

LETTERS TO THE EDITOR

CONSTRUCTION OF APPROXIMATE ANALYTICAL SOLUTIONS TO A NEW CLASS OF NON-LINEAR OSCILLATOR EQUATIONS

1. INTRODUCTION

The work of many authors, principally in this century, has given many approximate methods of solution of the non-linear oscillator equation:

$$\ddot{x} + x = \varepsilon f(x, \dot{x}), \quad (1.1)$$

where $0 < \varepsilon \ll 1$ and f is a polynomial function of its arguments. Mickens and Oyediji in a recent letter [1] have investigated a new class of non-linear oscillator,

$$\ddot{x} + x^3 = \varepsilon f(x, \dot{x}) \quad (1.2)$$

with ε and f as before. If in the previous equations we put $\varepsilon = 0$, the resulting equations are called generating equations and their solutions are called generating solutions.

A century of work on equation (1.1) has been based on obtaining approximate solutions, where the circular functions play a central role. In reference [1] equation (1.2) is solved by using circular functions. Our group has applied the Jacobian elliptic functions to non-linear problems of relativistic quantum mechanics [2, 3] and non-linear oscillators [4, 5]. They are the natural extension to non-linear problems of circular functions. For the present problem we think it more appropriate to use the solution of the generating equation

$$\ddot{x} + x^3 = 0, \quad (1.3)$$

i.e., the Jacobian elliptic functions

$$x(t) = A \operatorname{cn}(\omega t + \phi, \mu^2 = \frac{1}{2}), \quad (1.4)$$

where $A = \omega$ and where A and ϕ are constants fixed by the initial conditions.

2. THE ELLIPTIC METHODS

In this letter we would like to show how to extend with Jacobian elliptic functions the methods of harmonic balance and slowly varying amplitude and phase. We use, when possible, the notation and presentation of reference [1].

For the first method (of harmonic balance) one assumes a solution of the form

$$x(t) = A \operatorname{cn}(\omega t, \frac{1}{2}), \quad (2.1)$$

where A and ω are constants to be determined. Substituting equation (2.1) into equation (1.2) gives

$$F_1(A, \omega, \alpha, \varepsilon) \cos z + F_2(A, \omega, \alpha, \varepsilon) \sin z + (\text{higher order harmonics}) = 0, \quad (2.2)$$

where z is the amplitude function of argument ωt and parameter $\frac{1}{2}$ [6], $z = \operatorname{am}(\omega t, \frac{1}{2})$, that is, $\cos z = \operatorname{cn}(\omega t, \frac{1}{2})$ and $\sin z = \operatorname{sn}(\omega t, \frac{1}{2})$ and where α collectively denotes any parameter which appears in the non-linear function $f(x, \dot{x})$. One now first takes $F_1 = 0$ and then $F_2 = 0$. Each solution $A(\alpha, \varepsilon)$ and $\omega(\alpha, \varepsilon)$ of this system corresponds to a possible steady state limit cycle and/or limit motion of equation (1.2).

In the second method (of slowly varying amplitude and phase) one assumes for equation (1.2) a solution of the form:

$$x(t) = A(t) \operatorname{cn} [\omega t + \phi(t), \mu^2 = \frac{1}{2}] = A(t) \operatorname{cn} [\psi(t), \frac{1}{2}] \equiv A \operatorname{cn}, \quad (2.3)$$

where ω is an unknown constant and $A(t)$ and $\phi(t)$ are functions to be determined. For this, one imposes two constraints on equation (2.3): constraint-1, equation (2.3) must be a solution of equation (1.2); constraint 2, the time derivative of equation (2.3) must have the same form as the time derivative of the generating solution, that is

$$\dot{x}(t) = -\omega A \operatorname{sn} \operatorname{dn}. \quad (2.4)$$

Differentiating equation (2.3) and using constraint 2 one has

$$\dot{A} \operatorname{cn} - A \dot{\phi} \operatorname{sn} \operatorname{dn} = 0. \quad (2.5)$$

It is known that

$$(d^2/d^2\psi) \operatorname{cn}(\psi, \mu^2) = -(1-2\mu^2) \operatorname{cn} - 2\mu^2 \operatorname{cn}^3. \quad (2.6)$$

Then, differentiating equation (2.3) twice, using equation (2.5) and substituting into equation (1.2) (constraint 1) gives

$$-\omega \dot{A} \operatorname{sn} \operatorname{dn} - \omega^2 A \operatorname{cn}^3 - \omega A \dot{\phi} \operatorname{cn}^3 + A^3 \operatorname{cn}^3 = \varepsilon f(A \operatorname{cn}, -\omega A \operatorname{sn} \operatorname{dn}). \quad (2.7)$$

Solving equations (2.5) and (2.7) one obtains the system

$$\begin{aligned} \dot{A} \omega (\operatorname{sn}^2 \operatorname{dn}^2 + \operatorname{cn}^4) + (\omega^2 A - A^3) \operatorname{cn}^4 \operatorname{sn} \operatorname{dn} &= -\varepsilon f \operatorname{sn} \operatorname{dn}, \\ \omega A \dot{\phi} (\operatorname{sn}^2 \operatorname{dn}^2 + \operatorname{cn}^4) + (\omega^2 A - A^3) \operatorname{cn}^4 &= -\varepsilon f \operatorname{cn}. \end{aligned} \quad (2.8)$$

