alpha)/\eta(1)$ as a function of the coefficient of restitution α for a three-dimensional equimolar mixture in the cases $\alpha_{11}=\alpha_{12}=\alpha_{22}=\alpha$ (case A) and $\alpha_{11}=\alpha$, $\alpha_{12}=(1+\alpha)/2$, $\alpha_{22}=(3+\alpha)/4$ (case B). The panels correspond, from top to bottom, to the systems $\{m_1/m_2,\sigma_1/\sigma_2\}=\{1,1\}$, $\{1,2\}$, $\{2,1\}$, and $\{2,2\}$, respectively. The solid lines are the Boltzmann results, while the dashed lines are the predictions of the model (2.11).

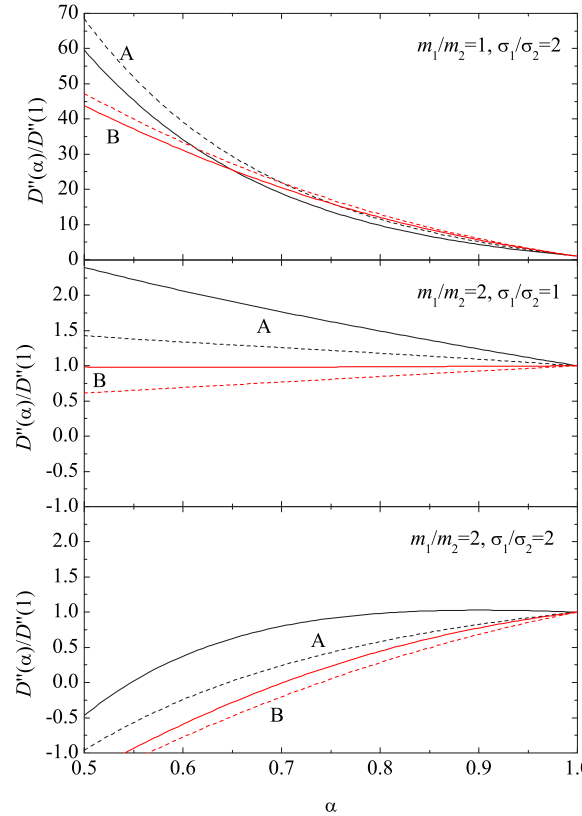

FIG. 2. (Color online) Plot of the reduced Dufour coefficient $D''(\alpha)/D''(1)$ as a function of the coefficient of restitution α for a three-dimensional equimolar mixture in the cases $\alpha_{11}=\alpha_{12}=\alpha_{22}=\alpha$ (case A) and $\alpha_{11}=\alpha$, $\alpha_{12}=(1+\alpha)/2$, $\alpha_{22}=(3+\alpha)/4$ (case B). The panels correspond, from top to bottom, to the systems $\{m_1/m_2,\sigma_1/\sigma_2\}=\{1,2\}$, $\{2,1\}$, and $\{2,2\}$, respectively. The solid lines are the Boltzmann results, while the dashed lines are the predictions of the model (2.11).

To focus on the influence of inelasticity on the transport coefficients, we fix the values of the mass and size ratios and plot the transport coefficients as functions of α. In addition, the coefficients are reduced with respect to their values in the elastic limit, except in the case of the pressure energy coefficient L, which vanishes for elastic collisions if $m_1=m_2$. In this case, the plotted quantity is $L(\alpha)n/\lambda(1)$, where $\lambda(1)$ denotes the elastic value of the thermal conductivity coefficient. As representative cases we have chosen $\{m_1/m_2,\sigma_1/\sigma_2\}=\{1,1\}$, $\{1,2\}$, $\{2,1\}$, and $\{2,2\}$. Note that the system $\{m_1/m_2,\sigma_1/\sigma_2\}=\{1,1\}$ corresponds to mechanically equivalent particles in case A.

The shear viscosity is shown in Fig. 1. We observe that, at least for the cases analyzed here, the model underestimates the Boltzmann values. Although the relative discrepancies increase with dissipation, they are practically insensitive to the mass and size ratios. As a consequence, the model captures well the influence of m_1/m_2 and σ_1/σ_2 on η. In particular, it is interesting to note that the ratio $\eta(\alpha)/\eta(1)$ for the system $\{m_1/m_2,\sigma_1/\sigma_2\}=\{1,1\}$ is practically indistinguishable from that of the system $\{m_1/m_2,\sigma_1/\sigma_2\}=\{2,2\}$, this effect being reproduced by the model. As expected, the influence of dissipation is less significant in case B than in case A.

Now we consider the three coefficients associated with the heat flux. We start with the Dufour coefficient D'', which is plotted in Fig. 2. The system $\{m_1/m_2,\sigma_1/\sigma_2\}=\{1,1\}$ is not displayed since $D''=0$ at any value of α for mechanically equivalent particles (case A). In case B, however, $D''\neq 0$ for the system $\{m_1/m_2,\sigma_1/\sigma_2\}=\{1,1\}$ since $\alpha_{11}\neq\alpha_{12}\neq\alpha_{22}$. We have checked in that case that the performance of the model is quite good. It is apparent from Fig. 2 that, for a given value of α, the ratio $D''(\alpha)/D''(1)$ has a significant dependence on m_1/m_2 and/or σ_1/σ_2. This effect is well accounted for by the model. It is noteworthy the dramatic influence of inelasticity on the value of the Dufour coefficient when $m_1=m_2$ and $\sigma_1\neq\sigma_2$. As shown in the top panel of Fig. 2, this feature is accurately predicted by the model.

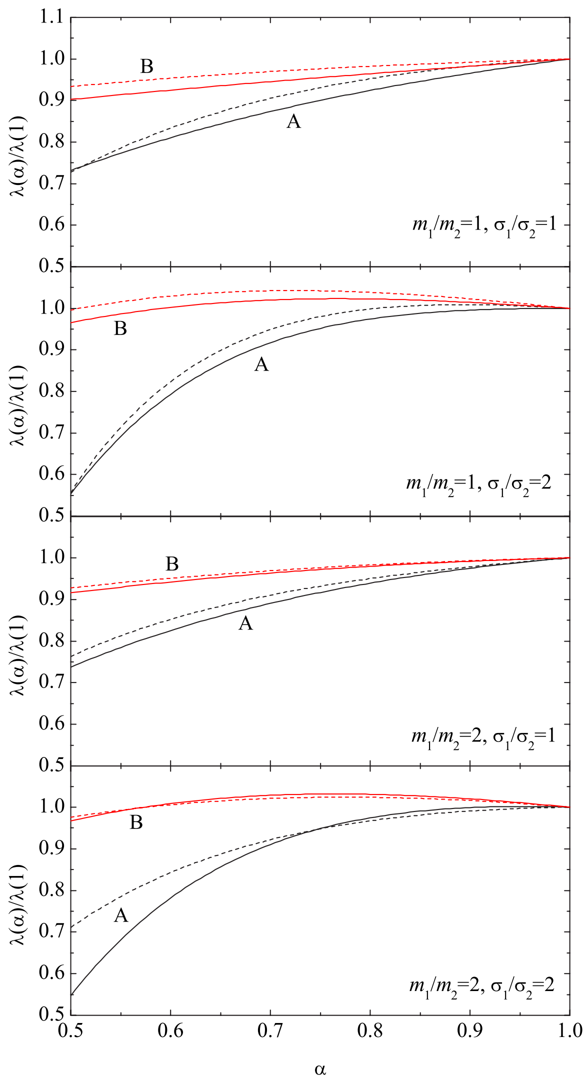

FIG. 3. (Color online) Plot of the reduced thermal conductivity coefficient $\lambda(\alpha)/\lambda(1)$ as a function of the coefficient of restitution α for a three-dimensional equimolar mixture in the cases $\alpha_{11}=\alpha_{12}=\alpha_{22}=\alpha$ (case A) and $\alpha_{11}=\alpha$, $\alpha_{12}=(1+\alpha)/2$, $\alpha_{22}=(3+\alpha)/4$ (case B). The panels correspond, from top to bottom, to the systems $\{m_1/m_2,\sigma_1/\sigma_2\}=\{1,1\}$, $\{1,2\}$, $\{2,1\}$, and $\{2,2\}$, respectively. The solid lines are the Boltzmann results, while the dashed lines are the predictions of the model (2.11).

