A simple model kinetic equation for inelastic Maxwell particles

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Outline

- The Boltzmann equation for Inelastic Maxwell Particles
- Model kinetic equation
- Results
- Conclusions

Model of Inelastic Hard Spheres (HCS)

- Smooth inelastic hard spheres (mass m, diameter σ, coefficient of normal restitution α)
- Post-collisional velocities:

$$\mathbf{v}' = \mathbf{v} - \frac{1+\alpha}{2}(\mathbf{g}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}$$

Relative velocity

$$\mathbf{v}_1' = \mathbf{v}_1 + \frac{1+\alpha}{2}(\mathbf{g}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}$$

Boltzmann equation for IHS

- Dilute granular gas
- Absence of velocity correlations before collision

$$\partial_t f + \mathbf{v} \cdot \nabla f = J[\mathbf{v}|f]$$

$$J[\mathbf{v}|f] = \sigma^2 \int d\mathbf{v}_1 \int d\widehat{\boldsymbol{\sigma}}\, \Theta(\mathbf{g} \cdot \widehat{\boldsymbol{\sigma}})(\mathbf{g} \cdot \widehat{\boldsymbol{\sigma}}) \times \left[\alpha^{-2} f(\mathbf{v}'') f(\mathbf{v}_1'') - f(\mathbf{v}) f(\mathbf{v}_1)\right]$$

pre-collisional

Model of Inelastic Maxwell Particles (IMP)

- Bobylev, Carrillo, Gamba, Cercignani (2000)

$$\mathrm{g} \cdot \hat{\boldsymbol{\sigma}} \to \mathrm{const}\sqrt{2T/m}\,\hat{\mathrm{g}} \cdot \hat{\boldsymbol{\sigma}}$$

- Ben-Naim, Krapivsky, Ernst, Brito (2002)

$$\mathrm{g} \cdot \hat{\boldsymbol{\sigma}} \to \mathrm{const}\sqrt{2T/m}$$

Boltzmann equation for IHS

$$\partial_t f + \mathbf{v}\cdot\nabla f = J[\mathbf{v}|f]$$

$$J[\mathbf{v}|f] = \sigma^2 \frac{5}{8\pi}\frac{\nu_0}{n}\int d\mathbf{v}_1 \int d\widehat{\boldsymbol{\sigma}}\, \Theta(\mathbf{g}\cdot\widehat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\widehat{\boldsymbol{\sigma}}) \times \left[\alpha^{-2} f(\mathbf{v}'') f(\mathbf{v}_1'') - f(\mathbf{v}) f(\mathbf{v}_1)\right]$$

$\nu_0 = \frac{16}{5} n\sigma^2\sqrt{T/m\pi}$: collision frequency

Boltzmann equation for IMP

- The IMP model is interesting by itself since it allows the derivation of some *exact* results.
- Those results show unambiguously the strong influence of inelasticity on the nonequilibrium properties of the gas.
- The model is useful to gain a broader perspective on the peculiar properties of dissipative gases.

Basic properties of the Boltzmann equation for IMP

- Cooling rate: $$\frac{m}{3n}\int \mathrm{d}\mathbf{v}\, V^2 J[f] = -\zeta(\alpha) T$$
- Collisional rates of change:

$$m\int \mathrm{d}\mathbf{v}\left(V_i V_j - \frac{1}{3}V^2\delta_{ij}\right) J[f] = -\nu_\eta(\alpha)\left(P_{ij} - p\delta_{ij}\right)$$

$$\frac{m}{2}\int \mathrm{d}\mathbf{v}\, V^2 \mathbf{V} J[f] = -\nu_\kappa(\alpha)\mathbf{q}$$

$$n^{-1}\int \mathrm{d}\mathbf{v}\, V^4 J[f] = -\nu_2(\alpha)\langle V^4\rangle + \lambda(\alpha)\nu_0(2T/m)^2$$

