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GENERALIZED FOURIER SERIES FOR THE STUDY OF LIMIT CYCLES

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The approximate solution, to first order, of non-linear differential equations is studied using the method of harmonic balance with generalized Fourier series and Jacobian elliptic functions. As an interesting use of the series, very good analytic approximations to the limit cycles of Liénard's ordinary differential equation (ODE), $\ddot{X}+g(X)=f(X)\dot{X}$, are presented. Specificially, it is shown that, contrary to an opinion given in a well-known textbook on non-linear oscillations, $g(X)$ not only modifies the period but influences the topology. In the generalized van der Pol equation with $f(X)=\varepsilon(1-X^2)$ and $g(X)=AX+2BX^3$ for $\varepsilon<0{\cdot}1$, the presence of zero, one, or three limit cycles is found to depend on the value of A/B.

1. INTRODUCTION

Simple harmonic oscillators (SHO) form the basis for understanding various ideal concepts, but many interesting features of real systems are a consequence of their anharmonic character.

The simplest possible non-linear extensions of the SHO are oscillators that obey the ODEs

$$\dot{X}^2+AX^2+BX^p=E \qquad \text{where } \dot{X}=\mathrm{d}X/\mathrm{d}t, \tag{1.1}$$

and where $p=4$ for the anharmonic symmetric and $p=3$ for the anharmonic asymmetric oscillator (ASO and AAO). We have used the Jacobi elliptic functions in a series of papers to study complete analytical solutions of these equations, and have applied the results to many different problems including the quantum ASO [1] and AAO [2], and such exotic phenomena as the evaporation of the primigenial microscopic black holes [3].

Our series [4] are also useful for finding approximate solutions in other branches of physics. For example, we have used [5, 6] these generalized Fourier series with a harmonic balance method to find an approximate solution of equations of the type

$$\ddot{X}+X^3=\varepsilon(1-X^2)\dot{X}. \tag{1.2}$$

In this paper we give an interesting example of very good analytic approximations to the limit cycles of Liénard's ODE

$$\ddot{X}+g(X)=\varepsilon f(X,\dot{X}). \tag{1.3}$$

To avoid mathematical difficulties as much as possible, we limit our study to the very simple case of $g(X)=AX+2BX^3$; i.e., to a generalized van der Pol equation

$$\ddot{X}+AX+2BX^3=\varepsilon(1-X^2)\dot{X}. \tag{1.4}$$

This relatively simple equation allows us to show how well the method works in a first approximation. As a bonus we found the unexpected result that, contrary to an opinion given in such a well-known textbook on non-linear oscillations as that of Minorsky [7],

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$g(X)$ not only modifies the period but also influences the topology. For $\varepsilon < 0{\cdot}1$, whether there are zero, one, or three limit cycles depends on the value of A/B. Equations (1.3) or (1.4) with $\varepsilon = 0$ are called generating equations, and their solutions generation solutions.

2. THE ELLIPTIC METHODS

To avoid computational difficulties we limit our discussion to a first order approximation. Nevertheless, the usual Fourier series (our special case of $m = 0$) for the elliptic functions contain higher harmonics, which is why this very simple approximation has proved to be quite good in the cases we have studied.

In the potentials we are considering, $V(X) = AX^2 + BX^4$ for $E < V_{max} = A^2/4B$, the generating solutions [1, 8] are of the form

$$X(t) = a \operatorname{pq}(\omega t; k^2), \tag{2.1}$$

with pq a convenient Jacobian elliptic function determined from the tables of references [1, 8]. Therefore, when we use the method of harmonic balance we assume a solution of the form (2.1) with a, ω and k^2 to be determined.

Substituting equation (2.1) into equation (1.4) gives

$$F_1(a, w, k^2, \varepsilon, \alpha) \cos \varphi + F_2(a, w, k^2, \varepsilon, \alpha) \sin \varphi + (\text{higher order harmonics}) = 0, \tag{2.2}$$

where φ is the generalized circular function of the case being considered [4], and α collectively denotes any parameter which appears in the non-linear function $f(X, \dot{X})$. We first take $F_1 = 0$ and $F_2 = 0$, using the method of harmonic balance [9].

We shall study the different cases of the quartic ASO with the form of the potential given above: i.e., with $E = 0$ for $X = 0$. We shall distinguish three cases of the quartic oscillator, as follows: (I) Consisting of two types of the potential, $B > 0$ and $A > 0$, and $B > 0$, $A < 0$ and $E > 0$; (II) $B < 0$ and $A > 0$; (III) $B > 0$, $A < 0$ and $E < 0$. We shall see that these types have as fundamental generating functions the three elliptic functions cn, sn and dn respectively.

