Granular Poiseuille flow

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Outline

- Gravity-driven Poiseuille flow for *conventional* gases.
- Newtonian description.
- Gravity-driven Poiseuille flow for *heated granular* gases.
- Kinetic theory description through second order in gravity.
- Results.
- Conclusions.

Jean-Louis Marie Poiseuille (1797-1869)

Poiseuille's law

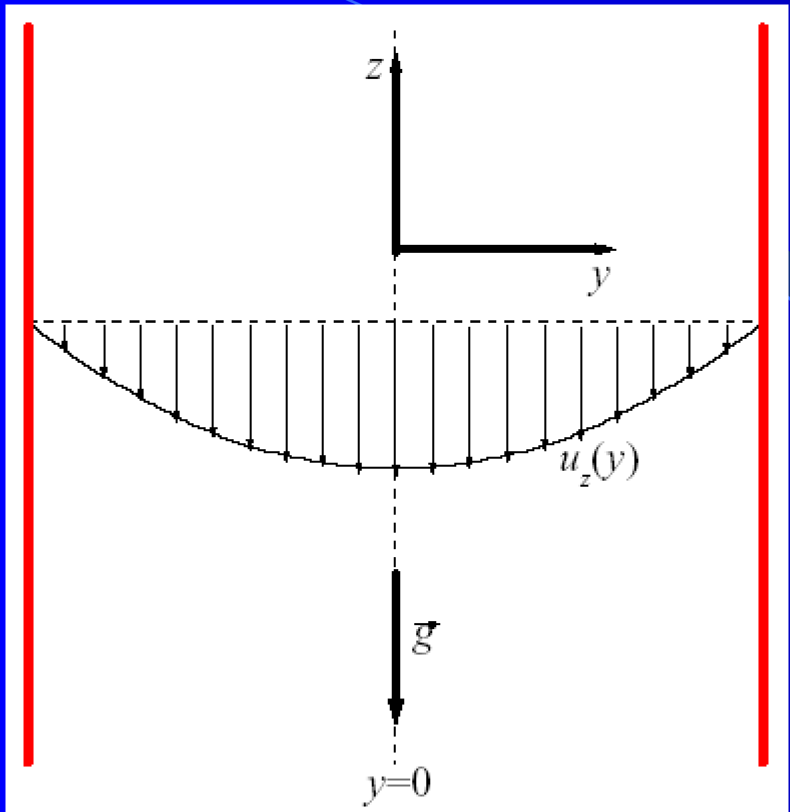
From Wikipedia, the free encyclopedia.

The **Poiseuille's law** (or the **Hagen-Poiseuille law** also named after Gotthilf Heinrich Ludwig Hagen (1797-1884) for his experiments in 1839) is the physical law concerning the voluminal laminar stationary flow Φ_V of incompressible uniform viscous liquid (so called Newtonian fluid) through a cylindrical tube with the constant circular cross-section, experimentally derived in 1838, formulated and published in 1840 and 1846 by Jean Louis Marie Poiseuille (1797-1869), and defined by:

$$\Phi_V = \frac{dV}{dt} = v_s \pi r^2 = \frac{\pi r^4}{8\eta}\left(-\frac{dp^\star}{dz}\right) = \frac{\pi r^4}{8\eta}\frac{\triangle p^\star}{l},$$

where V is a volume of the liquid, poured in the time unit t, v_s median fluid velocity along the axial cylindrical coordinate z, r internal radius of the tube, Δp^* the preasure drop at the two ends, η dynamic fluid viscosity and l characteristic length along z, a linear dimension in a cross-section (in non-cylindrical tube).

Planar Poiseuille flow generated by a gravity field in a *conventional* gas



Conservation equations for momentum and energy

$$\frac{\partial P_{yy}}{\partial y} = 0$$

$$\frac{\partial P_{yz}}{\partial y} = -\rho g$$

$$P_{yz}\frac{\partial u_z}{\partial y} + \frac{\partial q_y}{\partial y} = 0$$

Navier-Stokes (Newtonian) description

$$P_{xx} = P_{yy} = P_{zz} = p$$

Equal normal stresses

$$P_{yz} = -\eta \frac{\partial u_z}{\partial y}$$

Newton's law

$$q_y = -\kappa \frac{\partial T}{\partial y}$$

Fourier's law

$$q_z = 0$$

No longitudinal heat flux

$$p(y) = p_0 = \text{const}$$

$$u_z(y) = u_0 + \frac{\rho_0 g}{2\eta_0} y^2 + \mathcal{O}(g^3)$$

$$T(y) = T_0 - \frac{\rho_0^2 g^2}{12\eta_0 \kappa_0} y^4 + \mathcal{O}(g^4)$$

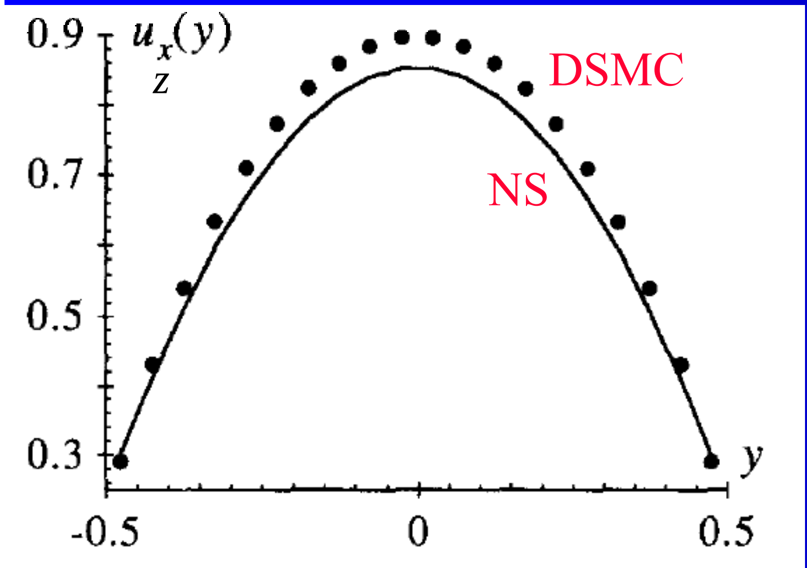
Temperature is *maximal* at the central layer (*y*=0)

Do NS predictions agree with computer simulations?

