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CÉPADUÈS ÉDITIONS

On the Influence of Gravity on the Thermal Conductivity

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1 Introduction

The simplest heat flow problem consists of a gas enclosed between two infinite, parallel plates kept at different temperatures. In the continuum limit, the Fourier law establishes a linear relation between the heat flux and the thermal gradient, i.e., $q = -\kappa \nabla T$, where κ is the thermal conductivity coefficient. The continuum description applies when $\lambda/L \ll 1$ and $\lambda/\ell \ll 1$, where λ is the mean free path, L is the separation between the plates, and $\ell \sim T|\nabla T|^{-1}$ is the characteristic length over which the temperature changes. Nevertheless, exact results from the Boltzmann equation for Maxwell molecules and from the BGK model for general interactions [1, 2] show that the Fourier law still holds for large gradients ($\lambda \sim \ell$), provided that the Knudsen number $Kn = \lambda/L$ is small.

In this paper we are interested in studying the influence on the heat flux of an external field g (e.g., gravity) normal to the plates. We will also assume that the flow velocity vanishes. This absence of convection is possible if the Rayleigh number is smaller than its critical value ($Ra < 1700$) [3]. This precludes the existence of the Rayleigh-Bénard instability, that has been studied for dilute gases by other authors [4]. In addition to λ , L , and ℓ , the presence of gravity introduces a fourth characteristic length, namely $h \sim k_B T/mg$, which represents the vertical distance over which the field produces a significant effect. In ordinary laboratory conditions, h is several orders of magnitude larger than λ and ℓ , so that the constitutive equations are not affected by the action of gravity. However, discrepancies with respect to the Navier-Stokes predictions can be expected if h is not extremely large. According to a recent perturbation solution of the Boltzmann equation through second order in g [5], one can estimate that the heat flux deviates from the Fourier law as much as 10% if $\lambda \sim \ell \sim 0.01h$. The aim of

this paper is to go beyond the second order in g by using the BGK model of the Boltzmann equation. Specifically, we will obtain the hydrodynamic profiles as well as the pressure tensor and the heat flux through sixth order in gravity.

2 Description of the problem

Let us consider a dilute gas described by the BGK kinetic equation [6]:

$$\frac{\partial}{\partial t} f + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \frac{\partial}{\partial \mathbf{v}} f = -\nu(f - f_L). \quad (1)$$

Here, $f(\mathbf{r}, \mathbf{v}, t)$ is the one-particle distribution function, \mathbf{F} is an external force, m is the mass of a particle, ν is a collision frequency, and f_L is the local equilibrium distribution function, that is characterized by the local density n , the local velocity \mathbf{u} , and the local temperature T , defined as

$$\{n, n\mathbf{u}, 3nk_B T\} = \int d\mathbf{v} \{1, \mathbf{v}, m(\mathbf{v} - \mathbf{u})^2\} f, \quad (2)$$

where k_B is the Boltzmann constant. The collision frequency is proportional to the density and its dependence on the temperature models the influence of the interaction potential. For instance, for Maxwell molecules $\nu \propto n$, while $\nu \propto nT^{1/2}$ for hard spheres.

The problem we want to investigate is that of a gas enclosed in a slab between two plates at different temperatures. We assume the existence of a stationary state with spatial variation along the normal direction z only and a constant external field $\mathbf{F} = -mg\hat{\mathbf{z}}$ along that direction. The constant g can be interpreted as the gravitational acceleration. In addition, we assume that there is no convection, i.e., $\mathbf{u} = 0$. In order to ease the notation, it is convenient to introduce dimensionless quantities. To that end, we choose an arbitrary point z_0 in the bulk as the origin and take the quantities at that point (denoted by a subscript 0) as reference values. Therefore, we define $T^* \equiv T/T_0$, $p^* \equiv p/p_0$, $\mathbf{v}^* \equiv \mathbf{v}/v_0$, $f^* \equiv (k_B T_0/p_0)v_0^3 f$, $g^* \equiv g/v_0\nu_0$, where $p = nk_B T$ is the hydrostatic pressure and $v_0 \equiv (k_B T_0/m)^{1/2}$ is a thermal velocity. In these units, $g^* = \lambda_0/h_0$, where we define the mean free path as $\lambda_0 = v_0/\nu_0$. In the case of the spatial variable z , it is convenient to rescale it in a nonlinear way that takes into account the local dependence of the collision frequency. Consequently, we define