Basic properties of the Boltzmann equation for IMP

- Uniform, free cooling state
- Scaling solution: homogeneous cooling state (HCS):

$$\partial_t f(\mathbf{v}) = J[\mathbf{v}|f], \quad \partial_t T = -\zeta T$$

$$f(\mathbf{v},t) = n\left[\frac{m}{2T(t)}\right]^{3/2} f^*_{\mathrm{hcs}}(\mathbf{c}), \quad \mathbf{c} = \frac{\mathbf{v}}{\sqrt{2T(t)/m}}$$

$$\frac{\zeta^*}{2}\partial_{\mathbf{c}} \cdot \mathbf{c} f^*_{\mathrm{hcs}}(\mathbf{c}) = J^*[\mathbf{c}|f^*_{\mathrm{hcs}}]$$

Basic properties of the Boltzmann equation for IMP

- Homogeneous cooling state (HCS):
  - Approach to the HCS (Bobylev, Cercignani, Toscani, 2003)
  - Fourth cumulant (kurtosis):

$$a_2(\alpha) = \frac{4}{15}\langle c^4 \rangle_{\text{hcs}} - 1$$

  - Algebraic high-energy tail:

$$c \gg 1 \Rightarrow f^*_{\text{hcs}}(\mathbf{c}) \sim c^{-3-s(\alpha)}$$

Basic properties of the Boltzmann equation for IMP

- Navier-Stokes transport coefficients:

$$P_{ij} = p\delta_{ij} - \eta(\alpha)\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{3}\nabla\cdot\mathbf{u}\delta_{ij}\right)$$

$$\mathbf{q} = -\kappa(\alpha)\nabla T - \mu(\alpha)\nabla n$$

$\kappa(\alpha)$ and $\mu(\alpha)$ are negative for $\alpha < 1/9$

Why a model kinetic equation for IMP?

- The Boltzmann equation for IMP is more manageable than for IHS and some important properties are accessible in an exact way.
- However, its explicit solution $f(\mathbf{v})$ is not known, even for the HCS.
- Is it possible to construct a simple (and accurate) generalization of the well-known BGK model kinetic equation to the case of IMP?

Model kinetic equation for IMP

Effective collision frequency

$$J[\mathbf{v}|f] \to \widetilde{J}[\mathbf{v}|f] \equiv -\beta(\alpha)\nu_0 \left[f(\mathbf{v}) - f_0(\mathbf{v})\right] + \gamma(\alpha)\nu_0 \partial_{\mathbf{v}} \cdot \mathbf{V} f(\mathbf{v})$$

Friction coefficient

$$f_0(\mathbf{v}) = n \left(\frac{m}{2\pi\theta(\alpha)T}\right)^{3/2} e^{-mV^2/2\theta(\alpha)T}$$

