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An Explicit Difference Scheme for the Fractional Cable Equation ⋆

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Abstract: An explicit difference method to solve the fractional cable equation in the Riemann-Liouville form is studied. The numerical scheme is an extension of other schemes studied previously by the authors in which the Riemann-Liouville derivative is approximated by means of the Grünwald-Letnikov formula. The accuracy and stability of the method is considered. The stability analysis is carried out by means of a kind of von Neumann method adapted to fractional equations. The stability bound is checked numerically.

Keywords: Fractional cable equation, numerical algorithms, stability

1. INTRODUCTION

Fractional calculus is increasingly becoming a tool of key importance to solve a broad range of scientific problems. Biology, economics, physics or engineering are some of the disciplines that, in the last years, have profited from this mathematical field. For example, the fractional formalism is widely used to study anomalous diffusion problems described by the Continuous Time Random Walk model [1]. In particular, during the last years, one of the authors (SBY) has used fractional ideas and techniques in the study of reaction kinetics problems where reactions are limited by subdiffusion [2–5]. Many other examples can be found in Refs. [1,6–10]

A recent and quite interesting application of the fractional calculus is the modeling of neuronal dynamics. In the last few years, varied experiments involving physical and biological systems have reported on anomalous diffusion. The heterogeneous nature of the neuronal tissue is supposed to be the origin of this evidence, according certain works. In these models, the collisions between the messenger ions and other structures of short mobility make the ions slow down [11]. There is another alternative explanation for anomalous diffusion. According this theory, ions would be trapped by buffering proteins or indicator proteins used in the experiments. The trapping of ions will carry a subsequent reduction of the mobility [12].

The core conductor concept and associated cable equation are the basis for a macroscopic explanation of the electrophysiological behavior in neuronal processes. Some authors [13] have proved that under certain simplifying conditions, the Nernst-Planck equation and the cable equation are equivalent. This supports models that incorporate anomalous diffusion to describe this kind of neuronal processes [14]. On this respect, the works of Langlands et al. [15, 16] and Henry et al. [17], are particularly interesting. The resulting fractional cable equation proposed by these authors is

$$\frac{\partial u}{\partial t} = \frac{\partial^{1-\gamma_1}}{\partial t^{1-\gamma_1}} \left(K \frac{\partial^2 u}{\partial x^2} \right) - \mu^2 \frac{\partial^{1-\gamma_2} u}{\partial t^{1-\gamma_2}} \tag{1}$$

where

$$\frac{\partial^\gamma}{\partial t^\gamma} f(t) \equiv \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{d\tau^n} \int_0^t d\tau \frac{f(\tau)}{(t-\tau)^{1+\gamma-n}}, \qquad n-1 < \gamma < n \qquad (n = integer) \tag{2}$$

is the fractional derivative in Riemann-Liouville's sense.

Many different numerical methods for solving many classes of fractional equations have been proposed and studied along the last years [16–34]. The aim of this communication is to present an explicit finite difference scheme for solving the above fractional cable equation. This method is close to the methods studied in Refs. [18] and [19]. Besides, we study its accuracy and stability. The explicit methods enjoy some characteristics that make them quite widely appreciated [18, 21]: flexibility, simplicity, small computational demand, and easy generalization to spatial dimensions higher than one. Unfortunately, they can become unstable in some cases, so that it is necessary to determine the conditions under which these methods are stable.

In order to carry out the numerical comparisons, we consider the fractional cable equation (1) defined in the interval $-L/2 \le x \le L/2$, with absorbing boundary conditions, $u(x = -L/2, t) = u(x = L/2, t) = 0$, and where the initial condition is a Dirac's delta centered at $x = 0$: $u(x, 0) = \delta(x)$. The exact solution of this problem for $L \to \infty$ is [15, 16]

$$u(x,t) = \frac{1}{\sqrt{4t^{\gamma_1}\pi}} \sum_{k=0}^{\infty} \frac{(-\mu^2 t^{\gamma_2})^k}{k!} H_{1,2}^{2,0} \left[\frac{x^2}{4t^{\gamma_1}} \middle|_{(0,1),(1/2+k,1)}^{(1-\gamma_1/2+\gamma_2 k,\gamma_1)} \right] \tag{3}$$

where H denotes the Fox H function.