$$s = v_0^{-1} \int_{z_0}^z dz' \nu(z'). \quad (3)$$

Thus, the stationary BGK equation reads

$$\left(1 + v_2^* \partial_s - g^* \frac{T^*}{p^*} D_v\right) f^* = f_L^*, \quad (4)$$

where $\partial_s \equiv \partial/\partial s$ and $D_v \equiv \partial/\partial v_z^*$. Furthermore, for the sake of concreteness, we have restricted ourselves to the case of Maxwell molecules (i.e., $\nu \propto p/T$). In the geometry of the problem, the relevant velocity moments are defined as

$$M_{\alpha\beta} = \int d\mathbf{v}^* v^{*2\alpha} v_z^{*\beta} f^*. \quad (5)$$

In particular, the corresponding moments at local equilibrium are

$$M_{\alpha\beta}^L = \frac{(2\alpha + \beta + 1)!!}{\beta + 1} p^* T^{*\alpha-1+\beta/2} \quad (6)$$

for β even, being zero otherwise. Conservation of momentum and energy implies that $\partial_s M_{02} = -g^*$ and $\partial_s M_{11} = 0$.

In the absence of gravitation ($g = 0$), Eq. (4) has an *exact* solution [2] characterized by a constant pressure, $p^* = 1$, and a "linear" temperature profile, $T^* = 1 + \epsilon s$, that applies to arbitrary values of the reduced thermal gradient $\epsilon = \lambda_0/\ell_0$. The velocity moments are *polynomials* in both s and ϵ . Their explicit expression for $\beta + 2(\alpha - 1) \geq 0$ is [2]

$$M_{\alpha\beta} = (-1)^\beta \sum_{\substack{r=0 \\ (r+\beta)\text{even}}}^{\beta+2(\alpha-1)} \frac{(2\alpha + \beta + r + 1)!! (\alpha - 1 + \frac{\beta+r}{2})!}{(\alpha - 1 + \frac{\beta-r}{2})! (\beta + r + 1)} \epsilon^r (1 + \epsilon s)^{\alpha-1+(\beta-r)/2}. \quad (7)$$

In particular, $M_{11} = -5\epsilon$, which means that the Fourier law holds even for large thermal gradients.

The motivation of this paper is to analyze the influence of gravitation on the profiles and transport properties of the above steady Fourier flow. However, the presence of the operator D_v in Eq. (4) complicates the problem significantly. A convenient strategy is to take the pure steady Fourier flow corresponding to a value of ϵ equal to the actual thermal gradient at the point $s = 0$ as a reference state. Consequently, we will carry out a perturbation expansion in powers of g :

$$f^* = f^{(0)} + f^{(1)} g^* + f^{(2)} g^{*2} + \dots, \quad (8)$$

$$M_{\alpha\beta} = M_{\alpha\beta}^{(0)} + M_{\alpha\beta}^{(1)} g^* + M_{\alpha\beta}^{(2)} g^{*2} + \dots, \quad (9)$$

$$p^* = p^{(0)} + p^{(1)} g^* + p^{(2)} g^{*2} + \dots, \quad (10)$$

$$T^* = T^{(0)} + T^{(1)} g^* + T^{(2)} g^{*2} + \dots, \quad (11)$$

where $M_{\alpha\beta}^{(0)}$ is given by Eq. (7), $p^{(0)} = 1$, and $T^{(0)} = 1 + \epsilon s$. By definition, $p^{(k)}(0) = T^{(k)}(0) = \partial_s T^{(k)}|_{s=0} = 0$ for $k \geq 1$. It must be emphasized that the terms of order g^{*k} are *nonlinear* functions of ϵ since no restriction to the order on ϵ exists.