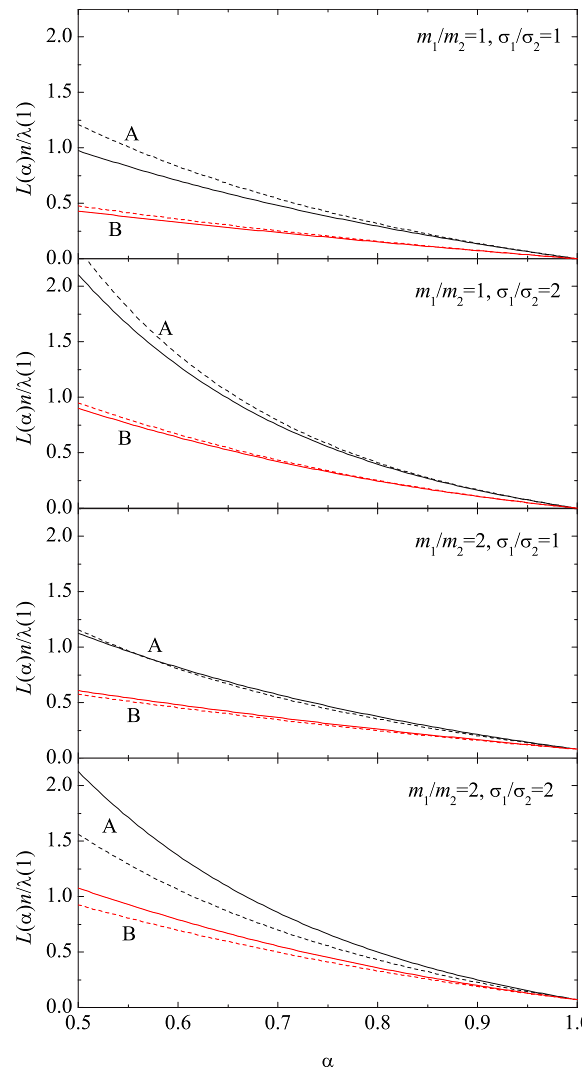

FIG. 4. (Color online) Plot of the reduced pressure energy coefficient $L(\alpha)n/\lambda(1)$ as a function of the coefficient of restitution α for a three-dimensional equimolar mixture in the cases $\alpha_{11}=\alpha_{12}=\alpha_{22}=\alpha$ (case A) and $\alpha_{11}=\alpha$, $\alpha_{12}=(1+\alpha)/2$, $\alpha_{22}=(3+\alpha)/4$ (case B). The panels correspond, from top to bottom, to the systems $\{m_1/m_2,\sigma_1/\sigma_2\}=\{1,1\}$, $\{1,2\}$, $\{2,1\}$, and $\{2,2\}$, respectively. The solid lines are the Boltzmann results, while the dashed lines are the predictions of the model (2.11).

The thermal conductivity and the pressure energy coefficients are displayed in Figs. 3 and 4, respectively. In contrast to D'', these coefficients are meaningful in the case of mechanically equivalent particles. The model performs quite good a job, even for strong dissipation, except perhaps for the most disparate mixture $\{m_1/m_2,\sigma_1/\sigma_2\}=\{2,2\}$ in case A.

In summary, it is fair to say that the model of elastic spheres subject to a drag force mimics, at least at a semiquantitative level, the influence of inelasticity on the transport coefficients of a binary mixture of inelastic hard spheres. More specifically, the mass flux coefficients (D, D', and D_p) are the same in both systems, while the shear viscosity η is underestimated. In the least favorable case (case A), the discrepancies of the values of η are about 7% at α=0.8 and 13% at α=0.6 for the systems studied here. With respect to the heat flux coefficients (D'', λ, and L), the general agreement is, paradoxically, better than in the case of the shear viscosity. For instance, the deviations of the thermal conductivity are less than 3% at α=0.8 and 8% at α=0.6 for the systems studied here in case A. It is also interesting to remark that the reliability of the transport coefficients predicted by the model is practically independent of the disparity in mass and size ratio.

IV. DISCUSSION

The study of multicomponent granular systems is of paramount importance from a practical point of view, but also at a fundamental level. In the low density regime, granular gases are well described by the inelastic Boltzmann equations with constant coefficients of restitution $\alpha_{ij} \leq 1$. However, the intricacy of the inelastic Boltzmann collision operator J_{ij}^{inel} makes it difficult to extract explicit results, especially at moderate or strong dissipation. This motivates the search for models that, while retaining the basic properties of J_{ij}^{inel}, provide a simpler framework for granular mixtures.

In this paper, we have proposed a model of elastic hard spheres under the action of an external drag force. The role of the force is to mimic the collisional energy loss in the true gas of inelastic spheres. In addition, the collision rate for elastic collisions is modified by a factor β_{ij} with respect to the one for inelastic collisions. Accordingly, in this model the collision operator J_{ij}^{inel} is replaced by the operator K_{ij} defined in Eq. (2.11). While the dependence of J_{ij}^{inel} on α_{ij} appears in the gain term and through the collision rules [see Eqs. (2.1) and (2.2)], such a dependence appears explicitly in K_{ij} through the parameters β_{ij} and ζ_{ij}. These parameters are determined by imposing that the model operator K_{ij} reproduces the correct collisional transfer of momentum and energy, Eqs. (2.16) and (2.17). The former condition gives the simple expression (2.22) for the parameter β_{ij} modifying the collision rate. The energy condition (2.17) shows that the drag coefficient ζ_{ij} is a functional of the distribution functions f_i and f_j that is simply proportional to $1-\alpha_{ij}^2$, Eq. (2.24). Thus, ζ_{ij} can be interpreted as the cooling rate of species i due to collisions with particles of species j, as indicated by Eq. (2.28). Since ζ_{ij} requires the average $\langle g_{12}^3\rangle_{ij}$ defined by Eq. (2.25), the Gaussian forms (2.29) parametrized by the partial velocities and temperatures are used to estimate ζ_{ij}, yielding Eq. (2.30).

By construction, the model leads to the same results for the temperature ratios in homogeneous states as those obtained from the inelastic Boltzmann equation in the multitemperature Maxwellian approximation [4,5], which is known to compare quite well with computer simulations [5–7]. To measure the performance of the model in inhomogeneous situations, the seven NS transport coefficients for a binary mixture predicted by the model have been computed and compared with previous results from the inelastic Boltzmann equation [12,18,21]. The three transport coefficients associated with the mass flux are identical in both descriptions in the first Sonine approximation, while the remaining four coefficients differ, as illustrated in Figs. 1–4. We found that the model captures reasonably well the dependence of the shear viscosity η and the three heat flux coefficients D'', λ, and L on dissipation. This agreement is especially remarkable in the case of the thermal conductivity λ. The degree of agreement between the model and the inelastic Boltzmann transport coefficients has been found to be practically independent of the disparity of mass and/or size, being similar to the one found for the one-component case [17]. Beyond the NS domain, where the spatial gradients are not sufficiently small, it is uncertain whether or not the model predictions are close to the inelastic Boltzmann ones. However, at least in the uniform shear flow problem, the simulation results in the one-component case are practically indistinguishable in both systems [22]. We expect that such a good agreement is also kept for multicomponent systems, given that the discrepancies between the model and the inelastic Boltzmann transport coefficients are not significantly affected by the mass and size ratios.

