# Comment on "Mean first passage time for anomalous diffusion"

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We correct a previously erroneous calculation [Phys. Rev. E **62**, 6065 (2000)] of the mean first passage time of a subdiffusive process to reach either end of a finite interval in one dimension. The mean first passage time is in fact infinite.

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Anomalous diffusion is commonly characterized by the behavior of the mean squared displacement as a function of time at long times [1–3],

$$\langle x^2\rangle \sim \frac{2K_\alpha}{\Gamma(1+\alpha)} t^\alpha, \tag{1}$$

where $K_\alpha$ is a generalized diffusion constant. Ordinary diffusion corresponds to $\alpha=1$ and, in the more usual notation, $K_1=D$. *Superdiffusion* is associated with motion that is faster than diffusive, $\alpha>1$, while *subdiffusion* occurs when $\alpha<1$. In a recent paper [4], the mean first passage times to the ends of an interval for a superdiffusive and a subdiffusive random walker on a line were calculated, and these results have even more recently been applied to the problem of anomalous heat conduction in such a line in the presence of a temperature gradient [5]. However, there is an error in the calculation for the *subdiffusive* problem from which one concludes that the mean first passage time reported in Ref. [4] for this case are incorrect. *In fact, the mean first passage time for the subdiffusive problem is infinite.*

To support this observation it is compelling to note that a continuous time subdiffusive nearest neighbor random walk (CTRW) [3] with a waiting time distribution which has a long tail, i.e., a walk in which the probability density that a particle takes the next step at a time $t\to\infty$ after the previous step is $\psi(t)\sim C_\alpha/t^{1+\alpha}$ leads exactly to Eq. (1) ($C_\alpha$ is a constant). The mean time for the particle to make *a single* jump is

$$T_1=\lim_{t\to\infty} T_1(t), \tag{2}$$

where

$$T_1(t)=\int_0^t \tau\psi(\tau)\mathrm{d}\tau. \tag{3}$$

For large $t$,

$$T_1(t)\propto \int^t \frac{\mathrm{d}\tau}{\tau^\alpha} \propto t^{1-\alpha}, \tag{4}$$

so that one obtains the well-known result $T_1=\infty$, i.e., the mean time to go from any one location to another even in a single jump is infinite. However, this argument might generate issues about the waiting time for the first step of the process, since one of the differences between a CTRW and the fractional diffusion equation lies precisely in the assumptions associated with this first step. In a CTRW there is a singular contribution to the probability density that the particle is still at the origin $x_0$ at time $t$,

$$P(x,t)\sim \frac{C_\alpha}{\alpha} t^{-\alpha}\delta(x_0) + (\text{other terms}), \tag{5}$$

which does not appear in the solution of the fractional diffusion equation [6].

To sidestep this problem and show that the divergence of the mean first passage time does not arise only from this term, we also obtain the divergent result starting with the fractional diffusion equation that was the starting point in Ref. [4]. Although that general formulation was for an arbitrary starting site in the interval $(0,L)$ and in the presence of an external constant force, the explicit final result was presented for a particular initial location, $x=L/2$, and with no external force. This explicit result is also the one used in Ref. [5]. We thus restrict our presentation to this specific case.

The mean first passage time from $x=L/2$ to either $x=0$ or $x=L$ is given by [7]

$$T=\int_0^L \mathrm{d}x\int_0^\infty \mathrm{d}t P(x,t)=\int_0^\infty \mathrm{d}t S(t), \tag{6}$$

where

$$S(t)=\int_0^L \mathrm{d}x P(x,t) \tag{7}$$

and $P(x,t)$ is the solution of the fractional diffusion equation [3,8,9]

$$\frac{\partial}{\partial t}P(x,t)=K_\alpha\, {}_0D_t^{1-\alpha}\frac{\partial^2}{\partial x^2}P(x,t), \tag{8}$$

 

with absorbing boundary conditions $P(0,t)=P(L,t)=0$ and initial condition $P(x,t=0)=\delta(x-L/2)$. Here ${}_0D_t^{1-\alpha}$ is the Riemann-Liouville operator,

$$ {}_0D_t^{1-\alpha}P(x,t)=\frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t d\tau \frac{P(x,\tau)}{(t-\tau)^{1-\alpha}}, \tag{9} $$

and $K_\alpha$ is the generalized diffusion coefficient in Eq. (1). The quantity $S(t)$ is called the survival probability because, as one sees from Eq. (7), $S(t)$ is just the probability that the particle has not been absorbed by the boundaries at $x=0$ and $x=L$ during the time interval $[0,t]$.