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2. THE NUMERICAL METHOD

Henceforth, we use the notation $x_j = j\Delta x$, $t_m = m\Delta t$, and $u(x_j, t_m) \simeq U_j^{(m)}$, where $U_j^{(m)}$ is the numerical estimate of the exact solution $u(x,t)$ for $x = x_j$ and $t = t_m$.

We approximate (discretize) the Riemann-Liouville derivative by means of the Grünwald-Letnikov's formula [2, 22]

$$\left.\frac{\partial^\gamma f}{\partial t^\gamma}\right|_{t_m} \simeq \frac{1}{(\Delta t)^\gamma}\sum_{k=0}^{m}\omega_k^\gamma f(t_{m-k}) \tag{4}$$

where

$$\omega_k^\gamma = \left(1 - \frac{1+\gamma}{k}\right)\omega_{k-1}^\gamma\,. \tag{5}$$

Using (4) in equation (1), and approximating the second-order space derivative by the usual three-point centered formula

$$\frac{\partial^2}{\partial x^2}u(x_j,t_m) = \frac{u(x_{j+1},t_m) - 2u(x_j,t_m) + u(x_{j-1},t_m)}{(\Delta x)^2} + O(\Delta x)^2 \tag{6}$$

we get a difference scheme for the fractional cable equation:

$$U_j^{(m+1)} = U_j^{(m)} + S\sum_{k=0}^{m}\omega_k^{1-\gamma_1}\left(U_{j+1}^{(m-k)} - 2U_j^{(m-k)} + U_{j-1}^{(m-k)}\right) - \mu^2\,(\Delta t)^{\gamma_2}\sum_{k=0}^{m}\omega_k^{1-\gamma_2}U_j^{(m-k)}, \tag{7}$$

with

$$S = K\frac{(\Delta t)^{\gamma_1}}{(\Delta x)^2}, \tag{8}$$

and where the error of discretization is order $O(\Delta t) + O(\Delta x)^2$.

We check this explicit difference scheme comparing the analytical solution and the numerical solution of the problem described before Eq. (3) for several cases with different values of γ_1 and γ_2. Some illustrative cases are shown in figure 1 where $\gamma_1 = 1$. In the numerical procedure, the exact initial condition $u(x,0) = \delta(x)$ is approximated by

$$u(x_j,0) = \begin{cases} 1/\Delta x, & j = 0 \\ 0, & j \neq 0 \end{cases} \tag{9}$$

The differences between the exact and the numerical solution is shown in figure 2. One sees that, except for very short times, the agreement is quite good. The large value of the error for small times is due, in part, to the approximation embodied by Eq. (9). In figure 3 we compare the analytical and numerical solution of the fractional cable when $\gamma_1 = 1/2$. The error is shown in figure 4. The results here are similar to those shown in figures 1 and 2 for the case with $\gamma_1 = 1$.

For the cases with $\gamma_1 = 1/2$ we have used a smaller value of Δt (10^{-5}) and, simultaneously, a larger value of Δx than for the cases with $\gamma_1 = 1$. This is necessary in order to keep stable the numerical scheme. This issue will be discussed in section 3.

3. STABILITY

The explicit difference scheme (7) we are considering is not always stable. In fact, for any given value of γ there

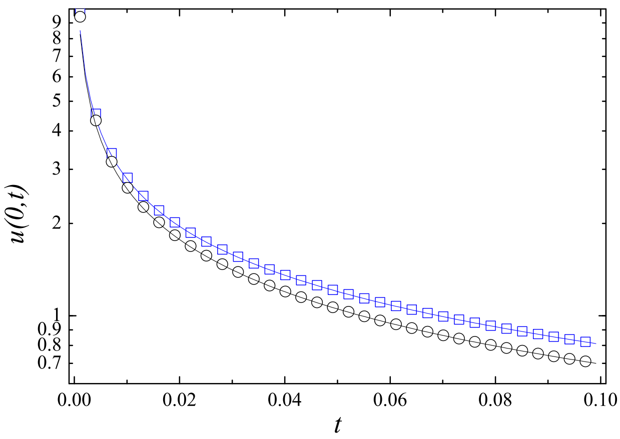

Fig. 1. Numerical solution at the mid-point $x = 0$ of the fractional cable equation described in the main text for $\gamma_1 = 1$ and $\gamma_2 = 1$ (squares) and $\gamma_2 = 1/2$ (circles). Lines: exact solution given by Eq. (3). We have used $\Delta x = 1/20$, $\Delta t = 10^{-4}$, $K = 1$ and $\mu = 1$.