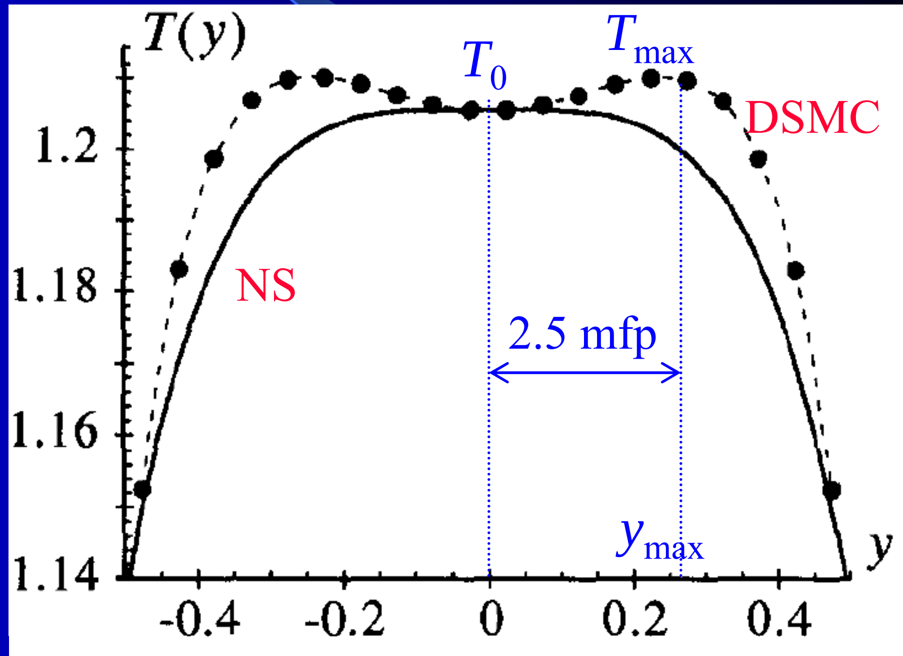
On the validity of hydrodynamics in plane Poiseuille flows

M. Malek Mansour[a,*], F. Baras[a], Alejandro L. Garcia[b,1]

Physica A 240 (1997) 255–267



but ...



A Burnett-order effect?

PHYSICAL REVIEW E VOLUME 60, NUMBER 4 OCTOBER 1999

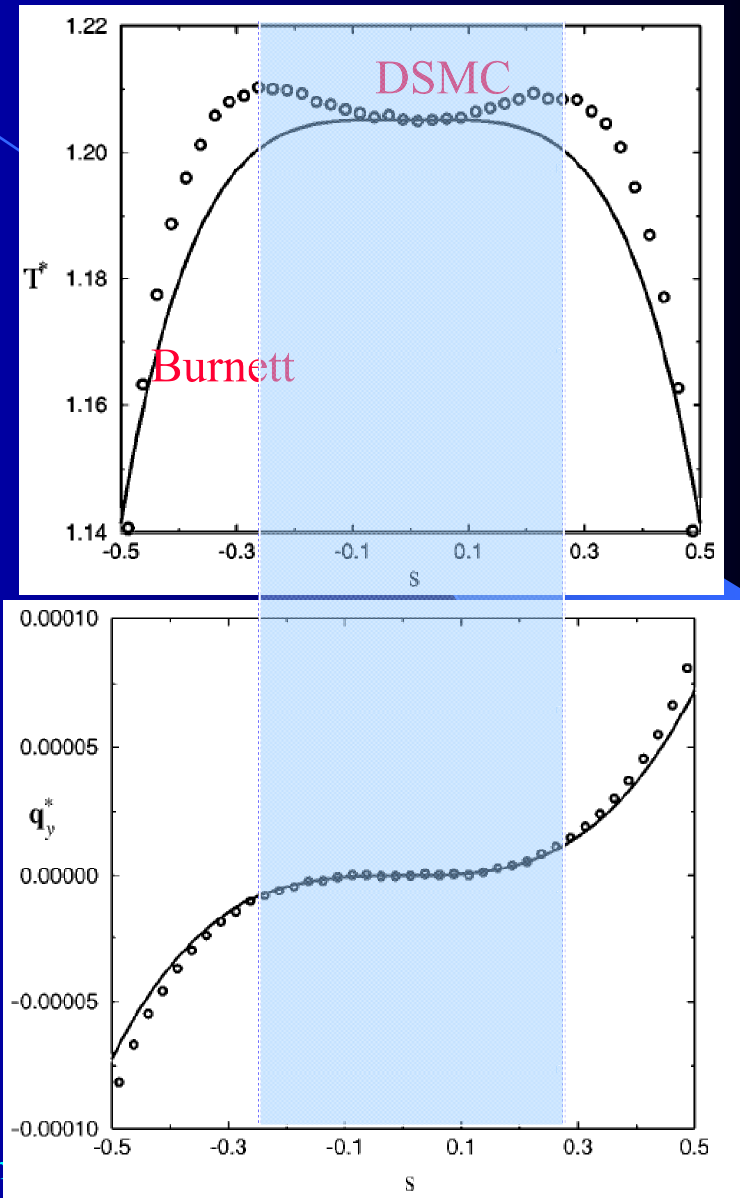
Burnett description for plane Poiseuille flow