3. STUDY OF THE THREE TYPES OF QUARTIC OSCILLATOR

3.1. OSCILLATOR TYPE I

For this type of oscillator, following references [1, 8], one can take a generating function

$$X = a \operatorname{cn}(wt; k^2), \tag{3.1}$$

where a, w and $k^2 = m$ are constants to be determined. Substituting expression (3.1) into equation (1.4) gives

$$(2Ba^3 - 2aw^2 m) \operatorname{cn}^3 - \varepsilon a^3 w \operatorname{cn}^2 \operatorname{sn} \operatorname{dn} + \varepsilon aw \operatorname{sn} \operatorname{dn} + [Aa + aw^2(2m - 1)] \operatorname{cn} = 0. \tag{3.2}$$

Fourier expanding each of the quantities up to the first harmonic only gives

$$\begin{aligned} &\{(3/4)(2Ba^3 - 2aw^2 m) + [Aa + aw^2(2m - 1)]\} \cos \varphi \\ &\quad + \{\varepsilon aw([4(2m - 1)E(m) + 4\mathrm{K}(m)m_1]/3\pi m) \\ &\quad - \varepsilon a^3 w([4\,\mathrm{K}(m)m_1(m - 2) + 8(m^2 + m_1)\,\mathrm{E}(m)]/15\pi m^2)\} \sin \varphi \\ &\quad + (\text{higher harmonics}) = 0, \end{aligned} \tag{3.3}$$

where $\mathrm{K}(m)$ and $\mathrm{E}(m)$ are the complete elliptic integrals of the first and second kind respectively, and $m_1 = k'^2 = 1 - m$ is the complementary parameter. This is the usual treatment in terms of elliptic functions, where the argument of $\sin\varphi$ and $\cos\varphi$ in the Fourier expansion is the amplitude function $\varphi = \text{am}(\omega t; m)$, so that $\cos\varphi = \text{cn}(\omega t; m) = \text{cn}\, u$ and $\sin\varphi = \text{sn}(\omega t; m) = \text{sn}\, u$. Setting the coefficient of $\sin\varphi$ to zero one obtains

$$a^2 = 5m[(2m-1)\,\text{E}(m) + m_1\text{K}(m)]/[m_1(m-2)\text{K}(m) + 2(m^2+m_1)\text{E}(m)]. \quad (3.4)$$

The coefficient of $\cos\varphi$ can be most simply made zero if the two terms in the bracket are set to zero: i.e., if X is a generating solution. One then obtains $w^2 = Ba^2/m$ and $w^2 = A/(1-2m)$, and from these two equations

$$A/B = (1-2m)a^2/m. \quad (3.5)$$

We have used as generating solution for X the periodic solution of the generating equation corresponding to the initial condition, with a equal to the maximum amplitude of the oscillation. For initial conditions corresponding to a known initial velocity the suitable elliptic function according to references [1, 8] is sd.. When using $X = ak'\,\text{sd}\,(wt; m)$, the only change necessary in the foregoing formulae is to the left side of equation (3.4) which is now a^2m_1.

3.2. OSCILLATOR TYPE II

For this type of oscillator the generating solution corresponding to the initial condition with a equal to the maximum amplitude of the oscillation is [1, 8] $X = a\,\text{cd}\,(wt; m)$. For an initial condition of known velocity one uses instead

$$X = a\,\text{sn}\,(wt; m). \quad (3.6)$$

Substituting expression (3.6) into equation (1.4) and, with the same approximation as before, one now obtains a formula similar to formula (3.3). Setting the coefficient of $\sin\varphi$ to zero, one obtains

$$a^2 = 5m[(1+m)\text{E}(m) - m_1\text{K}(m)]/[m_1(m-2)\text{K}(m) + 2(m^2+m_1)\text{E}(m)]. \quad (3.7)$$

The coefficient of $\cos\varphi$ can be most simply made zero if the two terms in the bracket are set to zero: i.e., if X is a generating solution. One then obtains $\omega^2 = -Ba^2/m$ and $\omega^2 = A/(1+m)$, and from these two equations

$$A/B = -(1+m)a^2/m.$$

If one uses cd for the generating solution, no change is necessary in the foregoing formulae.