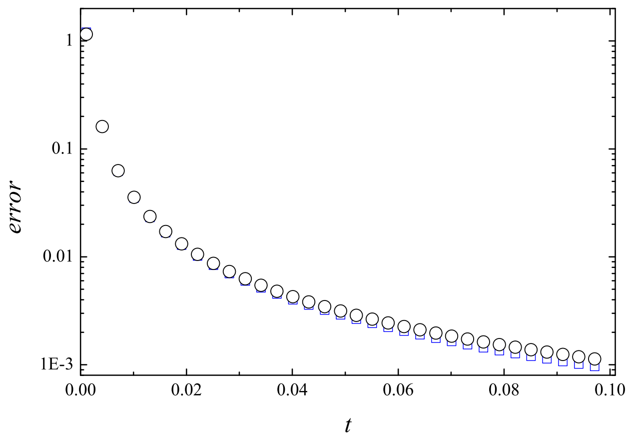

Fig. 2. Error of the numerical method for the problems considered in figure 1. Squares: $\gamma_2 = 1$; circles: $\gamma_2 = 1/2$.

are choices of Δx and Δt for which the numerical scheme becomes unstable. Therefore, it is important to determine the conditions, if any, under which these explicit methods are stable. Here we are going to use the fractional von Neumann stability analysis employed in [18] and [19, 20] for standard fractional diffusion equations. A question we partially address here is up to what extent this procedure is valid for more complicate fractional equations where there appear fractional derivatives of different order.

We start by assuming a solution in the form of a subdiffusive mode, $u_j^{(m)} = \zeta_m e^{iqj\Delta x}$, where q is a real spatial wave number. Inserting this expression into (7) one gets

$$\zeta_{m+1} = \zeta_m + S\sum_{k=0}^{m}\omega_k^{1-\gamma_1}\left(e^{iq\Delta x} - 2 + e^{-iq\Delta x}\right)\zeta_{m-k} - \mu^2\,(\Delta t)^{\gamma_2}\sum_{k=0}^{m}\omega_k^{1-\gamma_2}\zeta_{m-k}. \tag{10}$$

The stability is determined by the behavior of ζ_m. Writing

$$\zeta_{m+1} = \xi\zeta_m \tag{11}$$

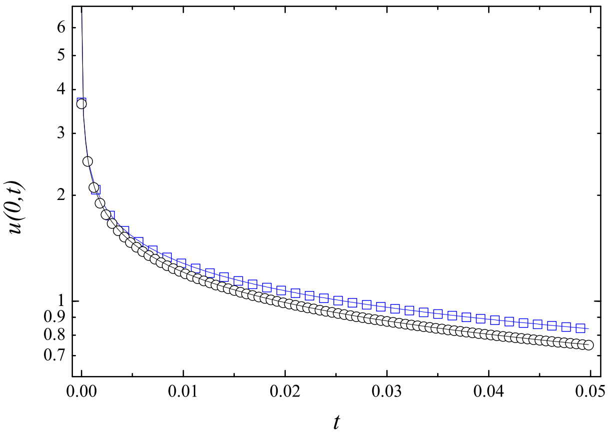

Fig. 3. Numerical solution at the mid-point $x = 0$ of the fractional cable equation described in the main text for $\gamma_1 = 1/2$ and $\gamma_2 = 1$ (squares) and $\gamma_2 = 1/2$ (circles). Lines: exact solution given by Eq. (3). We have used $\Delta x = 1/10$, $\Delta t = 10^{-5}$, $K = 1$ and $\mu = 1$.

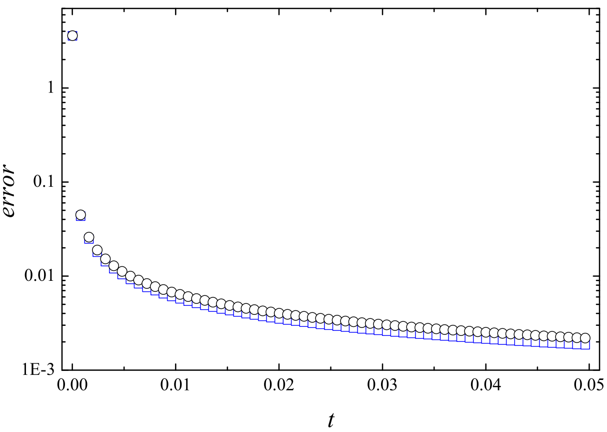

Fig. 4. Error of the numerical method for the problem considered in figure 3. Squares: $\gamma_2 = 1$; circles: $\gamma_2 = 1/2$.