F. J. Uribe Alejandro L. Garcia*

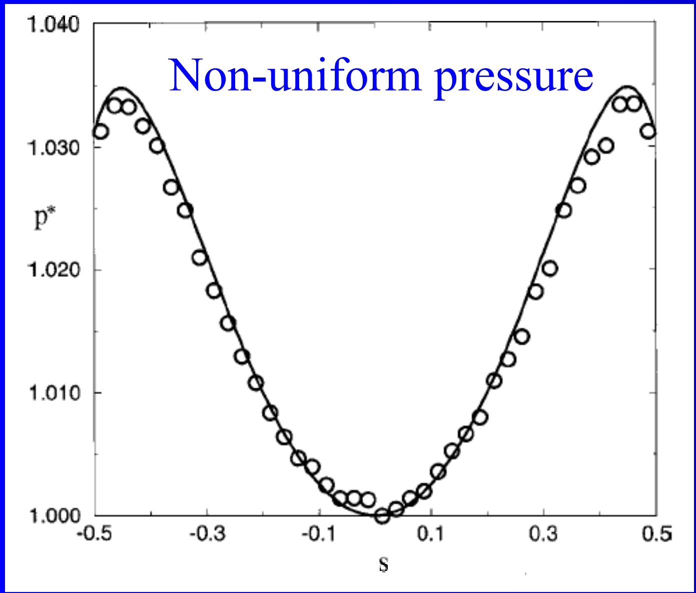
In the slab $y<|y_{max}|$,

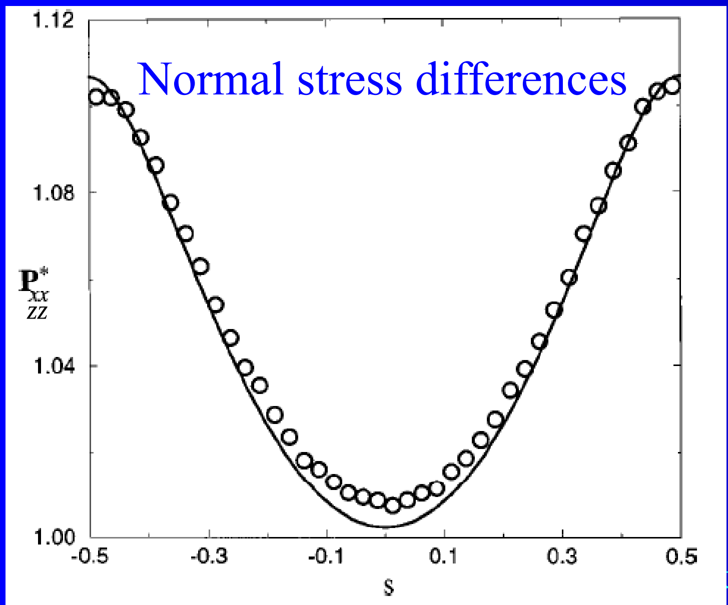
$$\boxed{\mathrm{sgn}\, q_y = \mathrm{sgn}\, \partial T/\partial y}$$

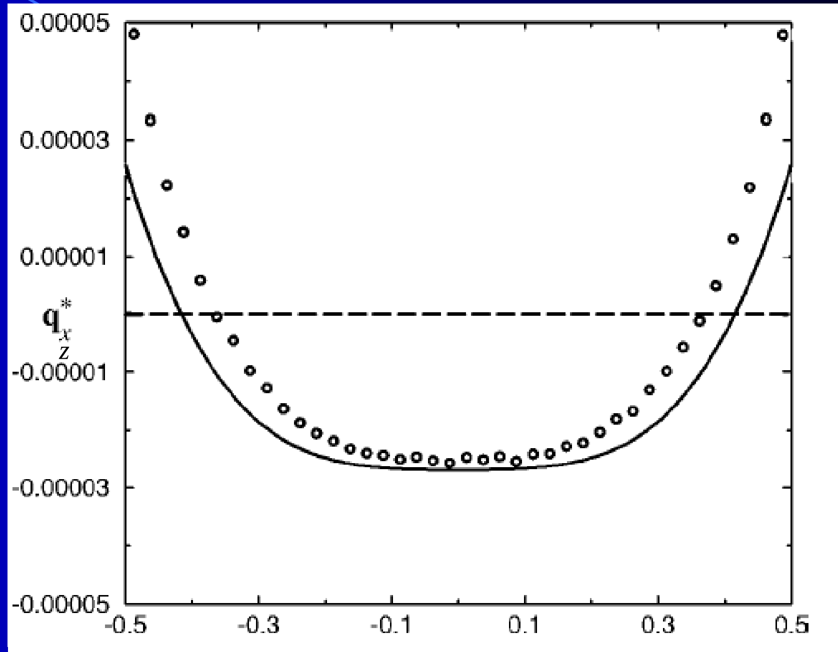
Heat flows from the colder to the hotter layers!!



Other Non-Newtonian properties







Longitudinal component of the heat flux
(but no longitudinal thermal gradient!)

These Non-Newtonian effects are well accounted for by kinetic theory tools:

- Perturbative solution of the BGK and Boltzmann-Maxwell kinetic equations (M. Tij, M. Sabbane, A.S.).
- Grad's method applied to the Boltzman equation for hard spheres (S. Hess, M. Malek Mansour, D. Risso, P. Cordero).
- Asymptotic analysis of the BGK model for small Knudsen numbers (K. Aoki, S. Takata, T. Nakanishi).

Is the gravity-driven Poiseuille flow relevant to real gases?

$$\frac{T_{\max} - T_0}{T_0} \gtrsim 10^{-2} \Rightarrow g\frac{\lambda}{v_{\text{th}}^2} \gtrsim 2 \times 10^{-2}$$

λ: mean free path; v_{th}: thermal velocity

Argon at room conditions $\left\{ \begin{array}{c} g = 9.8 \text{ m/s}^2 \\ \lambda \sim 700 \text{ Å} \\ v_{\text{th}} \sim 400 \text{ m/s} \end{array} \right\}$ $\boxed{g\lambda/v_{\text{th}}^2 \sim 10^{-12} \text{ !!}}$

Fluidized *granular* particles

They are *mesoscopic* particles ($\sigma \sim 1$ mm)

Some typical values $\left\{ \begin{array}{c} g = 9.8 \text{ m/s}^2 \\ \lambda \approx 1 \text{ mm-1cm} \\ v_{th} \gtrsim 1 \text{ m/s} \end{array} \right\}$ $\boxed{g\lambda/v_{th}^2 \sim 10^{-3}\text{-}10^{-1}}$

The dimensionless parameter $g\lambda/v_{th}^2$ measures the strength of gravity between collisions. It can be:

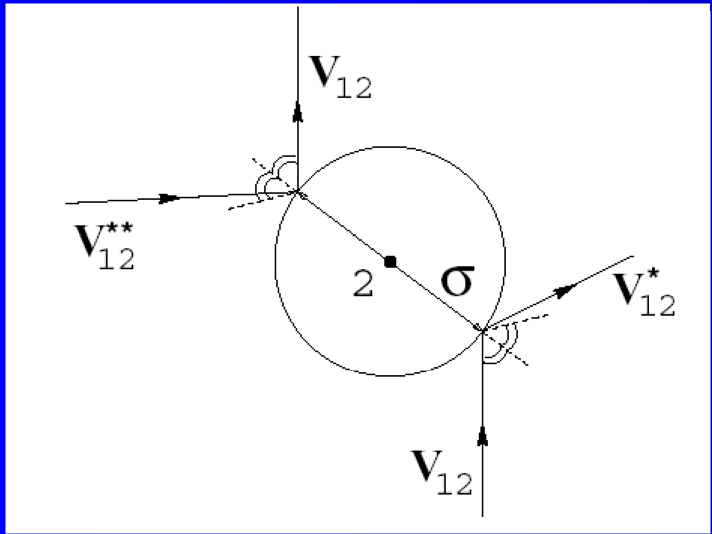
- Large enough as to produce measurable effects.
- Small enough as to allow for a perturbative treatment.