Effective reference temperature

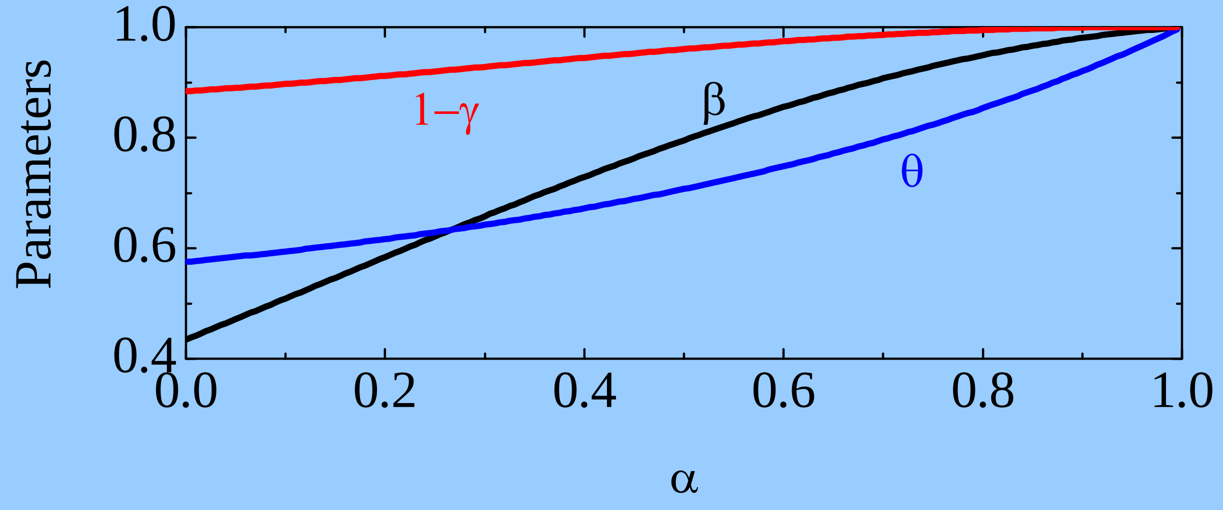
Expressions for the main quantities

Quantity	Boltzmann equation	Kinetic model
$\zeta^* \equiv \zeta/\nu_0$	$\frac{5}{12}(1-\alpha^2)$	$\beta(1-\theta)+2\gamma$
$\nu_\eta^* \equiv \nu_\eta/\nu_0$	$\frac{1}{4}(1+\alpha)^2+\zeta^*$	$\beta\theta+\zeta^*$
$\nu_\kappa^* \equiv \nu_\kappa/\nu_0$	$\frac{1}{6}(1+\alpha)^2+\frac{3}{2}\zeta^*$	$\frac{1}{2}\beta\,(3\theta-1)+\frac{3}{2}\zeta^*$
$\nu_2^* \equiv \nu_2/\nu_0$	$\frac{1}{48}(1+\alpha)^2(5+6\alpha-3\alpha^2)+2\zeta^*$	$\beta(2\theta-1)+2\zeta^*$
λ	$\frac{5}{64}(1+\alpha)^2(11-6\alpha+3\alpha^2)$	$\frac{15}{4}\beta\theta^2$
a_2	$6(1-\alpha)^2/(5+6\alpha-3\alpha^2)$	$(1-\theta)^2/(2\theta-1)$
s	Transcendental eqn.	$2/(1-\theta)$

$\beta(\alpha)$, $\theta(\alpha)$, and $\gamma(\alpha)$ are determined by requir- ing the kinetic model to reproduce the correct $\zeta(\alpha)$, $\nu_\eta(\alpha)$, and $a_2(\alpha)$