3 Perturbation expansion

In this section we obtain the hydrodynamic profiles $p^{(k)}$ and $T^{(k)}$, the momentum flux $M_{02}^{(k)}$, and the heat flux $M_{11}^{(k)}$ through order $k = 6$. Insertion of Eq. (8) into Eq. (4) yields

$$f^{(k)} = \sum_{j=0}^{\infty} (-v_z^* \partial_s)^j \left[f_L^{(k)} + D_v \sum_{i=0}^{k-1} \left(\frac{T^*}{p^*} \right)^{(i)} f^{(k-i-1)} \right]. \quad (12)$$

This is a *formal* solution, since $f_L^{(k)}$ is a functional of $f^{(k)}$ through its dependence on the pressure and temperature. Taking moments in Eq. (12), one has

$$\begin{aligned} \Delta M_{\alpha\beta}^{(k)} &\equiv M_{\alpha\beta}^{(k)} - M_{\alpha\beta}^{L(k)} = \sum_{j=1}^{\infty} (-\partial_s)^j M_{\alpha,\beta+j}^{L(k)} - \sum_{j=0}^{\infty} (-\partial_s)^j \sum_{i=0}^{k-1} \left(\frac{T^*}{p^*} \right)^{(i)} \\ &\quad \times \left(2\alpha M_{\alpha-1,\beta+j+1}^{(k-i-1)} + (\beta+j) M_{\alpha,\beta+j-1}^{(k-i-1)} \right). \end{aligned} \quad (13)$$

The fact that f and f_L have the same hydrodynamic moments leads to the consistency conditions

$$\Delta M_{00}^{(k)} = \Delta M_{01}^{(k)} = \Delta M_{10}^{(k)} = 0, \quad (14)$$

for any k . In order to convert Eq. (13) into an explicit equation that can be solved recursively, we need to know the spatial dependence of $p^{(k)}$ and $T^{(k)}$. It turns out that $p^{(k)}$ is a polynomial of degree $k - 2$ in s (except $p^{(1)}$, that is linear in s), while $T^{(k)}$ is a polynomial of degree $k + 1$, the coefficients being nonlinear functions of the reduced thermal gradient ϵ . Thus, at a given order k , insertion of these polynomials into Eq. (13) and application of the consistency requirements (14) allow one to determine the unknown coefficients and the problem can be recursively solved. Notice that, seen as functions of the actual space variable z , $p^{(k)}$ and $T^{(k)}$ are much more complicated than just polynomials. In fact, in the absence of gravitational force, the relationship between s and z is nonlinear: $z = \lambda_0(s + \frac{1}{2}\epsilon s^2)$. In general, such a relationship can be obtained inverting Eq. (3) as

$$z = z_0 + \lambda_0 \int_0^s ds' \frac{T^*(s')}{p^*(s')}. \quad (15)$$

The above scheme is straightforward but tedious to carry out. Since all the manipulations are algebraic, they render themselves to the use of symbolic programming languages. In this paper, we have evaluated the perturbation expansion through sixth order in the field. Since the expressions of the hydrodynamic profiles become progressively longer, here we only give the

explicit results through order $k = 4$:

$$p^* = 1 - sg^* - \frac{276}{5}\epsilon^2 s g^{*3} - \frac{1}{5}s \left[\frac{12}{5}\epsilon (112973\epsilon^2 + 30) + 588\epsilon^2 s \right] g^{*4} + \mathcal{O}(g^{*5}), \quad (16)$$

$$\begin{aligned} T^* &= 1 + \epsilon s + \frac{1}{2}\epsilon s^2 g^* - \epsilon s^2 \left(\frac{66}{5}\epsilon - \frac{1}{3}s \right) g^{*2} - \epsilon s^2 \left[\frac{16}{25}(6624\epsilon^2 + 5) \right. \\ &\quad \left. + \frac{346}{15}\epsilon s - \frac{1}{4}s^2 \right] g^{*3} - \frac{1}{5}\epsilon s^2 \left[\frac{12}{25}\epsilon (50765962\epsilon^2 + 31445) \right. \\ &\quad \left. + \frac{2}{15}(399621\epsilon^2 + 200)s + \frac{971}{6}\epsilon s^2 - s^3 \right] g^{*4} + \mathcal{O}(g^{*5}). \end{aligned} \quad (17)$$