and assuming that $\xi \equiv \xi(q)$ is independent of time, we obtain this equation

$$\begin{aligned}\xi =& 1 + S\sum_{k=0}^{m}\omega_k^{1-\gamma_1}(e^{iq\Delta x} - 2 + e^{-iq\Delta x})\xi^{-k} \\ &- \mu^2\,(\Delta t)^{\gamma_2}\sum_{k=0}^{m}\omega_k^{1-\gamma_2}\xi^{-k} \end{aligned} \tag{12}$$

for the amplification factor ξ of the subdiffusive mode. If $|\xi| > 1$ for some q, the temporal factor of the solution grows to infinity [c.f., equation (11)] and the mode is unstable. Considering the extreme value $\xi = -1$, we get from (12) that the numerical method is stable if this inequality holds:

$$S\sin^2\left(\frac{q\Delta x}{2}\right) \le S_\times^m = \frac{-2 + \mu^2\,(\Delta t)^{\gamma_2}\sum_{k=0}^{m}\omega_k^{1-\gamma_2}(-1)^k}{-4\sum_{k=0}^{m}\omega_k^{1-\gamma_1}(-1)^k} \tag{13}$$

If we define $S_\times = \lim_{x\to\infty} S_\times^m$, we get

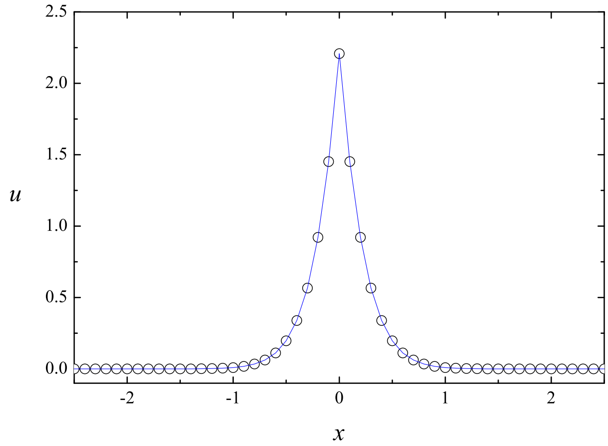

Fig. 5. Numerical solution (circles) provided by our method for the fractional cable equation with $\gamma_1 = 0.5$ and $\gamma_2 = 0.5$ after 100 timesteps when $\Delta x = 1/10$, $\Delta t = 10^{-5}$ and $S = (\Delta t)^{\gamma_1}/(\Delta x)^2 = 0.316$. Note that this value of S is smaller than the stability bound $S_\times = (2^{\gamma_2} - \mu^2\,(\Delta t)^{\gamma_2})/(2^{2+\gamma_2-\gamma_1}) \simeq 0.352\ldots$ provided by equation (15), so that we are inside the stability region.The solid line is the exact solution.

$$S\sin^2\left(\frac{q\Delta x}{2}\right) \le S_\times = \frac{-2 + \mu^2\,(\Delta t)^{\gamma_2}\sum_{k=0}^{\infty}\omega_k^{1-\gamma_2}(-1)^k}{-4\sum_{k=0}^{\infty}\omega_k^{1-\gamma_1}(-1)^k} \tag{14}$$

But $\sum_{k=1}^{\infty}\omega_k^{1-\gamma} = 2^{1-\gamma}$, so that we finally obtain the following relation for the stability bound on S

$$S = \frac{(\Delta t)^{\gamma_1}}{(\Delta x)^2} \le S_\times = \frac{2^{\gamma_2} - \mu^2\,(\Delta t)^{\gamma_2}}{2^{2+\gamma_2-\gamma_1}}. \tag{15}$$

In figures 5 and 6 we show two representative examples corresponding to the problem of figure 3 but for two values of S respectively larger and smaller than the stability bound provided by (15). We see that the value of S is crucial: when S is smaller than $S_\times$, we are inside the stable region and we get a sensible numerical solution (fig. 5); otherwise we get an evidently unstable and nonsensical solution (fig. 6).