Our main goal is:

- Call attention to the fact that non-Newtonian properties in the gravity-driven Poiseuille flow can be observable on granular gases under laboratory conditions.

- Assess the influence of inelasticity on the hydrodynamic fields and their fluxes.
  E.g., is $(T_{max}-T_0)/T_0$ enhanced or inhibited by inelasticity?

A gas of (smooth) *inelastic* hard spheres



(After T.P.C. van Noije & M.H. Ernst)

α: coefficient of (normal) restitution

Direct collision

$$\mathbf{v}_1' = \mathbf{v}_1 - \frac{1+\alpha}{2}(\mathbf{v}_{12}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}, \quad \mathbf{v}_2' = \mathbf{v}_2 + \frac{1+\alpha}{2}(\mathbf{v}_{12}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}$$

Restituting collision

$$\mathbf{v}_1'' = \mathbf{v}_1 - \frac{1+\alpha}{2\alpha}(\mathbf{v}_{12}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}, \quad \mathbf{v}_2'' = \mathbf{v}_2 + \frac{1+\alpha}{2\alpha}(\mathbf{v}_{12}\cdot\widehat{\boldsymbol{\sigma}})\widehat{\boldsymbol{\sigma}}$$

Boltzmann equation

$$\left(\partial_t + \mathbf{v}\cdot\nabla + \underbrace{\mathbf{g}\cdot\frac{\partial}{\partial \mathbf{v}}}_{\text{Gravity}} + \underbrace{\mathcal{F}}_{\text{External driving}}\right) f = \underbrace{J[f,f]}_{\text{Inelastic collisions}}$$

$$J[f,f] = \sigma^2 \int d\mathbf{v}_1 \int d\widehat{\boldsymbol{\sigma}}\, \Theta((\mathbf{v}-\mathbf{v}_1)\cdot\widehat{\boldsymbol{\sigma}})[(\mathbf{v}-\mathbf{v}_1)\cdot\widehat{\boldsymbol{\sigma}}]\left[\alpha^{-2} f(\mathbf{v}'')f(\mathbf{v}_1'') - f(\mathbf{v})f(\mathbf{v}_1)\right]$$

Collisional "cooling"

$$\frac{m}{3}\int d\mathbf{v}\, V^2 J[f,f] = -\zeta n T$$

Cooling rate

External "heating"
(e.g., vibrations)

$$\frac{m}{3}\int d\mathbf{v}\, V^2 \mathcal{F} f(\mathbf{v}) = -\gamma n T$$

Heating rate

$$\mathbf{V} \equiv \mathbf{v} - \mathbf{u}$$ (peculiar velocity)

$$\zeta \simeq \nu \frac{5}{12}(1-\alpha^2)$$

Gaussian approximation

$$\nu = \frac{16}{5} n\sigma^2 \left(\frac{\pi T}{m}\right)^{1/2}$$

Effective collision frequency

White noise driving

It is a *bulk* heating mechanism that intends to mimic the effect of boundary driving (e.g., vibrations).

Each particle is subjected to the action of a stochastic force with white noise properties:

$$\langle \mathbf{F}^{\mathrm{wn}}(t) \rangle = \mathbf{0}, \quad \langle F_\alpha^{\mathrm{wn}}(t) F_\beta^{\mathrm{wn}}(t') \rangle = m^2 \xi^2 \delta_{\alpha\beta} \delta(t - t')$$

During a small time step Δt, each particle receives a "kick," so its velocity is incremented by a random amount $\Delta \mathbf{v}$

$$\Delta t \Rightarrow |\Delta \mathbf{v}| \sim \xi \sqrt{\Delta t}$$

Diffusion in velocity space: $\mathcal{F} = -\frac{\xi^2}{2} \left(\frac{\partial}{\partial \mathbf{v}} \right)^2 \Rightarrow \gamma = \frac{m\xi^2}{T}$

Heating rate

Our choice: The white noise compensates *locally* for the collisional cooling.

$$\gamma = \zeta \Rightarrow \frac{|\Delta \mathbf{v}|}{v_{\mathrm{th}}} \sim \sqrt{\nu \Delta t (1 - \alpha^2)}$$

The *relative* magnitude of the kick scales with (the square root of) the (local) probability of a collision.