One of the main features of the model (2.11) is that it allows, in an approximate way, to "disentangle" in J_{ij}^{inel} the purely dissipative effects (represented by the drag-force term) from those (represented by $\beta_{ij}J_{ij}^{\text{el}}$) which are essentially present in the elastic case. Taking advantage of this decomposition, one can extend to the inelastic case any model kinetic equation proposed for ordinary multicomponent gases. This is quite an important issue since, even for elastic collisions, the Boltzmann equation is too complex to study far from equilibrium situations [23], for which the NS description fails. For one-component ordinary gases, the prototype model kinetic equation is the Bhatnagar-Gross-Krook (BGK) model [24,25]. Several extensions of this model to the case of one-component inelastic hard spheres have been proposed [26–28]. Furthermore, some kinetic models for ordinary gas mixtures inspired on the BGK model can be found in the literature [23,29–31]. The common structure of the latter models is

$$J_{ij}^{\text{el}}[\mathbf{v}|f_i,f_j] \to -\nu_{ij}[f_i(\mathbf{v})-f_{ij}(\mathbf{v})], \tag{4.1}$$

where ν_{ij} is a velocity-independent effective collision frequency of a particle of species i with particles of species j and $f_{ij}(\mathbf{v})$ is a reference distribution function whose velocity dependence is explicit and that involves a number of parameters to be determined by imposing that Eq. (4.1) retains the main physical properties of the Boltzmann operator J_{ij}^{el}. Using the mapping (2.11), any kinetic model of the form (4.1) can be easily extended to the case of inelastic collisions as

$$J_{ij}^{\text{inel}}[\mathbf{v}|f_i,f_j] \to -\beta_{ij}\nu_{ij}[f_i(\mathbf{v})-f_{ij}(\mathbf{v})]+\frac{\zeta_{ij}}{2}\frac{\partial}{\partial\mathbf{v}}\cdot[(\mathbf{v}-\mathbf{u}_i)f_i(\mathbf{v})], \tag{4.2}$$

where β_{ij} and ζ_{ij} are given by Eqs. (2.22) and (2.30), respectively. A specific choice for f_{ij} based on the Gross-Krook kinetic model [29] is worked out in Appendix C. Several applications of the kinetic model (4.2) to non-Newtonian flows are under way and will be published elsewhere.

ACKNOWLEDGMENTS

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APPENDIX A: EVALUATION OF $\langle g_{12}^3\rangle_{ij}$, $\langle g_{12}\mathbf{g}_{12}\rangle_{ij}$, AND $\langle g_{12}\mathbf{g}_{12}\cdot\mathbf{G}_{12}\rangle_{ij}$

The aim of this Appendix is to evaluate the averages $\langle g_{12}^3\rangle_{ij}$, $\langle g_{12}\mathbf{g}_{12}\rangle_{ij}$, and $\langle g_{12}\mathbf{g}_{12}\cdot\mathbf{G}_{12}\rangle_{ij}$ by assuming the Gaussian forms (2.29).

Let us start with the evaluation of $\langle g_{12}^3\rangle_{ij}$. Inserting Eq. (2.29) into Eq. (2.25), one obtains

$$\langle g_{12}^3\rangle_{ij} = \left(\frac{m_i}{2\pi\widetilde{T}_i}\right)^{d/2}\left(\frac{m_j}{2\pi\widetilde{T}_j}\right)^{d/2}\int d\mathbf{w}_1\int d\mathbf{w}_2 \times\exp\left(-\frac{m_i w_1^2}{2\widetilde{T}_i}-\frac{m_j w_2^2}{2\widetilde{T}_j}\right)|\mathbf{w}_{12}+\mathbf{u}_i-\mathbf{u}_j|^3, \quad (A1)$$

where $\mathbf{w}_1=\mathbf{v}_1-\mathbf{u}_i$, $\mathbf{w}_2=\mathbf{v}_2-\mathbf{u}_j$, and $\mathbf{w}_{12}=\mathbf{w}_1-\mathbf{w}_2$. Let us also introduce the variable

$$\mathbf{W}_{12} = \left(\frac{m_i}{\widetilde{T}_i}+\frac{m_j}{\widetilde{T}_j}\right)^{-1}\left(\frac{m_i}{\widetilde{T}_i}\mathbf{w}_1+\frac{m_j}{\widetilde{T}_j}\mathbf{w}_2\right), \quad (A2)$$

so that $d\mathbf{w}_1 d\mathbf{w}_2=d\mathbf{w}_{12}d\mathbf{W}_{12}$ and

$$\frac{m_i w_1^2}{\widetilde{T}_i}+\frac{m_j w_2^2}{\widetilde{T}_j} = \left(\frac{m_i}{\widetilde{T}_i}+\frac{m_j}{\widetilde{T}_j}\right)W_{12}^2+\left(\frac{\widetilde{T}_i}{m_i}+\frac{\widetilde{T}_j}{m_j}\right)^{-1}w_{12}^2. \quad (A3)$$

Thus, after making the changes $(\mathbf{v}_1,\mathbf{v}_2)\to(\mathbf{w}_{12},\mathbf{W}_{12})$ and integrating over $\mathbf{W}_{12}$, Eq. (A1) becomes

$$\langle g_{12}^3\rangle_{ij} = \pi^{-d/2}\left(\frac{2\widetilde{T}_i}{m_i}+\frac{2\widetilde{T}_j}{m_j}\right)^{-d/2}\int d\mathbf{w}_{12} \times\exp\left[-\left(\frac{2\widetilde{T}_i}{m_i}+\frac{2\widetilde{T}_j}{m_j}\right)^{-1}w_{12}^2\right]|\mathbf{w}_{12}+\mathbf{u}_i-\mathbf{u}_j|^3 = \left(\frac{2\widetilde{T}_i}{m_i}+\frac{2\widetilde{T}_j}{m_j}\right)^{3/2}\pi^{-d/2}\int d\mathbf{c}e^{-c^2}|\mathbf{c}+\mathbf{u}_{ij}^*|^3, \quad (A4)$$

where in the second step we have made the change of variable $\mathbf{c}=(2\widetilde{T}_i/m_i+2\widetilde{T}_j/m_j)^{-1/2}\mathbf{w}_{12}$ and have defined $\mathbf{u}_{ij}^*=(2\widetilde{T}_i/m_i+2\widetilde{T}_j/m_j)^{-1/2}(\mathbf{u}_i-\mathbf{u}_j)$. The Gaussian integral in Eq. (A4) cannot be evaluated analytically for arbitrary $\mathbf{u}_{ij}^*$. Here we will evaluate it by neglecting terms of fourth and higher order in $\mathbf{u}_{ij}^*$. In that case, one has

$$|\mathbf{c}+\mathbf{u}_{ij}^*|^3 = c^3+3c\mathbf{c}\cdot\mathbf{u}_{ij}^*+\frac{3}{2}cu_{ij}^{*2}+\frac{3}{2}c^{-1}(\mathbf{c}\cdot\mathbf{u}_{ij}^*)^2+\cdots, \quad (A5)$$

where we have taken into account that the third order terms do not contribute to the integral. As a consequence,

$$\pi^{-d/2}\int d\mathbf{c}e^{-c^2}|\mathbf{c}+\mathbf{u}_{ij}^*|^3 \to \pi^{-d/2}\int d\mathbf{c}e^{-c^2}\left(c^3+\frac{3}{2}\frac{d+1}{d}cu_{ij}^{*2}\right) = \frac{\Gamma[(d+1)/2]}{\Gamma(d/2)}\frac{d+1}{2}\left(1+\frac{3}{d}u_{ij}^{*2}\right). \quad (A6)$$