4. NUMERICAL CHECK OF THE STABILITY ANALYSIS

Here we carry out a comprehensive check of the validity of our stability bound (15) by using many different values of the parameters γ_1, γ_2, μ, Δt, and Δx. This stability check is carried out in the following way. First, we choose a set of values of γ_1, γ_2, μ, Δx and S and integrate the corresponding fractional cable equation. Then we say that the method is unstable when, at any position j, $\left|u_j^{m-1} - u_j^m\right|$ is larger than 10 within the first 1000 integration steps. Otherwise, we label the method as stable. In tables 1–3 we show the results obtained for a large set of values. In all cases the method turns out to be stable (unstable) when S is smaller (larger) than $S_\times$, in agreement with Eq. (15).

5. CONCLUSIONS

We have considered an explicit method for solving fractional cable equations where the fractional Riemann-Liouville derivatives are approximated by means of the

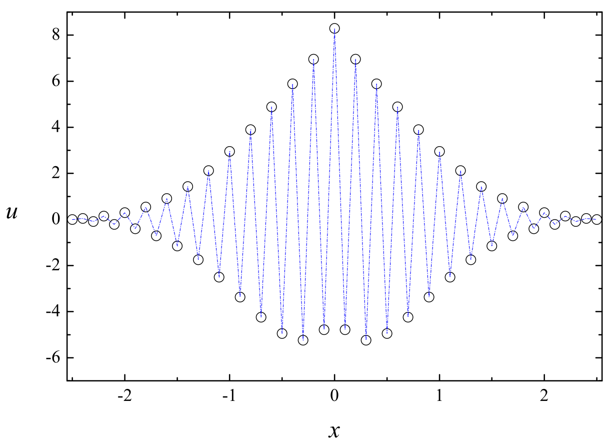

Fig. 6. Numerical solution (circles) provided by our explicit method for the fractional cable equation with $\gamma_1 = 0.5$ and $\gamma_2 = 0.5$ after 100 timesteps when $\Delta x = 1/10$, $\Delta t = 1.3 \times 10^{-5}$ and $S = (\Delta t)^{\gamma}/(\Delta x)^2 = 0.36$. Note that this value is larger than the stability bound $S_\times = (2^{\gamma_2} - \mu^2 (\Delta t)^{\gamma_2})/(2^{2+\gamma_2-\gamma_1}) \simeq 0.352\ldots$ provided by equation (15). The broken line is to guide the eye.

Δx	γ_1	γ_2	S	$S_\times$	Stable
1/10	0.25	1	0.289	0.297	YES
1/10	0.25	1	0.308	0.297	NO
1/20	0.5	1	0.335	0.353	YES
1/20	0.5	1	0.358	0.353	NO
1/20	0.75	1	0.400	0.420	YES
1/20	0.75	1	0.429	0.420	NO
1/20	1	1	0.480	0.499	YES
1/20	1	1	0.500	0.499	NO
1/10	0.25	0.75	0.289	0.297	YES
1/10	0.25	0.75	0.308	0.297	NO
1/20	0.5	0.75	0.335	0.353	YES
1/20	0.5	0.75	0.358	0.353	NO
1/20	0.75	0.75	0.400	0.420	YES
1/20	0.75	0.75	0.429	0.420	NO
1/20	1	0.75	0.480	0.498	YES
1/20	1	0.75	0.500	0.498	NO
1/10	0.25	0.5	0.289	0.297	YES
1/10	0.25	0.5	0.308	0.297	NO
1/20	0.5	0.5	0.335	0.353	YES
1/20	0.5	0.5	0.358	0.353	NO
1/20	0.75	0.5	0.400	0.417	YES
1/20	0.75	0.5	0.429	0.417	NO
1/20	1	0.5	0.480	0.488	YES
1/20	1	0.5	0.500	0.488	NO
1/10	0.25	0.25	0.289	0.297	YES
1/10	0.25	0.25	0.308	0.297	NO
1/20	0.5	0.25	0.335	0.345	YES
1/20	0.5	0.25	0.358	0.345	NO
1/20	0.75	0.25	0.369	0.386	YES
1/20	0.75	0.25	0.400	0.385	NO
1/20	1	0.25	0.400	0.425	YES
1/20	0.25	1	0.440	0.423	NO

Table 1. Check of stability for several values of γ_1, γ_2, Δx and S for $\mu = 1$. Note that the method is stable ("YES") when $S < S_\times$ and unstable otherwise

Grünwald-Letnikov formula. The method has been used to solve the fractional cable equation with free boundary conditions, Dirac's delta initial condition, and different fractional exponents. We have found that the error of the numerical method is compatible with the truncating error, which is of order $O(\Delta t) + O(\Delta x)^2$. Finally, by means of a kind of von-Neumann stability analysis, we have obtained the conditions under which the method is stable. This stability bound has been checked numerically.