The solution of Eq. (8) with the given boundary and initial conditions can be found by the method of separation of variables [3,10]:

$$ P(x,t)=\frac{2}{L}\sum_{n=0}^{\infty}(-1)^n \sin\frac{(2n+1)\pi x}{L} \times E_\alpha(-K_\alpha(2n+1)^2\pi^2 t^\alpha/L^2). \tag{10} $$

Here $E_\alpha(-z)$ is the Mittag-Leffler function [for $\alpha=1$ it reduces to the exponential $\exp(-z)$ and thus yields the usual solution for the diffusive problem]. It then follows that

$$ S(t)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}E_\alpha(-K_\alpha(2n+1)^2\pi^2 t^\alpha/L^2). \tag{11} $$

The mean first passage time to $x=0$ or $L$ is then $T=\lim_{t\to\infty}T(t)$, where

$$ T(t)=\int_0^t d\tau\, S(\tau). \tag{12} $$

To address the convergence of $T(t)$ for $t\to\infty$ we need to analyze the long-time behavior of $S(t)$. Note that $S(t)$ is well behaved for finite times (the survival probability goes to 1 for $t\to 0$), so that the divergence of $T$ is due to the behavior at long times. For large $z$ the Mittag-Leffler function behaves as

$$ E_\alpha(-z)\sim\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{\Gamma(1-\alpha m)}z^{-m}, \tag{13} $$

and consequently, for large $t$,

$$ S(t)\sim\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}L^{2m}}{\Gamma(1-\alpha m)[K_\alpha(2n+1)^2\pi^2t^\alpha]^m} \sim\sum_{m=1}^{\infty}\frac{(-1)^{m+1}L^{2m}}{\Gamma(1-\alpha m)[\pi^2K_\alpha t^\alpha]^m}Z(m), \tag{14} $$

where $Z(m)=\Sigma_{n=0}^{\infty}(-1)^n/(2n+1)^{2m+1}$. For $t\to\infty$, and using the fact that $Z(1)=\pi^3/32$, we then have

$$ S(t)\sim\frac{1}{8\Gamma(1-\alpha)}\frac{L^2}{K_\alpha t^\alpha}. \tag{15} $$

It then follows that for large $t$ we have

$$ T(t)=\int_0^t d\tau\, S(\tau)\sim\frac{1}{8(1-\alpha)\Gamma(1-\alpha)}\frac{L^2}{K_\alpha t^{\alpha-1}}, \tag{16} $$

i.e., $T(t)\propto t^{1-\alpha}$, exactly as in Eq. (4). We thus conclude that $T(t)\to\infty$ when $t\to\infty$ for any $\alpha<1$.

We have thus shown that the mean first passage time for a subdiffusive process described by the fractional diffusion equation (or, for that matter, by a continuous time random walk) to reach the boundaries of a one-dimensional interval is infinite. We note that our Eq. (11) appears as Eq. (40) in Ref. [11], but the connection between the survival probability and the mean first passage time is never made in that work so a user of the result in Ref. [4] would not necessarily discover the connection. An expression for the first passage time density involving the Mittag-Leffler function appears as Eq. (3.87) in Ref. [10], but, again, they do not go on to calculate the mean first passage time, nor do they do the necessary asymptotic analysis of the Mittag-Leffler function that would allow them to do so. That these results would not necessarily lead a reader to conclude that the mean first passage time is infinite is reinforced by the fact that both of these references appear in Ref. [4]. Finally, we note that the validity of the result in Ref. [4] for the mean first passage time in the superdiffusive regime has also been questioned recently because it violates a theorem due to Sparre Andersen [12,13].

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