Insertion of Eq. (A6) into Eq. (A4) yields

$$\langle g_{12}^3\rangle_{ij} = (d+1)\frac{\Gamma[(d+1)/2]}{\Gamma(d/2)}\left(\frac{2\widetilde{T}_i}{m_i}+\frac{2\widetilde{T}_j}{m_j}\right)^{1/2} \times\left[\frac{\widetilde{T}_i}{m_i}+\frac{\widetilde{T}_j}{m_j}+\frac{3}{2d}(\mathbf{u}_i-\mathbf{u}_j)^2\right]. \quad (A7)$$

Following similar steps, it is straightforward to obtain

$$\langle g_{12}\mathbf{g}_{12}\rangle_{ij} = \left(\frac{2\widetilde{T}_i}{m_i}+\frac{2\widetilde{T}_j}{m_j}\right)\pi^{-d/2}\int d\mathbf{c}e^{-c^2}|\mathbf{c}+\mathbf{u}_{ij}^*|(\mathbf{c}+\mathbf{u}_{ij}^*). \quad (A8)$$

Next, one has

$$|\mathbf{c}+\mathbf{u}_{ij}^*|(\mathbf{c}+\mathbf{u}_{ij}^*) = c(\mathbf{u}_{ij}^*+c^{-2}\mathbf{u}_{ij}^*\cdot\mathbf{c}\mathbf{c}+\cdots), \quad (A9)$$

where the ellipsis denote terms which are odd in $\mathbf{c}$ or are of at least of third order in $\mathbf{u}_{ij}^*$. Thus,

$$\pi^{-d/2}\int d\mathbf{c}e^{-c^2}|\mathbf{c}+\mathbf{u}_{ij}^*|(\mathbf{c}+\mathbf{u}_{ij}^*) \to \mathbf{u}_{ij}^*\frac{d+1}{d}\pi^{-d/2}\int d\mathbf{c}e^{-c^2}c = \frac{d+1}{d}\frac{\Gamma[(d+1)/2]}{\Gamma(d/2)}\mathbf{u}_{ij}^*. \quad (A10)$$

From Eqs. (A8) and (A9) it follows that

$$\langle g_{12}\mathbf{g}_{12}\rangle_{ij} = \frac{d+1}{d}\frac{\Gamma[(d+1)/2]}{\Gamma(d/2)}\left(\frac{2\widetilde{T}_i}{m_i}+\frac{2\widetilde{T}_j}{m_j}\right)^{1/2}(\mathbf{u}_i-\mathbf{u}_j). \quad (A11)$$

Finally, we consider the evaluation of $\langle g_{12}\mathbf{g}_{12}\cdot\mathbf{G}_{12}\rangle_{ij}$. Taking into account Eq. (A2) and the definition of $\mathbf{G}_{12}$ [see below Eq. (2.23)], it is easy to obtain the relationship

$$\mathbf{G}_{12} = \mathbf{W}_{12}+\left(\frac{m_i}{\widetilde{T}_i}+\frac{m_j}{\widetilde{T}_j}\right)^{-1}\left[\left(\frac{m_i}{\widetilde{T}_i}\mathbf{u}_i+\frac{m_j}{\widetilde{T}_j}\mathbf{u}_j\right)-\frac{m_i m_j}{m_i+m_j}(\widetilde{T}_i^{-1}-\widetilde{T}_j^{-1})\mathbf{g}_{12}\right]. \quad (A12)$$

As a consequence,

$$\langle g_{12}\mathbf{g}_{12}\cdot\mathbf{G}_{12}\rangle_{ij} = \left(\frac{m_i}{\widetilde{T}_i}+\frac{m_j}{\widetilde{T}_j}\right)^{-1}\left[\langle g_{12}\mathbf{g}_{12}\rangle_{ij}\cdot\left(\frac{m_i}{\widetilde{T}_i}\mathbf{u}_i+\frac{m_j}{\widetilde{T}_j}\mathbf{u}_j\right)-\frac{m_i m_j}{m_i+m_j}(\widetilde{T}_i^{-1}-\widetilde{T}_j^{-1})\langle g_{12}^3\rangle_{ij}\right], \quad (A13)$$

where, by symmetry, $\langle g_{12}\mathbf{g}_{12}\cdot\mathbf{W}_{12}\rangle_{ij}=0$ when evaluated with the distributions (2.29). Making use of Eqs. (A7) and (A11) one obtains

$$\langle g_{12}\mathbf{g}_{12}\cdot\mathbf{G}_{12}\rangle_{ij} = \frac{d+1}{d}\frac{\Gamma[(d+1)/2]}{\Gamma(d/2)}\left(\frac{2\widetilde{T}_i}{m_i}+\frac{2\widetilde{T}_j}{m_j}\right)^{1/2} \times\left(\frac{m_i}{\widetilde{T}_i}+\frac{m_j}{\widetilde{T}_j}\right)^{-1}\left\{\left(\frac{m_i}{\widetilde{T}_i}\mathbf{u}_i+\frac{m_j}{\widetilde{T}_j}\mathbf{u}_j\right)\cdot(\mathbf{u}_i-\mathbf{u}_j)-d\frac{m_i m_j}{m_i+m_j}(\widetilde{T}_i^{-1}-\widetilde{T}_j^{-1}) \times\left[\frac{\widetilde{T}_i}{m_i}+\frac{\widetilde{T}_j}{m_j}+\frac{3}{2d}(\mathbf{u}_i-\mathbf{u}_j)^2\right]\right\}. \quad (A14)$$

APPENDIX B: EXPLICIT EXPRESSIONS FOR THE TRANSPORT COEFFICIENTS

In this Appendix, we take advantage of the results derived from the original Boltzmann equation [12,21] to determine the expressions corresponding to the model (2.11), by using the forms (3.7) and (3.8) for the operators $\mathcal{L}_i$ and $\mathcal{M}_i$.

1. Mass flux

In the case of the transport coefficients associated with the mass flux, the results are [12,21]

$$D=\frac{\rho}{m_1m_2n}\left(\nu-\frac{1}{2}\zeta\right)^{-1}\left[p\left(\frac{\partial}{\partial x_1}x_1\gamma_1\right)_{p,T}+\rho\left(\frac{\partial\zeta}{\partial x_1}\right)_{p,T}(D_p+D')\right], \tag{B1}$$

$$D_p=\frac{n_1T}{\rho}\left(\gamma_1-\frac{m_1n}{\rho}\right)\left(\nu-\frac{3}{2}\zeta+\frac{\zeta^2}{2\nu}\right)^{-1}, \tag{B2}$$

$$D'=-\frac{\zeta}{2\nu}D_p. \tag{B3}$$

Here, $\zeta=\zeta_1=\zeta_2$ is the cooling rate of the mixture in the HCS [see Eq. (2.38)] and the collision frequency ν is given by

$$\nu=\frac{m_1}{dn_1T\gamma_1}\int d\mathbf{v}\mathbf{V}\cdot\left(\mathcal{L}_1f_{1,M}\mathbf{V}-\frac{x_1\gamma_1}{x_2\gamma_2}\mathcal{M}_1f_{2,M}\mathbf{V}\right). \tag{B4}$$

The result is

$$\nu=\nu_0\frac{2\pi^{(d-1)/2}}{d\Gamma\left(\frac{d}{2}\right)}(1+\alpha_{12})\left(\frac{\theta_1+\theta_2}{\theta_1\theta_2}\right)^{1/2}(x_2\mu_{21}+x_1\mu_{12}), \tag{B5}$$

where $\nu_0\equiv n\sigma_{12}^{d-1}\sqrt{2T(m_1+m_2)/m_1m_2}$, $\theta_1\equiv(\mu_{21}\gamma_1)^{-1}$, $\theta_2\equiv(\mu_{12}\gamma_2)^{-1}$, and we recall that $\gamma_i\equiv T_i/T$. The expressions of the transport coefficients D, D_p, and D' are exactly the same as those obtained from the true Boltzmann equation.