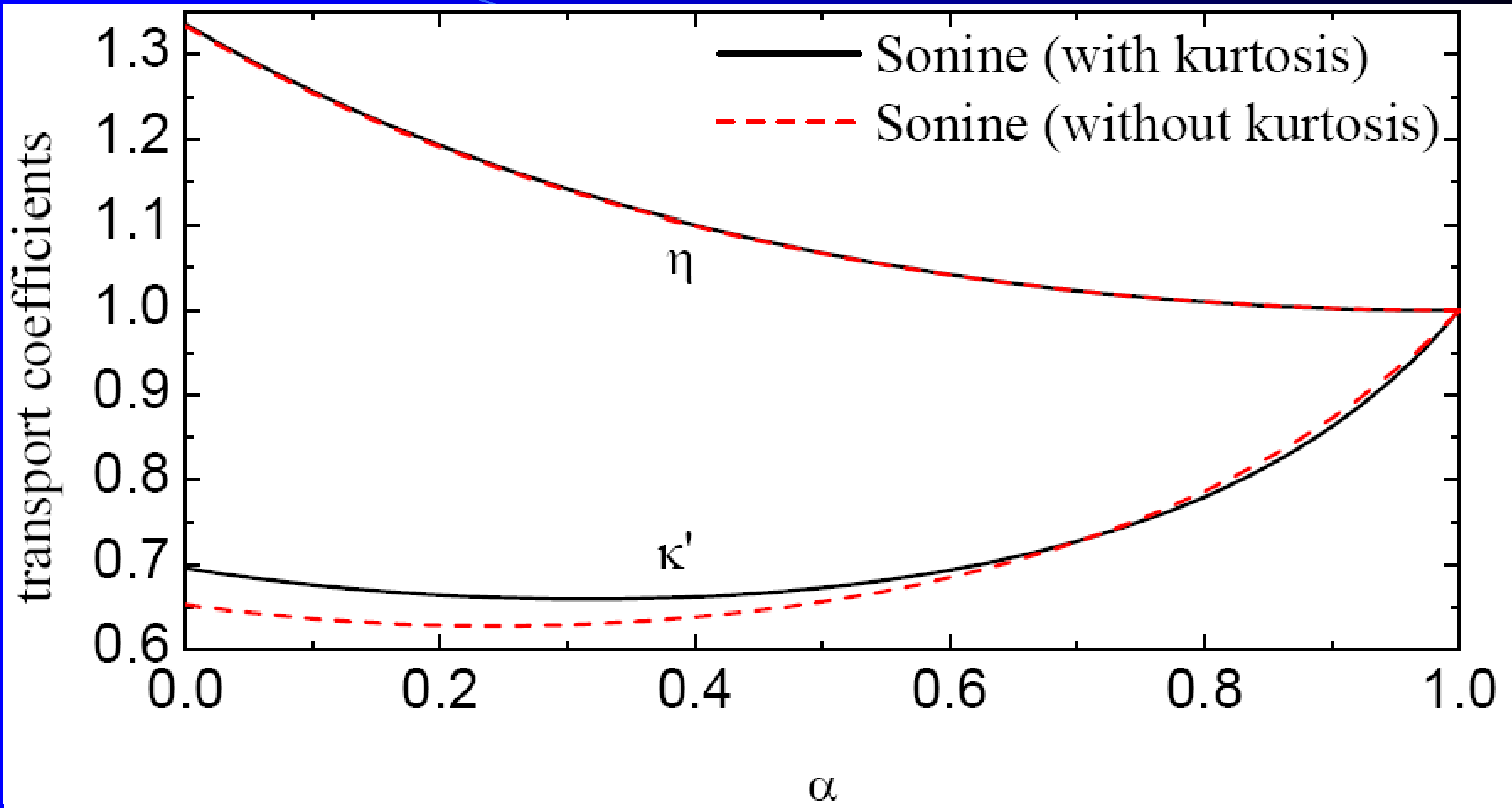
Associated NS transport coefficients:

(Garzó & Montanero, 2002)

$$\eta \simeq \frac{p}{\nu} \frac{4}{(1+\alpha)(3-\alpha)}, \quad \kappa \simeq \frac{5p}{2m\nu} \frac{48}{(1+\alpha)(49-33\alpha)}$$

Increases with inelasticity

Decreases with inelasticity ($\alpha \gtrsim 0.4$)
Increases with inelasticity ($\alpha \lesssim 0.4$)



Sonine (with kurtosis)
Sonine (without kurtosis)
transport coefficients
η
κ'
α
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.0
0.2
0.4
0.6
0.8
1.0

Stationary Boltzmann equation

$$\left(-\underbrace{\frac{\zeta T}{2m}\frac{\partial^2}{\partial \mathbf{v}^2}}_{\text{White noise heating}} - \underbrace{g\frac{\partial}{\partial v_z}}_{\text{Gravity}} + v_y\frac{\partial}{\partial y}\right) f = \underbrace{J[f,f]}_{\text{Inelastic collisions}}$$

BGK-like kinetic model:
(Brey, Dufty, A.S.)

$$J[f,f] \to -\underbrace{\beta(\alpha)\nu}_{\text{Modified collision frequency}}(f-f_\ell) + \underbrace{\frac{\zeta}{2}\frac{\partial}{\partial \mathbf{v}}\cdot\left[(\mathbf{v}-\mathbf{u})\, f\right]}_{\text{Effective drag force: mimics cooling}}$$

$$f_\ell(\mathbf{r},\mathbf{v};t) = n(\mathbf{r},t)\left[\frac{m}{2\pi T(\mathbf{r},t)}\right]^{3/2} \exp\left[-\frac{m\left(\mathbf{v}-\mathbf{u}(\mathbf{r},t)\right)^2}{2T(\mathbf{r},t)}\right]$$

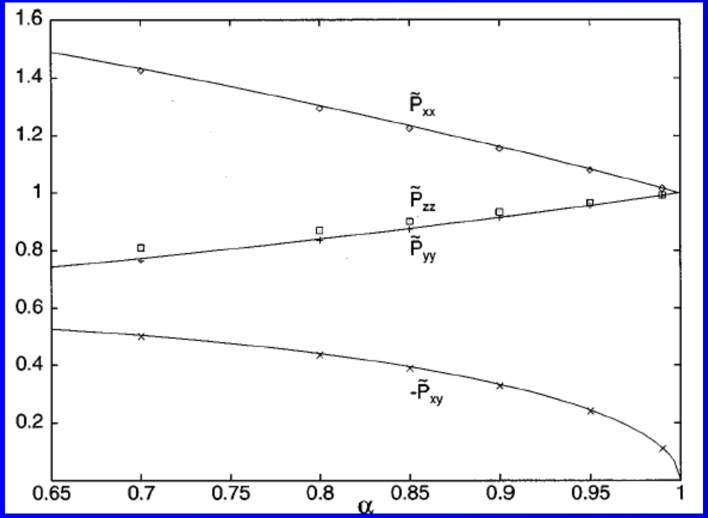
Local Gaussian distribution

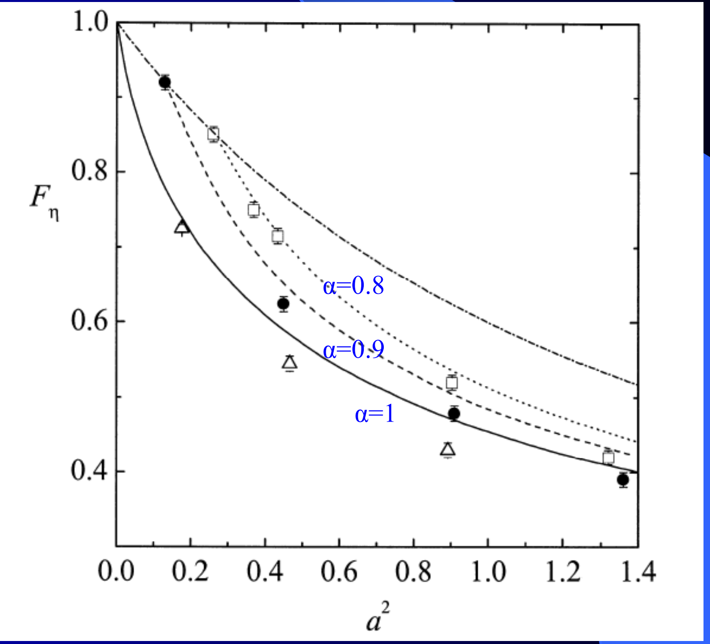
Digression: How reliable is the BGK-like model?