Δx	γ_1	γ_2	S	S_x	Stable
1/10	0.25	1	0.289	0.297	YES
1/10	0.25	1	0.308	0.297	NO
1/20	0.5	1	0.335	0.353	YES
1/20	0.5	1	0.358	0.353	NO
1/20	1	2	0.400	0.420	YES
1/20	0.75	1	0.429	0.420	NO
1/20	1	1	0.480	0.499	YES
1/20	1	1	0.500	0.499	NO
1/10	0.25	0.75	0.289	0.297	YES
1/10	0.25	0.75	0.308	0.297	NO
1/20	0.5	0.75	0.335	0.353	YES
1/20	0.5	0.75	0.358	0.353	NO
1/20	0.75	2	0.400	0.419	YES
1/20	0.75	0.75	0.429	0.419	NO
1/20	1	0.75	0.480	0.492	YES
1/20	1	0.75	0.500	0.492	NO
1/10	0.25	0.5	0.289	0.297	YES
1/10	0.25	0.5	0.308	0.297	NO
1/20	0.5	0.5	0.335	0.353	YES
1/20	0.5	0.5	0.358	0.353	NO
1/20	0.5	2	0.400	0.408	YES
1/20	0.75	0.5	0.429	0.408	NO
1/20	1	0.5	0.440	0.453	YES
1/20	1	0.5	0.460	0.452	NO
1/10	0.25	0.25	0.289	0.294	YES
1/10	0.25	0.25	0.308	0.294	NO
1/20	0.5	0.25	0.309	0.320	YES
1/20	0.5	0.25	0.322	0.320	NO
1/20	0.75	0.25	0.289	0.293	YES
1/20	0.75	0.25	0.306	0.291	NO
1/20	1	0.25	0.230	0.239	YES
1/20	1	0.25	0.240	0.237	NO

Table 2. Check of stability for several values of γ_1, γ_2, Δx and S for $\mu = 2$. Note that the method is stable ("YES") when $S < S_\times$ and unstable otherwise

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Δx	γ_1	γ_2	S	$S_\times$	Stable
1/10	0.25	1	0.289	0.297	YES
1/10	0.25	1	0.308	0.297	NO
1/20	0.5	1	0.335	0.353	YES
1/20	0.5	1	0.358	0.353	NO
1/20	0.75	1	0.400	0.42	YES
1/20	0.75	1	0.429	0.42	NO
1/20	1	1	0.480	0.499	YES
1/20	1	1	0.500	0.499	NO
1/10	0.25	0.75	0.289	0.297	YES
1/10	0.25	0.75	0.308	0.297	NO
1/20	0.5	0.75	0.335	0.353	YES
1/20	0.5	0.75	0.358	0.353	NO
1/20	0.75	0.75	0.400	0.419	YES
1/20	0.75	0.75	0.429	0.419	NO
1/20	1	0.75	0.480	0.499	YES
1/20	1	0.75	0.500	0.499	NO
1/10	0.25	0.5	0.289	0.297	YES
1/10	0.25	0.5	0.308	0.297	NO
1/20	0.5	0.5	0.335	0.353	YES
1/20	0.5	0.5	0.358	0.353	NO
1/20	0.75	0.5	0.400	0.420	YES
1/20	0.75	0.5	0.429	0.420	NO
1/20	1	0.5	0.440	0.497	YES
1/20	1	0.5	0.460	0.497	NO
1/10	0.25	0.25	0.289	0.294	YES
1/10	0.25	0.25	0.308	0.294	NO
1/20	0.5	0.25	0.309	0.351	YES
1/20	0.5	0.25	0.322	0.351	NO
1/20	0.75	0.25	0.400	0.411	YES
1/20	0.75	0.25	0.429	0.411	NO
1/20	1	0.25	0.460	0.481	YES
1/20	1	0.25	0.500	0.480	NO

Table 3. Check of stability for several values of γ_1, γ_2, Δx and S for $\mu = 1/2$. Note that the method is stable ("YES") when $S < S_\times$ and unstable otherwise

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