2. Pressure tensor

The shear viscosity coefficient η can be written as

$$\eta=\frac{p}{\nu_0}(x_1\gamma_1^2\eta_1^*+x_2\gamma_2^2\eta_2^*), \tag{B6}$$

where the expression of the (dimensionless) partial contribution η_1^* is

$$\eta_1^*=2\frac{\gamma_2(2\tau_{22}-\zeta^*)-2\gamma_1\tau_{12}}{\gamma_1\gamma_2[\zeta^*-2\zeta^*(\tau_{11}+\tau_{22})+4(\tau_{11}\tau_{22}-\tau_{12}\tau_{21})]}. \tag{B7}$$

Here, $\zeta^*\equiv\zeta/\nu_0$ and we have introduced the (reduced) collision frequencies

$$\tau_{11}=\frac{1}{(d-1)(d+2)}\frac{1}{n_1T^2\gamma_1^2\nu_0}\int d\mathbf{v}R_{1,k\ell}\mathcal{L}_1(f_{1,M}R_{1,k\ell}), \tag{B8}$$

$$\tau_{12}=\frac{1}{(d-1)(d+2)}\frac{1}{n_1T^2\gamma_1^2\nu_0}\int d\mathbf{v}R_{1,k\ell}\mathcal{M}_1(f_{2,M}R_{2,k\ell}), \tag{B9}$$

where

$$R_{1,k\ell}(\mathbf{V})=m_1\left(V_kV_\ell-\frac{1}{d}V^2\delta_{k\ell}\right). \tag{B10}$$

A similar expression can be obtained for η_2^* by just making the changes $1\leftrightarrow2$. Taking advantage of the results derived for the d-dimensional Boltzmann equation [21], one obtains for the model the expressions

$$\tau_{11}=\frac{4\pi^{(d-1)/2}}{d(d+2)\Gamma\left(\frac{d}{2}\right)}\left\{x_1\left(\frac{\sigma_1}{\sigma_{12}}\right)^{d-1}(2\theta_1)^{-1/2}d(1+\alpha_{11})+x_2\mu_{21}(1+\alpha_{12})\theta_1^{3/2}\theta_2^{-1/2}\left[(d+3)(\mu_{12}\theta_2-\mu_{21}\theta_1)\theta_1^{-2}(\theta_1+\theta_2)^{-1/2}+d\mu_{21}\theta_1^{-2}(\theta_1+\theta_2)^{1/2}+\frac{2d(d+1)-4}{2(d-1)}\theta_1^{-1}(\theta_1+\theta_2)^{-1/2}\right]\right\}+\zeta_{11}^*+\zeta_{12}^*, \tag{B11}$$

$$\tau_{12}=\frac{4\pi^{(d-1)/2}}{d(d+2)\Gamma\left(\frac{d}{2}\right)}x_2\frac{\mu_{21}^2}{\mu_{12}}\theta_1^{3/2}\theta_2^{-1/2}(1+\alpha_{12})\left[(d+3)(\mu_{12}\theta_2-\mu_{21}\theta_1)\theta_2^{-2}(\theta_1+\theta_2)^{-1/2}+d\mu_{21}\theta_2^{-2}(\theta_1+\theta_2)^{1/2}-\frac{2d(d+1)-4}{2(d-1)}\theta_2^{-1}(\theta_1+\theta_2)^{-1/2}\right], \tag{B12}$$

where $\zeta_{ij}^*\equiv\zeta_{ij}/\nu_0$ and ζ_j is given by Eq. (3.9).

3. Heat flux

The transport coefficients appearing in the heat flux (D'', L, and λ) can be written as [12,21]

$$D''=-\frac{d+2}{2}\frac{n}{(m_1+m_2)\nu_0}\left[\frac{x_1\gamma_1^3}{\mu_{12}}d_1^*+\frac{x_2\gamma_2^3}{\mu_{21}}d_2^*-\left(\frac{\gamma_1}{\mu_{12}}-\frac{\gamma_2}{\mu_{21}}\right)D^*\right], \tag{B13}$$

$$L=-\frac{d+2}{2}\frac{T}{(m_1+m_2)\nu_0}\left[\frac{x_1\gamma_1^3}{\mu_{12}}\ell_1^*+\frac{x_2\gamma_2^3}{\mu_{21}}\ell_2^*-\left(\frac{\gamma_1}{\mu_{12}}-\frac{\gamma_2}{\mu_{21}}\right)D_p^*\right], \tag{B14}$$

$$\lambda = -\frac{d+2}{2}\frac{nT}{(m_1+m_2)\nu_0}\left[\frac{x_1\gamma_1^3}{\mu_{12}}\lambda_1^* + \frac{x_2\gamma_2^3}{\mu_{21}}\lambda_2^* - \left(\frac{\gamma_1}{\mu_{12}} - \frac{\gamma_2}{\mu_{21}}\right)D'^*\right], \tag{B15}$$

where

$$D = \frac{\rho T}{m_1 m_2 \nu_0}D^*, \quad D_p = \frac{nT}{\rho\nu_0}D_p^*, \quad D' = \frac{nT}{\rho\nu_0}D'^*, \tag{B16}$$

the coefficients D, D_p, and D' being given by Eqs. (B1)–(B3), respectively. The expressions of the (dimensionless) coefficients d_i^*, ℓ_i^*, and λ_i^* are

$$d_1^* = \frac{1}{\Delta}\Bigg\{2[2\chi_{12}Y_2 - Y_1(2\chi_{22} - 3\zeta^*)][\chi_{12}\chi_{21} - \chi_{11}\chi_{22} + 2(\chi_{11}+\chi_{22})\zeta^* - 4\zeta^{*2}] + 2\left(\frac{\partial\zeta^*}{\partial x_1}\right)_{p,T}(Y_3+Y_5) \times[2\chi_{12}\chi_{21} + 2\chi_{22}^2 - \zeta^*(7\chi_{22} - 6\zeta^*)] - 2\chi_{12}\left(\frac{\partial\zeta^*}{\partial x_1}\right)_{p,T} \times(Y_4+Y_6)(2\chi_{11} + 2\chi_{22} - 7\zeta^*)\Bigg\}, \tag{B17}$$

$$\ell_1^* = \frac{1}{\Delta}\{-2Y_3[2(\chi_{12}\chi_{21} - \chi_{11}\chi_{22})\chi_{22} + \zeta^*(7\chi_{11}\chi_{22} - 5\chi_{12}\chi_{21} + 2\chi_{22}^2 - 6\chi_{11}\zeta^* - 7\chi_{22}\zeta^* + 6\zeta^{*2})] + 2Y_4\chi_{12}[2\chi_{12}\chi_{21} - 2\chi_{11}\chi_{22} + 2\zeta^*(\chi_{11}+\chi_{22}) - \zeta^{*2}] + 2Y_5\zeta^*[2\chi_{12}\chi_{21} + \chi_{22}(2\chi_{22} - 7\zeta^*) + 6\zeta^{*2}] - 2\chi_{12}\zeta^* Y_6[2(\chi_{11}+\chi_{22}) - 7\zeta^*]\}, \tag{B18}$$

$$\lambda_1^* = \frac{1}{\Delta}\{-Y_3\zeta^*[2\chi_{12}\chi_{21} + \chi_{22}(2\chi_{22} - 7\zeta^*) + 6\zeta^{*2}] + \chi_{12}\zeta^* Y_4[2(\chi_{11}+\chi_{22}) - 7\zeta^*] - Y_5[4\chi_{12}\chi_{21}(\chi_{22} - \zeta^*) + 2\chi_{22}^2(5\zeta^* - 2\chi_{11}) + 2\chi_{11}(7\chi_{22}\zeta^* - 6\zeta^{*2}) + 5\zeta^{*2}(6\zeta^* - 7\chi_{22})] + \chi_{12}Y_6[4\chi_{12}\chi_{21} + 2\chi_{11}(5\zeta^* - 2\chi_{22}) + \zeta^*(10\chi_{22} - 23\zeta^*)]\}. \tag{B19}$$