Once the hydrodynamic profiles are known, Eq. (13) can be used to obtain all the velocity moments at a given order. The most relevant moment is M_{11} , which is related to the heat flux, $q_z = (p_0 v_0/2)M_{11}$. Another important quantity is $M_{02} = P_{zz}/p_0$, which measures the anisotropy of the pressure tensor P . As said above, in the absence of gravitation the Fourier law applies exactly, i.e. $q_z^{(0)} = -\kappa \partial T/\partial z = -(5p_0 v_0/2)\epsilon$, and the pressure tensor is isotropic, i.e. $P_{zz}^{(0)} = p_0$. In order to characterize the deviations from these Navier-Stokes predictions due to gravity, we introduce the following reduced quantities:

$$\Lambda(\epsilon, g^*) = \frac{q_z}{q_z^{(0)}} = 1 + \sum_{k=1}^{\infty} \Lambda^{(k)}(\epsilon) g^{*k}, \quad (18)$$

$$\gamma(\epsilon, g^*) = \frac{P_{zz}|_{z=z_0}}{p_0} = 1 + \sum_{k=1}^{\infty} \gamma^{(k)}(\epsilon) g^{*k}. \quad (19)$$

The results show that $\Lambda^{(k)}$ and $\gamma^{(k)}$ are polynomials in ϵ of degree k and a defined parity,

$$\Lambda^{(k)}(\epsilon) = \sum_{\ell=0}^k \Lambda_{\ell}^{(k)} \epsilon^{\ell}, \quad \gamma^{(k)}(\epsilon) = \sum_{\ell=1}^k \gamma_{\ell}^{(k)} \epsilon^{\ell}. \quad (20)$$

To second order in g^* we get $\Lambda_1^{(1)} = \frac{58}{5}$, $\Lambda_0^{(2)} = \frac{16}{5}$, $\Lambda_2^{(2)} = \frac{47968}{25}$, $\gamma_1^{(1)} = 0$, and $\gamma_2^{(2)} = \frac{84}{5}$. The coefficients $\Lambda_{\ell}^{(k)}$ and $\gamma_{\ell}^{(k)}$ for $3 \leq k \leq 6$ are given in Tables 1 and 2, respectively.

4 Discussion

The numerical coefficients appearing in Eqs. (16) and (17), as well as in Tables 1 and 2 clearly indicate that the expansion in powers of g^* , Eq. (8),

ℓ	k			
	3	4	5	6
0	—	$\frac{4984}{25}$	—	$\frac{1488752}{25}$
1	$\frac{36788}{25}$	—	$\frac{84010272}{125}$	—
2	—	$\frac{240215772}{125}$	—	$\frac{1971403262472}{625}$
3	$\frac{171421552}{125}$	—	$\frac{3152117447716}{625}$	—
4	—	$\frac{1482769502032}{625}$	—	$\frac{14018843605272988}{625}$
5	—	—	$\frac{24150803847769024}{3125}$	—
6	—	—	—	$\frac{651597385814170947712}{15625}$

Table 1: Coefficients $\Lambda_\ell^{(k)}$ for $3 \leq k \leq 6$

ℓ	k			
	3	4	5	6
1	$\frac{24}{5}$	—	$\frac{30048}{25}$	—
2	—	$\frac{196272}{25}$	—	$\frac{860948304}{125}$
3	$\frac{236316}{25}$	—	$\frac{2282442504}{125}$	—
4	—	$\frac{1579897956}{125}$	—	$\frac{43675137132624}{625}$
5	—	—	$\frac{20495361514332}{625}$	—
6	—	—	—	$\frac{449964261460472196}{3125}$

Table 2: Coefficients $\gamma_\ell^{(k)}$ for $3 \leq k \leq 6$

is only asymptotic. This does not pose a serious problem, at least from a practical point of view, except for values of g^* (say, $g^* > 10^{-2}$) that correspond to gravitational fields unrealistically large. As a consequence, only the first few terms in the expansion are useful for small values of g^* . In the subsequent analysis, terms of order g^{*3} and higher will not be considered. This also allows us to make a closer comparison with results derived from the Boltzmann equation for Maxwell molecules [5]. Notwithstanding, the knowledge of the remaining terms might be useful to attempt to resum the infinite series and get the transport properties for arbitrary values of g^* . The solution to the Boltzmann equation [5] through order g^{*2} exhibits the same structure as in Eqs. (16)–(19), so that only the numerical coefficients differ. As a matter of fact, the coefficient $\frac{66}{5}$ appearing in Eq. (17) is replaced by $\frac{468}{45}$; in addition, the Boltzmann solution yields $\Lambda_1^{(1)} = \frac{46}{5}$, $\Lambda_0^{(2)} = \frac{12}{5}$, $\Lambda_2^{(2)} = 503.7$, $\gamma_1^{(1)} = 0$ and $\gamma_2^{(2)} = \frac{128}{45}$. The differences indicate that the influence of gravity is stronger in the BGK description than in the Boltzmann one, especially in the case of the pressure anisotropy. In order to carry out a more detailed comparison, it is convenient to construct Padé approximants for the generalized thermal conductivity coefficient Λ . More specifically, we