PHYSICAL REVIEW E VOLUME 55, NUMBER 3 MARCH 1997

Steady uniform shear flow in a low density granular gas

J. J. Brey, M. J. Ruiz-Montero, and F. Moreno





Journal of Statistical Physics, Vol. 103, Nos. 5/6, 2001

Nonlinear Couette Flow in a Low Density Granular Gas

M. Tij,[1] E. E. Tahiri,[2] J. M. Montanero,[3] V. Garzó,[4] A. Santos,[5] and J. W. Dufty[5]

Perturbation expansion

$$f(y,\mathbf{V}) = f_\ell(y,\mathbf{V})\left[1 + \Phi^{(1)}(y,\mathbf{V})g + \Phi^{(2)}(y,\mathbf{V})g^2 + \mathcal{O}(g^3)\right]$$

Velocity distribution function

$$\left.\begin{aligned} p(y) &= p_0 + p^{(2)}(y)g^2 + \mathcal{O}(g^4) \\ u_z(y) &= u_0 + u^{(1)}(y)g + \mathcal{O}(g^3) \\ T(y) &= T_0 + T^{(2)}(y)g^2 + \mathcal{O}(g^4) \end{aligned}\right\}$$

Hydrodynamic profiles

Structure of the solution through second order:

$$\Phi^{(1)}(y,\mathbf{V}) = V_z(a_0 + a_1 V_y^2 + a_2 V_y y)$$

$$\begin{aligned} \Phi^{(2)}(y,\mathbf{V}) \;=\; & b_0 + b_1 V_y^2 + b_2 V_y y + b_3 y^2 + b_4 V_y^4 + b_5 V_y^3 y + b_6 V_y^2 y^2 + b_7 V_y y^3 \\ & + \left(c_0 + c_1 V_y^2 + c_2 V_y y + c_3 y^2 + c_4 V_y^4 + c_5 V_y^3 y + c_6 V_y^2 y^2\right) V_z^2 \\ & + \left(d_0 + d_1 V_y^2 + d_2 V_y y + d_3 y^2 + d_4 V_y^4 + d_5 V_y^3 y + d_6 V_y^2 y^2 + d_7 V_y y^3\right) V^2 \end{aligned}$$

Hydrodynamic profiles

$$p(y) = p_0 \left[1 + \frac{6}{5} \left(\frac{mg}{T_0} \right)^2 y^2 \right] + \mathcal{O}(g^4)$$

$$u_z(y) = u_0 + \frac{\rho_0 g}{2\eta_0} y^2 + \mathcal{O}(g^3)$$

$$T(y) = T_0 \left[1 - \frac{\rho_0^2 g^2}{12\eta_0 \kappa_0 T_0} y^4 + \frac{1}{25} \frac{38 + 43\zeta_0^* + 17\zeta_0^{*2}}{(1+\zeta_0^*)(2+\zeta_0^*)} \left(\frac{mg}{T_0} \right)^2 y^2 \right] + \mathcal{O}(g^4)$$

NS terms

Extra terms

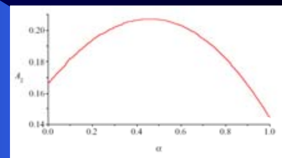
$$\zeta_0^* = \frac{\frac{5}{12}(1-\alpha^2)}{\beta(\alpha) + \frac{5}{12}(1-\alpha^2)}$$

Non-monotonic temperature profile

$$T = T_0 \left[1 - A_4(\alpha) \left(\frac{g\lambda_0}{v_0^2} \right)^2 \left(\frac{y}{\lambda_0} \right)^4 + A_2(\alpha) \left(\frac{g\lambda_0}{v_0^2} \right)^2 \left(\frac{y}{\lambda_0} \right)^2 \right] + \mathcal{O}(g^4)$$

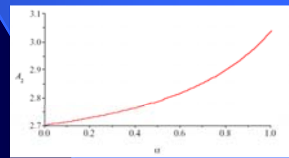
$$\eta, \kappa = \text{Boltzmann} \Rightarrow A_4(\alpha) = \frac{4}{1125\pi}(1+\alpha)^2(3-\alpha)(49-33\alpha)$$

NS term



$$\beta(\alpha) = (1+\alpha)\frac{2+\alpha}{6} \Rightarrow A_2(\alpha) = \frac{4}{25}\frac{2719 - 2741\alpha + 706\alpha^2}{(7-4\alpha)(23-11\alpha)}$$