In the above equations, we have introduced the quantities

$$\Delta \equiv [4(\chi_{12}\chi_{21} - \chi_{11}\chi_{22}) + 6\zeta^*(\chi_{11}+\chi_{22}) - 9\zeta^{*2}] \times[\chi_{12}\chi_{21} - \chi_{11}\chi_{22} + 2\zeta^*(\chi_{11}+\chi_{22}) - 4\zeta^{*2}], \tag{B20}$$

$$Y_1 = \frac{D^*}{x_1\gamma_1^2}(\omega_{12} - \zeta^*) - \frac{1}{\gamma_1^2}\left(\frac{\partial\gamma_1}{\partial x_1}\right)_{p,T}, \quad Y_2 = -\frac{D^*}{x_2\gamma_2^2}(\omega_{21} - \zeta^*) - \frac{1}{\gamma_2^2}\left(\frac{\partial\gamma_2}{\partial x_1}\right)_{p,T}, \tag{B21}$$

$$Y_3 = \frac{D_p^*}{x_1\gamma_1^2}(\omega_{12} - \zeta^*), \quad Y_4 = -\frac{D_p^*}{x_2\gamma_2^2}(\omega_{21} - \zeta^*), \tag{B22}$$

$$Y_5 = -\frac{1}{\gamma_1} + \frac{D'^*}{x_1\gamma_1^2}(\omega_{12} - \zeta^*), \quad Y_6 = -\frac{1}{\gamma_2} - \frac{D'^*}{x_2\gamma_2^2}(\omega_{21} - \zeta^*), \tag{B23}$$

$$\chi_{11} = \frac{2}{d(d+2)}\frac{m_1}{n_1 T^3\gamma_1^3\nu_0}\int d\mathbf{v}\mathbf{S}_1\cdot\mathcal{L}_1(f_{1,M}\mathbf{S}_1), \tag{B24}$$

$$\chi_{12} = \frac{2}{d(d+2)}\frac{m_1}{n_1 T^3\gamma_1^3\nu_0}\int d\mathbf{v}\mathbf{S}_1\cdot\mathcal{M}_1(f_{2,M}\mathbf{S}_2), \tag{B25}$$

$$\omega_{12} = \frac{2}{d(d+2)}\frac{m_1}{n_1 T^2\gamma_1^2\nu_0}\left[\int d\mathbf{v}\mathbf{S}_1\cdot\mathcal{L}_1(f_{1,M}\mathbf{V}_1) - \frac{x_1\gamma_1}{x_2\gamma_2}\int d\mathbf{v}\mathbf{S}_1\cdot\mathcal{M}_1(f_{2,M}\mathbf{V}_2)\right], \tag{B26}$$

where

$$\mathbf{S}_i(\mathbf{V}) = \left(\frac{1}{2}m_i V^2 - \frac{d+2}{2}T\gamma_i\right)\mathbf{V}. \tag{B27}$$

The explicit expressions for the coefficients χ_{ij} and ω_{ij} are

$$\chi_{11} = \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d}{2}\right)}\frac{4}{d(d+2)}\left(\frac{\sigma_1}{\sigma_{12}}\right)^{d-1}x_1(2\theta_1)^{-1/2}(1+\alpha_{11})(d-1) + \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d}{2}\right)}\frac{1}{d(d+2)}x_2\mu_{21}(1+\alpha_{12})\left(\frac{\theta_1}{\theta_2(\theta_1+\theta_2)}\right)^{3/2} \times\left[E - (d+2)\frac{\theta_1+\theta_2}{\theta_1}A\right] + \frac{3}{2}(\zeta_{11}^* + \zeta_{12}^*), \tag{B28}$$

$$\chi_{12} = -\frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d}{2}\right)}\frac{1}{d(d+2)}x_2\frac{\mu_{21}^2}{\mu_{12}}(1+\alpha_{12})\left(\frac{\theta_1}{\theta_2(\theta_1+\theta_2)}\right)^{3/2} \times\left[F + (d+2)\frac{\theta_1+\theta_2}{\theta_2}B\right], \tag{B29}$$

$$\omega_{12} = \frac{\pi^{(d-1)/2}}{\Gamma\left(\frac{d}{2}\right)}\frac{2}{d(d+2)}x_1\mu_{21}(1+\alpha_{12})(\theta_1 + \theta_2)^{-1/2}\theta_1^{1/2}\theta_2^{-3/2}\left(\frac{x_2}{x_1}A - \frac{\gamma_1}{\gamma_2}B\right) + \zeta_{11}^* + \zeta_{12}^*, \tag{B30}$$

where

$$A=(d+2)(2\phi_{12}+\theta_2)+4\mu_{21}(\theta_1+\theta_2)(d-1)\phi_{12}\theta_1^{-1}+3(d+3)\phi_{12}^2\theta_1^{-1}+\mu_{21}^2(d+3)\theta_1^{-1}(\theta_1+\theta_2)^2-(d+2)\theta_2\theta_1^{-1}(\theta_1+\theta_2), \tag{B31}$$

$$B=(d+2)(2\phi_{12}-\theta_1)-4\mu_{21}(\theta_1+\theta_2)(d-1)\phi_{12}\theta_2^{-1}-3(d+3)\phi_{12}^2\theta_2^{-1}-\mu_{21}^2(d+3)\theta_2^{-1}(\theta_1+\theta_2)^2+(d+2)(\theta_1+\theta_2), \tag{B32}$$

$$E=\mu_{21}^2\theta_1^{-2}(\theta_1+\theta_2)^2(d+3)[(d+2)\theta_1+(d+5)\theta_2]+4(d-1)\mu_{21}(\theta_1+\theta_2)\{\phi_{12}\theta_1^{-2}[(d+2)\theta_1+(d+5)\theta_2]+2\theta_2\theta_1^{-1}\}+3(d+3)\phi_{12}^2\theta_1^{-2}[(d+2)\theta_1+(d+5)\theta_2]+2\phi_{12}\theta_1^{-1}[(d+2)^2\theta_1+(24+11d+d^2)\theta_2]+(d+2)\theta_2\theta_1^{-1}[(d+8)\theta_1+(d+3)\theta_2]-(d+2)(\theta_1+\theta_2)\theta_1^{-2}\theta_2[(d+2)\theta_1+(d+3)\theta_2], \tag{B33}$$

$$F=\mu_{21}^2\theta_2^{-2}(\theta_1+\theta_2)^2(d+3)[(d+5)\theta_1+(d+2)\theta_2]+4(d-1)\mu_{21}(\theta_1+\theta_2)\{\phi_{12}\theta_2^{-2}[(d+5)\theta_1+(d+2)\theta_2]-2\theta_1\theta_2^{-1}\}+3(d+3)\phi_{12}^2\theta_2^{-2}[(d+5)\theta_1+(d+2)\theta_2]-2\phi_{12}\theta_2^{-1}[(24+11d+d^2)\theta_1+(d+2)^2\theta_2]+(d+2)\theta_1\theta_2^{-1}[(d+3)\theta_1+(d+8)\theta_2]-(d+2)(\theta_1+\theta_2)\theta_2^{-1}[(d+3)\theta_1+(d+2)\theta_2]. \tag{B34}$$

Here, $\phi_{12}=\mu_{12}\theta_2-\mu_{21}\theta_1$. The expressions for χ_{22}, χ_{21}, and ω_{21} can be easily obtained from Eqs. (B28)–(B30) by exchanging $1\leftrightarrow 2$.