consider the approximants

$$\Lambda_{[1,1]}(\epsilon, g^*) = \frac{\Lambda^{(1)}(\epsilon) + [\Lambda^{(1)2}(\epsilon) - \Lambda^{(2)}(\epsilon)] g^*}{\Lambda^{(1)}(\epsilon) - \Lambda^{(2)}(\epsilon) g^*}, \quad (21)$$

$$\Lambda_{[0,2]}(\epsilon, g^*) = \left\{ 1 - \Lambda^{(1)}(\epsilon) g^* + [\Lambda^{(1)2}(\epsilon) - \Lambda^{(2)}(\epsilon)] g^{*2} \right\}^{-1}. \quad (22)$$

In the case of the BGK equation, these two approximants differ less than

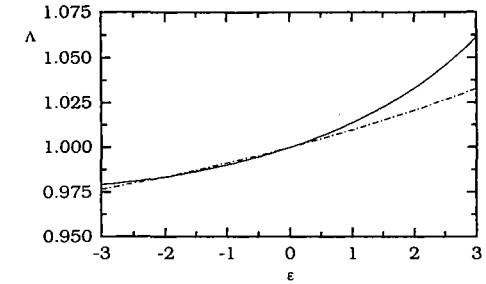


Figure 1: Plot of the (reduced) nonlinear thermal conductivity Λ versus the thermal gradient ϵ at $g^* = 10^{-3}$, as obtained from the BGK model (solid line) and the Boltzmann equation (dashed line).

2% if $g^* < 10^{-2}$ and $|\epsilon|g^* < 3 \times 10^{-3}$. The differences are smaller in the case of the Boltzmann equation. In this range of values for ϵ and g^* , a reliable approximation is $\Lambda \simeq \frac{1}{2} (\Lambda_{[1,1]} + \Lambda_{[0,2]})$. Figure 1 shows this approximation for Λ in the interval $-3 \leq \epsilon \leq 3$ at $g^* = 10^{-3}$, as given by the BGK and Boltzmann equations. We observe that the heat flux increases with respect to its Navier-Stokes value when one heats from above ($\epsilon > 0$), while the opposite happens when one heats from below ($\epsilon < 0$). This effect is not symmetric, since it is more significant if $\epsilon > 0$ than if $\epsilon < 0$. As said above, Fig. 1 also shows that the influence of gravity is less important in the Boltzmann description.

In summary, we have solved the BGK model for a gas simultaneously subjected to a thermal gradient and a parallel gravity field of magnitude g , in the absence of convection. The solution has been obtained by a perturbation expansion in powers of gravity, the reference state being the non-equilibrium pure Fourier flow with arbitrarily large thermal gradients. We have explicitly obtained the solution through sixth order in the field. The results clearly indicate that the expansion is not convergent, although it seems to be at least asymptotic. The work reported in this paper extends previous results

derived from the Boltzmann equation to second order in the field [5]. The similarity in the structure of the coefficients appearing in both descriptions suggests that the expansion obtained from the Boltzmann equation is also asymptotic. Nevertheless, given that at practical level, the values of g are small, the usefulness of the expansion is restricted to the first few terms.

The main results concerning the transport of momentum and energy are that the external field induces (i) anisotropy in the pressure tensor, $(P_{zz} - p)/p \simeq \frac{84}{5}\epsilon^2 g^{*2}$, and (ii) deviations from the Fourier law, $q_z/q_z^{(0)} - 1 \simeq \frac{58}{5}\epsilon g^*$. While the first effect is of second order, the correction to the heat flux is of first order, so that it depends on the sign of the thermal gradient. As a consequence, the heat transport is inhibited when the gas is heated from below ($\epsilon < 0$), while the opposite happens when the gas is heated from above ($\epsilon > 0$).

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