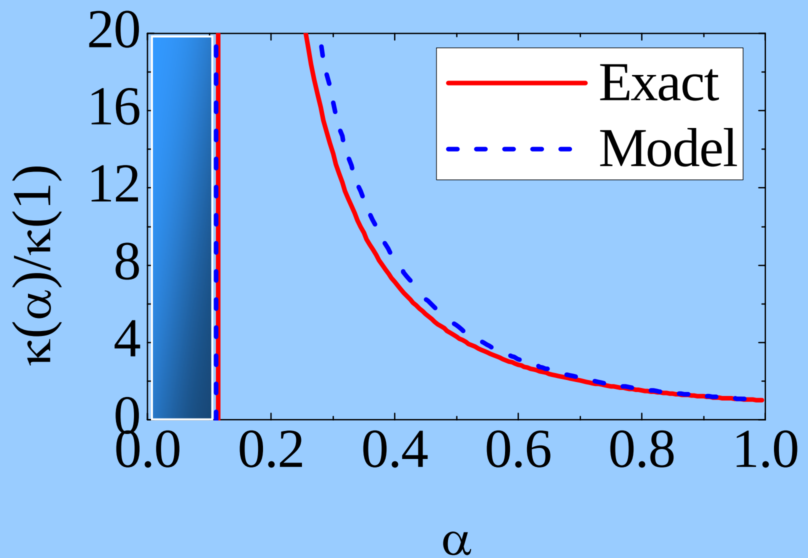
RESULTS

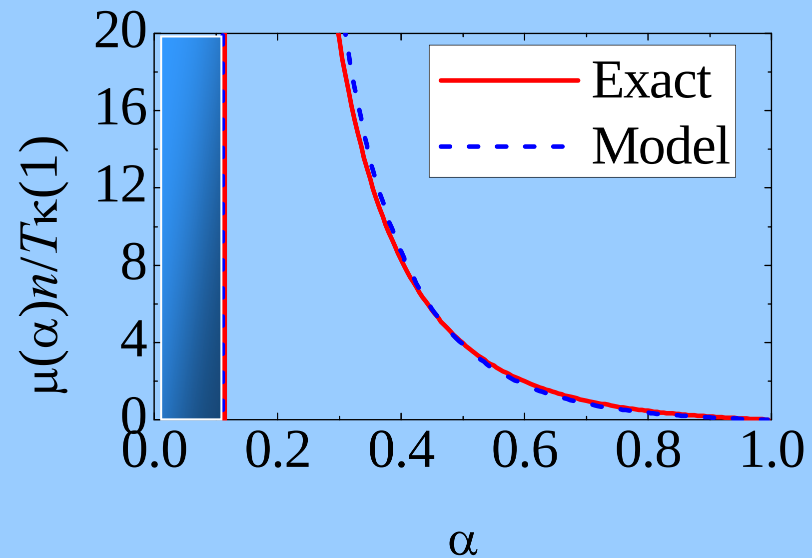
Parameters of the model



RESULTS

Transport coefficients



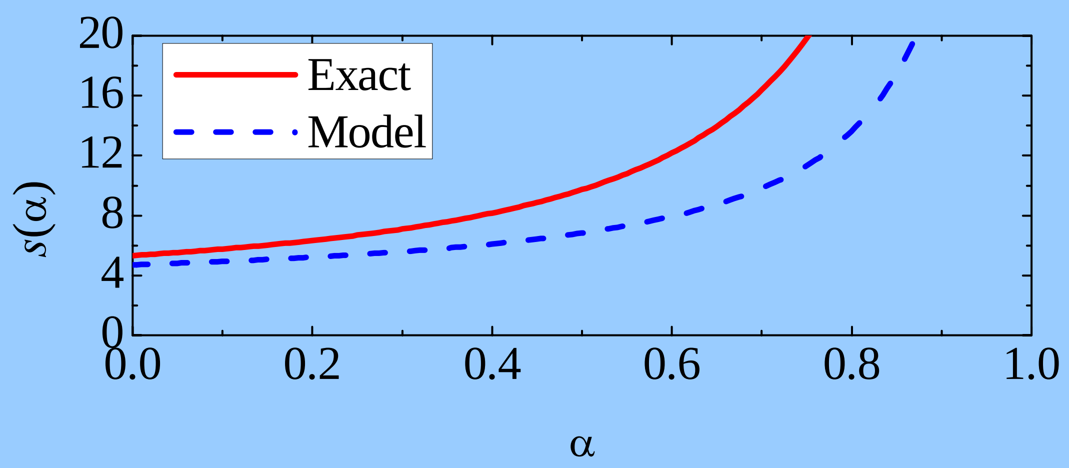


RESULTS

Homogeneous cooling state

$$f^*_{\mathrm{hcs}}(\mathbf{c}) = \frac{(1-\theta)^{-1}}{(\pi\theta)^{3/2}} \left(\frac{\theta}{c^2}\right)^{3/2+(1-\theta)^{-1}} \int_0^{c^2/\theta} dx\, x^{3/2+(1-\theta)^{-1}} e^{-x}$$