APPENDIX C: A MODEL KINETIC EQUATION FOR GRANULAR MIXTURES

This Appendix addresses the construction of the kinetic model (4.2) for a specific choice of the reference distribution function f_{ij}.

The most natural way of extending the BGK model to (elastic) mixtures is to assume a Gaussian form for the reference distribution f_{ij}, i.e.,

$$f_{ij}(\mathbf{v})=n_i\left(\frac{m_i}{2\pi T_{ij}}\right)^{d/2}\exp\left[-\frac{m_i}{2T_{ij}}(\mathbf{v}-\mathbf{u}_{ij})^2\right], \tag{C1}$$

where $\mathbf{u}_{ij}$ and T_{ij} are parameters to be determined. This is the form of the model proposed by Gross and Krook [29], which has been widely used in the literature. The usual criteria to determine the unknowns ν_{ij}, $\mathbf{u}_{ij}$, and T_{ij} are to impose that the kinetic model reproduces the collisional equations of momentum and energy of the original Boltzmann operator J_{ij}^{el}, namely,

$$\int d\mathbf{v}\mathbf{v}J_{ij}^{\text{el}}[\mathbf{v}|f_i,f_j]=-\nu_{ij}\int d\mathbf{v}\mathbf{v}[f_i(\mathbf{v})-f_{ij}(\mathbf{v})]=-\nu_{ij}n_i(\mathbf{u}_i-\mathbf{u}_{ij}), \tag{C2}$$

$$\int d\mathbf{v}v^2J_{ij}^{\text{el}}[\mathbf{v}|f_i,f_j]=-\nu_{ij}\int d\mathbf{v}v^2[f_i(\mathbf{v})-f_{ij}(\mathbf{v})]=-\nu_{ij}n_i\left[\frac{d}{m_i}(\widetilde{T}_i-T_{ij})+u_i^2-u_{ij}^2\right]. \tag{C3}$$

Conservation of total momentum in collisions ij implies that

$$\nu_{ij}n_im_i(\mathbf{u}_i-\mathbf{u}_{ij})+\nu_{ji}n_jm_j(\mathbf{u}_j-\mathbf{u}_{ji})=\mathbf{0}. \tag{C4}$$

For physical reasons it is assumed that $n_i\nu_{ij}=n_j\nu_{ji}$, as happens with the effective collision frequencies ξ_{ij} defined by Eq. (2.31). Moreover, the reference velocity $\mathbf{u}_{ij}$ is assumed to be symmetrical, i.e., $\mathbf{u}_{ij}=\mathbf{u}_{ji}$. Under the above conditions, Eq. (C4) yields

$$\mathbf{u}_{ij}=\mu_{ij}\mathbf{u}_i+\mu_{ji}\mathbf{u}_j. \tag{C5}$$

To identify the remaining parameters ν_{ij} and T_{ij} through Eqs. (C2) and (C3) we need to compute the corresponding collision integrals associated with J_{ij}^{el}. In the original model proposed by Gross and Krook [29], the left-hand sides of Eqs. (C2) and (C3) were evaluated by considering mixtures of Maxwell molecules, in which case the collision rate is independent of the relative velocity. Here, however, we want to keep in J_{ij}^{el} the velocity dependence of the collision rate characteristic of hard spheres.

Applying the property (2.20) and making use of Eqs. (2.21) and (2.23) with $\alpha_{ij}=1$, it is easy to obtain

$$\int d\mathbf{v}_1\mathbf{v}_1J_{ij}^{\text{el}}[\mathbf{v}_1|f_i,f_j]=-\frac{2\pi^{(d-1)/2}}{\Gamma[(d+3)/2]}\mu_{ji}\sigma_{ij}^{d-1}n_in_j\langle g_{12}\mathbf{g}_{12}\rangle_{ij}, \tag{C6}$$

$$\int d\mathbf{v}_1v_1^2J_{ij}^{\text{el}}[\mathbf{v}_1|f_i,f_j]=-\frac{4\pi^{(d-1)/2}}{\Gamma[(d+3)/2]}\times\mu_{ji}\sigma_{ij}^{d-1}n_in_j\langle g_{12}\mathbf{g}_{12}\cdot\mathbf{G}_{12}\rangle_{ij}, \tag{C7}$$

where the integrations over the solid angle have been performed. The averages appearing in Eqs. (C6) and (C7) can be evaluated, as in the case of $\langle g_{12}^3\rangle_{ij}$, by assuming Gaussian forms for f_i and f_j and neglecting terms of order third and higher in $\mathbf{u}_i-\mathbf{u}_j$. This is done in Appendix A with the results (A11) and (A14), so that Eqs. (C6) and (C7) become

$$\int d\mathbf{v}_1\mathbf{v}_1J_{ij}^{\text{el}}[\mathbf{v}_1|f_i,f_j]=-\xi_{ij}n_i\mu_{ji}(\mathbf{u}_i-\mathbf{u}_j), \tag{C8}$$

$$\int d\mathbf{v}_1 v_1^2 J_{ij}^{\mathrm{el}}[\mathbf{v}_1|f_i,f_j] = -2\xi_{ij}n_i\mu_{ji}\left(\frac{\widetilde{T}_i}{m_i}+\frac{\widetilde{T}_j}{m_j}\right)^{-1} \times\Bigg\{\left(\frac{\widetilde{T}_j}{m_j}\mathbf{u}_i+\frac{\widetilde{T}_i}{m_i}\mathbf{u}_j\right)\cdot(\mathbf{u}_i-\mathbf{u}_j) - \frac{d}{m_i+m_j}(\widetilde{T}_j-\widetilde{T}_i)\left[\frac{\widetilde{T}_i}{m_i}+\frac{\widetilde{T}_j}{m_j}+\frac{3}{2d}\times(\mathbf{u}_i-\mathbf{u}_j)^2\right]\Bigg\}, \tag{C9}$$

respectively, where ξ_{ij} is given by Eq. (2.31). Substitution of Eqs. (C8) and (C9) into Eqs. (C2) and (C3) yields

$$\nu_{ij}=\xi_{ij}, \tag{C10}$$

$$T_{ij}=\widetilde{T}_i+\frac{2m_im_j}{(m_i+m_j)^2}\left\{\widetilde{T}_j-\widetilde{T}_i+\frac{(\mathbf{u}_i-\mathbf{u}_j)^2}{2d}\times\left[m_j+\frac{\widetilde{T}_j-\widetilde{T}_i}{\widetilde{T}_i/m_i+\widetilde{T}_j/m_j}\right]\right\}. \tag{C11}$$

Note that $T_{ij}-\widetilde{T}_i+T_{ji}-\widetilde{T}_j=[m_im_j/(m_i+m_j)](\mathbf{u}_i-\mathbf{u}_j)^2/d$. It is also interesting to remark that the term proportional to $(\mathbf{u}_i-\mathbf{u}_j)^2(\widetilde{T}_j-\widetilde{T}_i)$ in Eq. (C11) is absent in the kinetic model based on the Boltzmann equation for Maxwell molecules [23].

Equations (C5), (C10), and (C11), along with Eq. (C1), close the construction of the kinetic model (4.1) for mixtures of elastic hard spheres. Next, the corresponding kinetic model for granular mixtures is defined by Eq. (4.2) This extended kinetic model has the same collisional transfer of momentum and energy as the true inelastic term J_{ij}^{inel}, as a consequence of Eqs. (2.16), (2.17), (C2), and (C3), at least in the Gaussian approximation (2.29). As a consequence, the temperature ratios $\gamma_i=T_i/T$ of the HCS are the same as those obtained from the inelastic Boltzmann equation.