The exact solution of this system will give the exact solution of equation (1.2). However, this does not seem easy. We give here only an approximate solution using the averaging method on equation (2.8). The averaging is over the Jacobian-elliptic-function-period, $4K(\mu^2)$, where $K(\mu^2)$ is the complete elliptic integral of the first kind (in our case $\mu^2 = \frac{1}{2}$). Then one obtains

$$\begin{aligned} \omega \dot{A} &= -\left(\frac{3\varepsilon}{8K}\right) \int_0^{4K} f(A \operatorname{cn}, -\omega A \operatorname{sn} \operatorname{dn}) \operatorname{sn} \operatorname{dn} \, d\psi, \\ \omega A \dot{\phi} &= -\left(\frac{3\varepsilon}{8K}\right) \int_0^{4K} f(A \operatorname{cn}, -\omega A \operatorname{sn} \operatorname{dn}) \operatorname{cn} \, d\psi - \frac{1}{3}(\omega^2 A - A^3), \end{aligned} \quad (2.9)$$

where $K = K(\frac{1}{2})$. Use has been made of (see reference [7])

$$\langle \operatorname{sn}^2 \operatorname{dn}^2 \rangle = (1/4K) \int_0^{4K} \operatorname{sn}^2 \operatorname{dn}^2 \, d\psi = \frac{1}{3}, \quad \langle \operatorname{cn}^4 \rangle = \frac{1}{3}, \quad \langle \operatorname{cn}^4 \operatorname{sn} \operatorname{dn} \rangle = 0. \quad (2.10)$$

3. AN EXAMPLE

We shall apply the two methods to the equation

$$\ddot{x} + x^3 = \varepsilon(1-x^2)\dot{x} \quad (3.1)$$

and compare the results with those in reference [1]. Substituting equation (2.1) into equation (3.1) gives

$$\begin{aligned} \frac{3}{4}(-\omega^2 A + A^3) \cos z + (A/4)(-\omega^2 A + A^3) \cos 3z \\ + \omega \varepsilon (A - A^3/4) \sin z \operatorname{dn} - (\varepsilon \omega / 4) A^3 \sin 3z \operatorname{dn} = 0. \end{aligned} \quad (3.2)$$

Here $dn = (1 - \frac{1}{2} \sin^2 z)^{1/2}$. Fourier expansion of $\sin z dn$ and $\sin 3z dn$ in equation (3.2) gives

$$\frac{3}{4}(-\omega^2 A + A^3) \cos z + (\varepsilon\omega/\pi) [\frac{4}{3}KA + \frac{4}{3}(K-2E)A^3] \sin z + (\text{higher order harmonics}) = 0, \quad (3.3)$$

where $K = K(\frac{1}{2})$ and $E = E(\frac{1}{2}) = E(\mu^2)$ is the complete elliptic integral of second kind. Setting the coefficients of $\cos z$ and $\sin z$ to zero, one gets a system with the solution

$$\omega = A_s, \quad A_s^2 = (\frac{5}{3})/[2(E/K) - 1] = 3.6474: \quad (3.4)$$

that is, $A_s = 1.9098$. Then, the steady state solution of equation (3.1) is

$$x(t) = A_s \operatorname{cn}(A_s t, \frac{1}{2}) \quad (3.5)$$

with A_s given by equation (3.4). The period is $\tau = 4K(\frac{1}{2})/\omega = 3.8833$. Mickens and Oyediji gave $x(t) = 2 \cos(\sqrt{3}t)$ with $\tau = 3.6276$. Solving equation (3.1) by numerical integration, we find an error in the steady state amplitude for $\varepsilon = 0.1$ of the order of 5% in reference [1] and less than 1% (that is more than 50 times smaller) with our method. For larger ε ($\varepsilon < 1$) the error of reference [1] is also about 5% and, at least, an order of magnitude smaller with our method.

In the second method, the integrals of equations (2.9) are [7]

$$\dot{A} = (\varepsilon/2)A\{1 + \frac{3}{5}(1 - 2E/K)A^2\} = (\varepsilon/2)A\{1 - (A/A_s)^2\}, \quad \dot{\phi} = (A^2 - \omega^2)/3\omega. \quad (3.6a, b)$$

It is easy to show that the non-zero steady state solution of this system (3.6) is the same as the steady state solution of the harmonic balance method. The transitory motion is obtained by solving equations (3.6). From equation (3.6a)

$$A^2(t) = A_s^2 A_0^2 / [A_0^2 + (A_s^2 - A_0^2) e^{-\varepsilon t}]: \quad (3.7)$$

that is, the same amplitude expression as in reference [1] (but with $A_s^2 = 4$). Equations (3.6b) and (3.7) give

$$\phi(t) = -(1/3\varepsilon A_s) \ln [1 + ((A_s^2/A_0^2) - 1) e^{-\varepsilon t}] + \phi_0. \quad (3.8)$$

The constants $A_0 \equiv A(0)$ and $\phi_0 \equiv \phi(0)$ are evaluated from the initial conditions. One concludes that the approximate solution to equation (3.1) is

$$x(t) = A(t) \operatorname{cn}(A_s t + \phi(t), \frac{1}{2}), \quad (3.9)$$

where $A(t)$, $\phi(t)$ and A_s are given by equations (3.7), (3.8) and (3.4).

4. CONCLUSIONS

We have generalized two methods of obtaining approximate solutions for the class of non-linear differential equations represented by equation (1.2), using Jacobi elliptic functions rather than circular functions. Following the same technique as reference [1], the generalized method of harmonic balance allows the determination of the parameter of the possible limit cycles and/or limit points, and the method of slowly varying amplitude and phase gives the transitory behaviour of the motion as the system approaches steady state. The precision is improved by at least an order of magnitude over the method of reference [1] when $\varepsilon \leq 1$ and orders of magnitude for $\varepsilon \leq 0.1$.

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