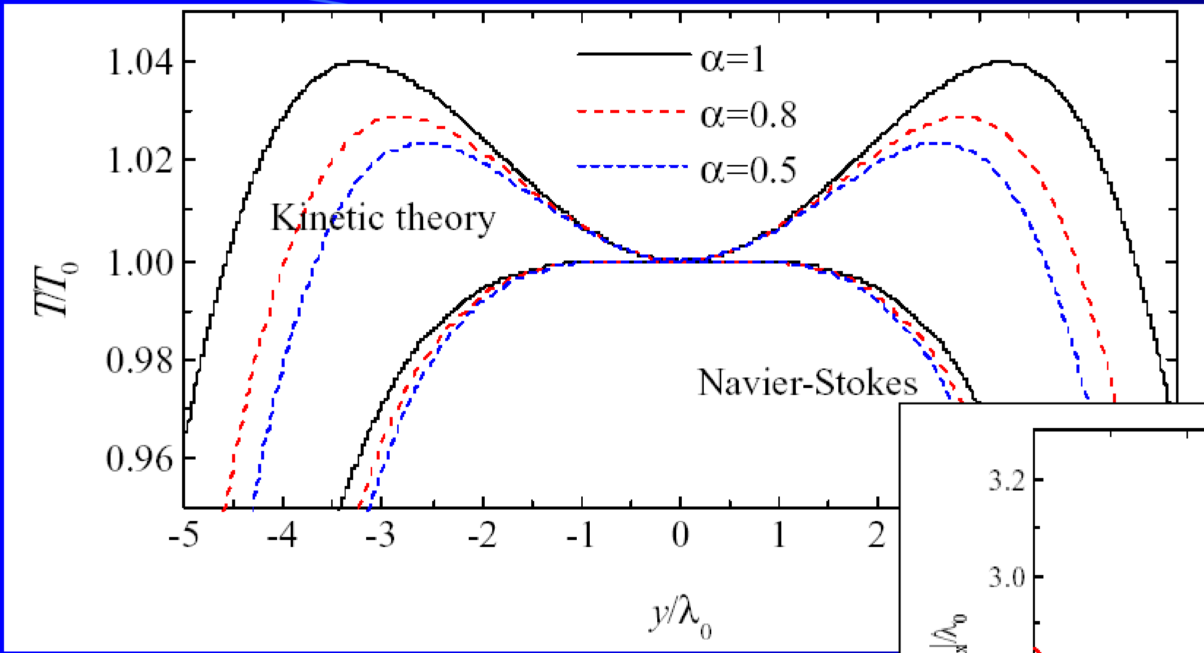
Extra term



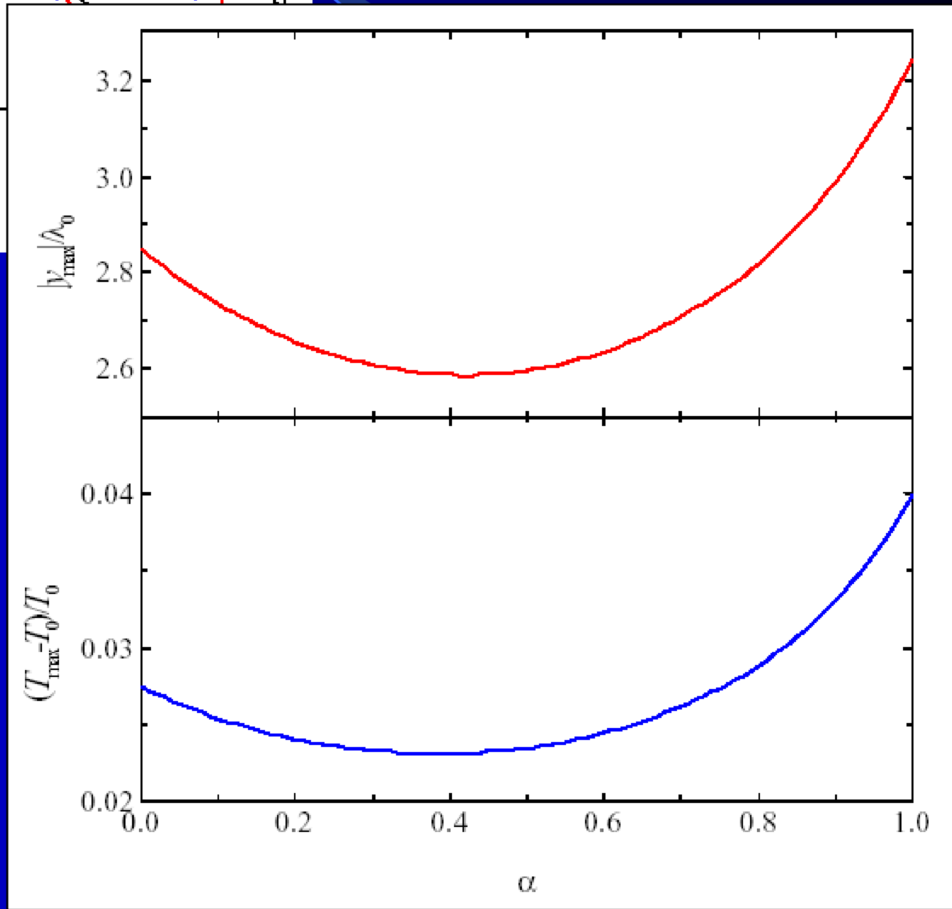
(independent of g)

$$y_{\max} = \pm\lambda_0 \sqrt{\frac{A_2(\alpha)}{2A_4(\alpha)}}$$

$$\frac{T_{\max} - T_0}{T_0} = \frac{A_2^2(\alpha)}{4A_4(\alpha)} \left(\frac{g\lambda_0}{v_0^2} \right)^2 + \mathcal{O}(g^4)$$



$g\lambda_0/v_0^2$=0.05



If α ≳ 0.4, the bi-modal shape of *T*(*y*) becomes (slightly) less pronounced as inelasticity increases.

However, the opposite behavior takes place if α ≲ 0.4.

Fluxes

$$P_{yz}(y) = -\rho_0 g y + \mathcal{O}(g^3)$$

$$P_{yy} = p_0 \left[1 - \frac{12}{25} \frac{102 + 87\zeta_0^* + 13\zeta_0^{*2}}{(1+\zeta_0^*)(2+\zeta_0^*)^2} \frac{\rho_0 \eta_0^2 g^2}{p_0^3} \right] + \mathcal{O}(g^4)$$

$$P_{zz}(y) = p_0 \left[1 + \frac{16}{25} \frac{82 + 67\zeta_0^* + 8\zeta_0^{*2}}{(1+\zeta_0^*)(2+\zeta_0^*)^2} \frac{\rho_0 \eta_0^2 g^2}{p_0^3} + \frac{14}{5} \left(\frac{mg}{T_0} \right)^2 y^2 \right] + \mathcal{O}(g^4)$$