The NS transport coefficients predicted by the kinetic model can be evaluated from the Chapman-Enskog method. Since $\mathbf{u}_i-\mathbf{u}=\mathbf{j}_i/m_in_i$, one has

$$\mathbf{u}_{ij}=\mathbf{u}+\frac{\mathbf{j}_i}{n_i(m_i+m_j)}+\frac{\mathbf{j}_j}{n_j(m_i+m_j)}. \tag{C12}$$

Furthermore, to NS order, $\widetilde{T}_i=T_i=\gamma_iT$ and

$$T_{ij}=T\left[\gamma_i+\frac{2m_im_j}{(m_i+m_j)^2}(\gamma_j-\gamma_i)\right]. \tag{C13}$$

As a consequence, the reference distribution f_{ij} becomes

$$f_{ij}(\mathbf{v})=f_{ij}^{(0)}(\mathbf{V})\left[1+\frac{\mu_{ij}}{T_{ij}}\mathbf{V}\cdot\left(\frac{\mathbf{j}_i}{n_i}+\frac{\mathbf{j}_j}{n_j}\right)\right], \tag{C14}$$

where

$$f_{ij}^{(0)}(\mathbf{V})=n_i\left(\frac{m_i}{2\pi T_{ij}}\right)^{d/2}\exp\left(-\frac{m_i}{2T_{ij}}V^2\right). \tag{C15}$$

Therefore, in the binary case, the linear operators $\mathcal{L}_1$ and $\mathcal{M}_1$ take the forms

$$\mathcal{L}_1f_1^{(1)}=\frac{1+\alpha_{11}}{2}\nu_{11}\left(f_1^{(1)}-f_{11}^{(0)}\mathbf{V}\cdot\frac{\mathbf{j}_1}{n_1\gamma_1T}\right)+\frac{1+\alpha_{12}}{2}\nu_{12}\times\left(f_1^{(1)}-f_{12}^{(0)}\frac{\mu_{12}}{T_{12}}\mathbf{V}\cdot\frac{\mathbf{j}_1}{n_1}\right)-\frac{\zeta_{11}+\zeta_{12}}{2}\times\left(\frac{\partial}{\partial\mathbf{v}}\cdot\mathbf{V}f_1^{(1)}-\frac{\mathbf{j}_1}{\rho_1}\cdot\frac{\partial}{\partial\mathbf{v}}f_1^{(0)}\right), \tag{C16}$$

$$\mathcal{M}_1f_2^{(1)}=-\frac{1+\alpha_{12}}{2}\nu_{12}f_{12}^{(0)}\frac{\mu_{12}}{T_{12}}\mathbf{V}\cdot\frac{\mathbf{j}_2}{n_2}. \tag{C17}$$

The transport coefficients are given again by Eqs. (B1)–(B3), (B6), (B7), and (B13)–(B15), but now the associated collision frequencies (B4), (B8), (B9), and (B24)–(B26) are computed by using the linear operators (C16) and (C17). In particular, the transport coefficients associated with the mass flux are the same as those obtained from the inelastic Boltzmann equation [12,21].

[1] I. Goldhirsch, Annu. Rev. Fluid Mech. **35**, 267 (2003).
[2] N. V. Brilliantov and T. Pöschel, *Kinetic Theory of Granular Gases* (Oxford University Press, Oxford, 2004).
[3] P. A. Martin and J. Piasecki, Europhys. Lett. **46**, 613 (1999).
[4] V. Garzó and J. W. Dufty, Phys. Rev. E **60**, 5706 (1999).
[5] A. Barrat and E. Trizac, Granular Matter **4**, 57 (2002).
[6] J. M. Montanero and V. Garzó, Granular Matter **4**, 17 (2002).
[7] S. R. Dahl, C. M. Hrenya, V. Garzó, and J. W. Dufty, Phys. Rev. E **66**, 041301 (2002).
[8] R. Pagnani, U. M. Bettolo Marconi, and A. Puglisi, Phys. Rev. E **66**, 051304 (2002); D. Paolotti, C. Cattuto, U. M. Bettolo Marconi, and A. Puglisi, Granular Matter **5**, 75 (2003); P. E. Krouskop and J. Talbot, Phys. Rev. E **68**, 021304 (2003); H. Q. Wang, G. J. Jin, and Y. Q. Ma, *ibid.* **68**, 031301 (2003); J. J. Brey, M. J. Ruiz-Montero, and F. Moreno, Phys. Rev. Lett. **95**, 098001 (2005).
[9] M. Schröter, S. Ulrich, J. Kreft, J. B. Swift, and H. L. Swinney, Phys. Rev. E **74**, 011307 (2006).
[10] R. D. Wildman and D. J. Parker, Phys. Rev. Lett. **88**, 064301 (2002); K. Feitosa and N. Menon, *ibid.* **88**, 198301 (2002).
[11] J. T. Jenkins and F. Mancini, Phys. Fluids A **1**, 2050 (1989); P. Zamankhan, Phys. Rev. E **52**, 4877 (1995); B. Arnarson and J. T. Willits, Phys. Fluids **10**, 1324 (1998); J. T. Willits and B. Arnarson, *ibid.* **11**, 3116 (1999); D. Serero, I. Goldhirsch, S. H. Noskowicz, and M.-L. Tan, J. Fluid Mech. **554**, 237 (2006).
[12] V. Garzó and J. W. Dufty, Phys. Fluids **14**, 1476 (2002).
[13] V. Garzó and J. M. Montanero, Phys. Rev. E **69**, 021301 (2004).

[14] J. J. Brey, M. J. Ruiz-Montero, D. Cubero, and R. García-Rojo, Phys. Fluids **12**, 876 (2000).
[15] J. M. Montanero and V. Garzó, Phys. Rev. E **67**, 021308 (2003).
[16] A. Santos, V. Garzó, and J. W. Dufty, Phys. Rev. E **69**, 061303 (2004).
[17] A. Santos and A. Astillero, Phys. Rev. E **72**, 031308 (2005).
[18] V. Garzó, J. M. Montanero, and J. W. Dufty, Phys. Fluids **18**, 083305 (2006).
[19] T. P. C. van Noije and M. H. Ernst, Granular Matter **1**, 57 (1998).
[20] See http://www.unex.es/eweb/fisteor/vicente/granular_files.html/
[21] V. Garzó and J. M. Montanero, e-print arXiv:cond-mat/0604552, J. Stat. Phys. (to be published).
[22] A. Astillero and A. Santos, Phys. Rev. E **72**, 031309 (2005).
[23] V. Garzó and A. Santos, *Kinetic Theory of Gases in Shear Flows. Nonlinear Transport* (Kluwer Academic, Dordrecht, 2003).
[24] P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954).
[25] P. Welander, Ark. Fys. **7**, 507 (1954).
[26] J. J. Brey, F. Moreno, and J. W. Dufty, Phys. Rev. E **54**, 445 (1996).
[27] J. J. Brey, J. W. Dufty, and A. Santos, J. Stat. Phys. **97**, 281 (1999).
[28] J. W. Dufty, A. Baskaran, and L. Zogaib, Phys. Rev. E **69**, 051301 (2004).
[29] E. P. Gross and M. Krook, Phys. Rev. **102**, 593 (1956).
[30] V. Garzó, A. Santos, and J. J. Brey, Phys. Fluids A **1**, 380 (1989).
[31] P. Andries, K. Aoki, and B. Perthame, J. Stat. Phys. **106**, 993 (2002).