$$f^*_{\mathrm{hcs}}(\mathbf{c}) \sim c^{-3-s(\alpha)}$$



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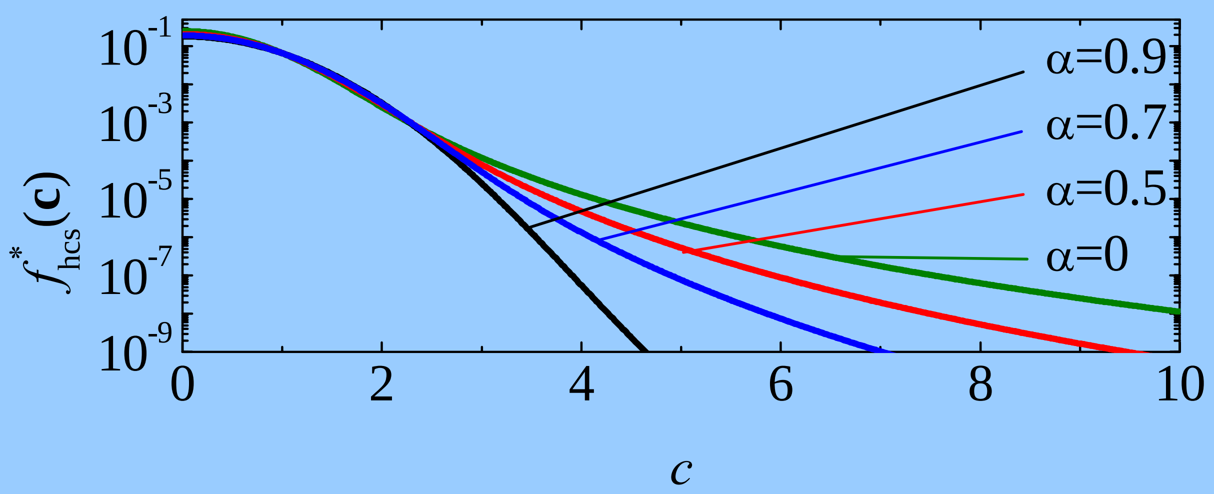
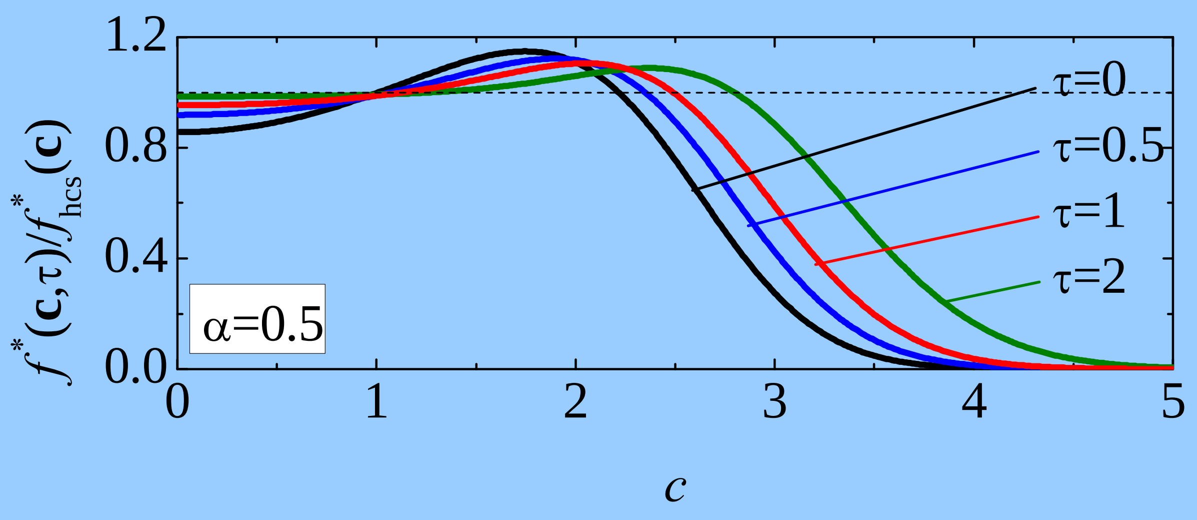


RESULTS

Approach to the HCS

$$\delta f^*(\mathbf{c},\tau) = e^{-\beta[1+3(1-\theta)/2]\tau}\delta f^*\left(e^{-\beta(1-\theta)\tau/2}\mathbf{c}, 0\right)$$



Conclusions

- The proposed kinetic model is a simple extension of the BGK model.
- The effect of the inelastic collisions is played by (i) a relaxation term toward a reference Maxwellian distribution plus (ii) a term representing the action of a friction force.
- The model contains three free parameters: a factor $\beta(\alpha)$ modifying the collision frequency, a factor $\theta(\alpha)$ modifying the temperature of the reference Maxweelian, and a friction coefficient $\gamma(\alpha)$.
- The parameters are determined by fitting the cooling rate, kurtosis, and shear viscosity of IMP.

Conclusions

- The kinetic model can be useful to have access, at least at a semi-quantitative way, to relevant information (such as the velocity distribution function itself) not directly available from the Boltzmann equation for IMP.
- The same philosophy can be applied to extensions of the ellipsoidal statistical kinetic model and to mixtures of IMP.

THANKS!

$$J[\mathbf{v}|f] \to \widetilde{J}[\mathbf{v}|f] \equiv -\beta(\alpha)\nu_0\left[f(\mathbf{v}) - f_0(\mathbf{v})\right] + \gamma(\alpha)\nu_0 \partial_{\mathbf{v}} \cdot \mathbf{V} f(\mathbf{v})$$