Normal stress differences

$$q_y(y) = \frac{\rho_0^2 g^2}{3\eta_0} y^3 + \mathcal{O}(g^4)$$

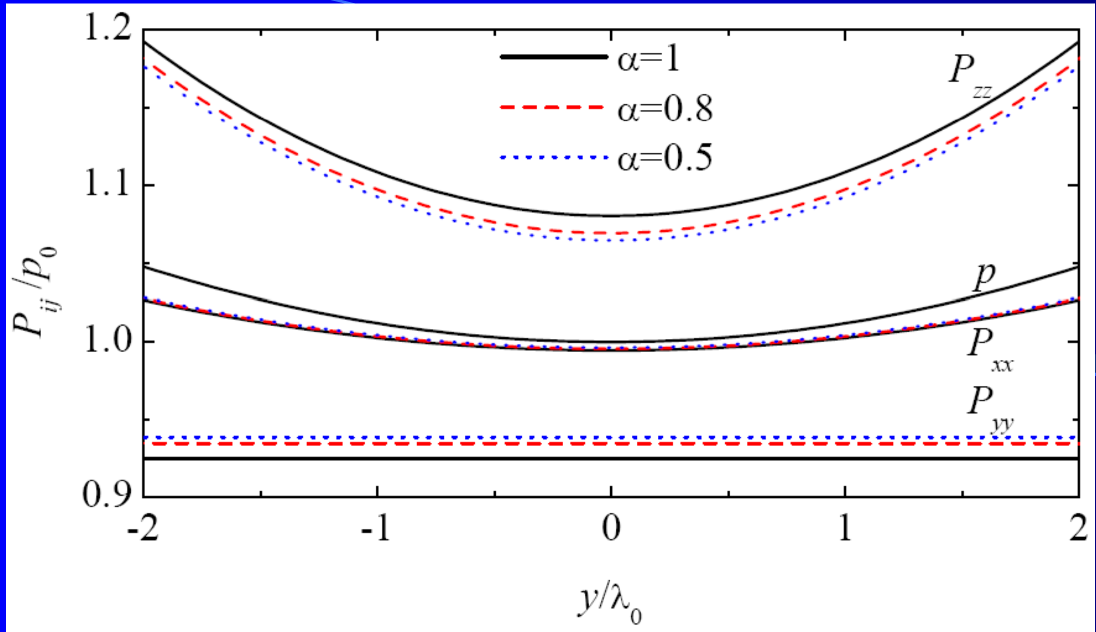
$$q_z = \frac{2}{5} m \kappa_0 g + \mathcal{O}(g^3)$$

Longitudinal heat flux

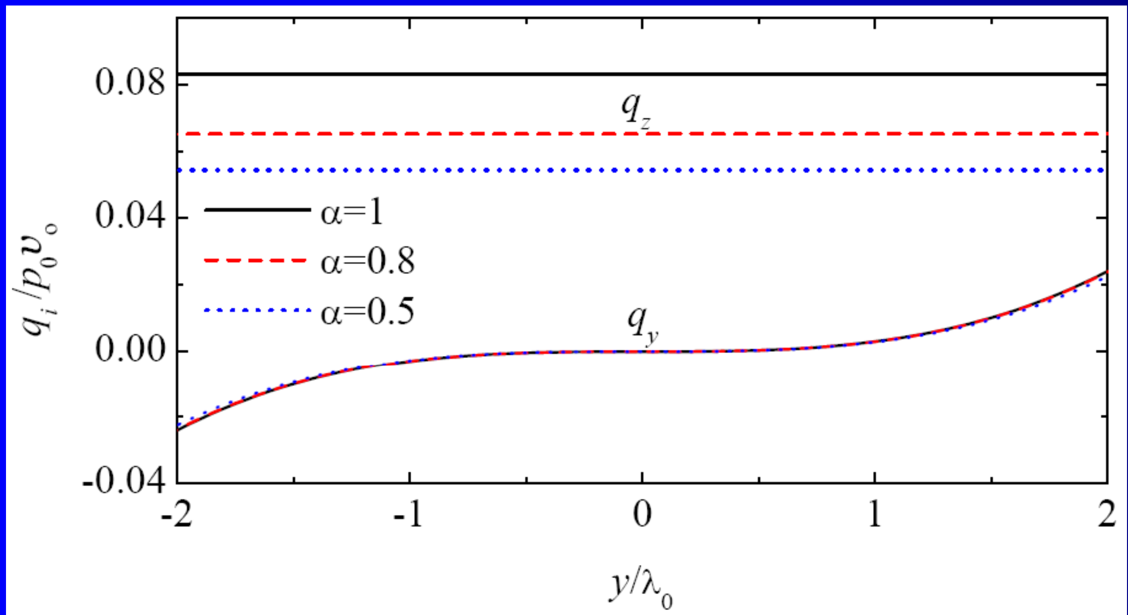
$$\begin{aligned} q_y &= -\kappa \frac{\partial}{\partial y} \left(T + \boxed{\frac{y_{\max}^2}{6} \nabla^2 T} \right) + \mathcal{O}(g^4) \\ &= -\kappa \frac{\partial T}{\partial y} + \boxed{\frac{y_{\max}^2}{6} \nabla^2 q_y} + \mathcal{O}(g^4) \end{aligned}$$

Super-Burnett

$g\lambda_0/v_0^2=0.05$



$P_{yy} < P_{xx} < p < P_{zz}$



$|q_y| < q_z$

Conclusions (I)

- Gravity-driven Poiseuille flow exhibits interesting (and even counter-intutitive) non-Newtonian properties which are accessible to granular gases.
- Non-uniform hydrostatic pressure.
- Non-isotropic normal stresses.
- Heat flux component normal to the thermal gradient.

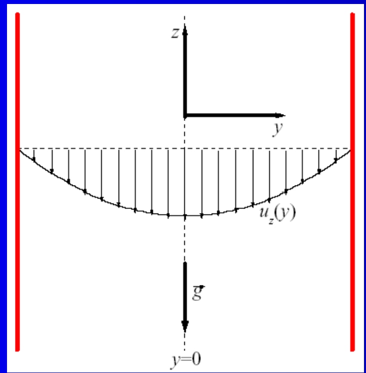
Conclusions (II)

- Bi-modal shape of the temperature profile:

  $|y_{max}| \approx 3$ mfp, $(T_{max}-T_0)/T_0 \approx 10\,(g\lambda/v_{th}^{2})^2$.
- For moderate or small inelasticity ($\alpha \gtrsim 0.4$), the larger the inelasticity, the less pronounced the bi-modal temperature profile.

  The reverse is true for large inelasticity ($\alpha \lesssim 0.4$).
- A similar influence of α on normal stress differences.
- Computer simulations (DSMC or MD) would be very welcome!

THANKS!



$$\left(-\frac{\zeta T}{2m}\frac{\partial^2}{\partial \mathbf{v}^2} - g\frac{\partial}{\partial v_z} + v_y\frac{\partial}{\partial y}\right) f = J[f,f].$$