3.3. OSCILLATOR TYPE III

For this type of oscillator the generating solutions corresponding to periodic motion are (following references [1, 8]) $X = a\,\text{dn}\,(\omega t; m)$ and $X = ak'\,\text{nd}\,(\omega t; m)$. Substituting the first of these generating solutions into equation (1.4) gives

$$(2Ba^3 - 2a\omega^2)\,\text{dn}^3 - \varepsilon\omega a^3 m\,\text{dn}^2\,\text{sn}\,\text{cn} + \varepsilon\omega a m\,\text{dn}\,\text{sn}\,\text{cn} + [Aa + a\omega^2(2-m)]\,\text{dn} = 0.$$

The Fourier expansion in terms of $\sin\varphi$ and $\cos\varphi$ is now calculated with the approximate φ for this case, i.e., $\varphi = k\int \text{cn}\, u\, \text{d}u$, which gives $\cos\varphi = \text{dn}\, u$ and $\sin\varphi = k\,\text{sn}\, u$ (for details, see reference [4]). If one limits, as before, the expansion to the first harmonic,

$$\begin{aligned}&\{(3/4)(2Ba^3 - 2a^2\omega^2) + [Aa + a\omega^2(2-m)]\}\cos\varphi\\ &\quad + \{a\varepsilon w([4(2-m)\text{E}(m) - 8m_1\text{K}(m)]/3\pi m)\\ &\quad \varepsilon a^3 w([4\text{K}(m)m_1(m-2) + 8(m^2+m_1)\text{E}(m)]/15\pi m^2)\}\sin\varphi\\ &\quad + (\text{higher harmonics}) = 0.\end{aligned}$$

Setting the coefficient of $\sin\varphi$ to zero one gets

$$a^2 = 5[(2-m)\text{E}(m) - 2m_1\text{K}(m)]/[m_1(m-2)\text{K}(m) + 2(m^2+m_1)\text{E}(m)]. \quad (3.8)$$

Assuming once more that X is a generating solution, one obtains now $w^2 = Ba^2$ and $w^2 = A/(m-2)$, and from these two equations

$$A/B = (m-2)a^2. \tag{3.9}$$

If one uses as generating solution $X = ak'\,\mathrm{nd}\,(wt; m)$ the only change necessary in the foregoing formulae is on the left side of equation (3.6) which is now $a^2 m_1$.

4. ANALYTICAL RESULTS

4.1. OSCILLATOR TYPE I

Figure 1 shows the ratio A/B versus m. For $A>0$ and $B>0$, one knows that $m<1/2$; for $A<0$ and $B>0$, $m>1/2$. When $m=0$, A/B tends to infinity as expected because in this case $B=0$. With $m=0$ one has the well-known case of the van der Pol oscillator with

$$X = a\,\mathrm{cn}\,(wt; m=0) = a\cos wt,$$

and our method gives the same results as the usual analysis with ordinary Fourier series. For $m=1/2$ one has the quartic oscillator we studied in references [5, 6]. If $A<0$ and $B>0$ there is no limit cycle in the upper part of the potential well if $A/B < -2{\cdot}6$. The possible cycles in the lower part of the well are discussed under oscillator type III.

2.2. OSCILLATOR TYPE II

In Figure 2, one sees that here $A/B<0$ for any values of $0<m<1$, and $A>0$ and $B<0$. When $m=0$, A/B tends to minus infinity, one has the van der Pol oscillator with

$$X = a\,\mathrm{sn}\,(wt; m=0) = a\sin wt,$$

and our method gives the same results as the usual Fourier series analysis with $a^2 = 4{\cdot}00$.

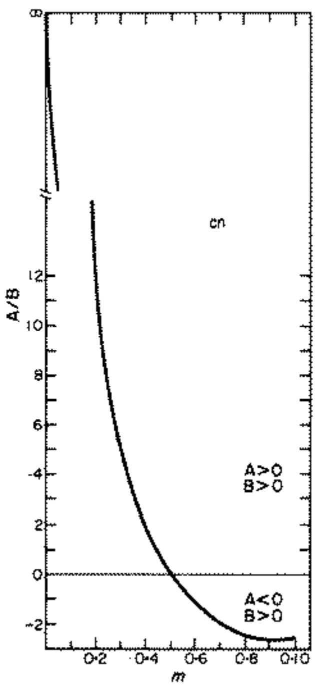

Figure 1. The ratio A/B versus m for the solution cn.

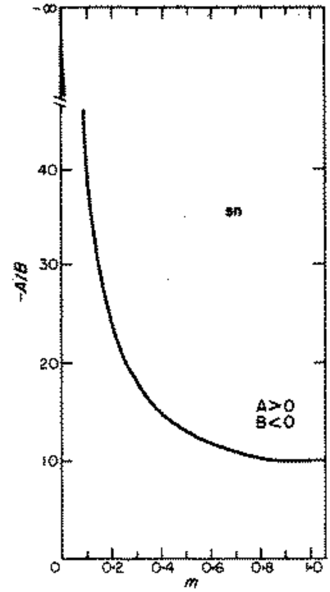

Figure 2. The ratio $-A/B$ versus m for the solution sn.

To have periodic motion, the limit cycle must lie below the potential maximum: i.e., $E < V_{max} = -A^2/4B$. In our first order approximation, one sees additionally in Figure 2 that there is no limit cycle for $A/B > -10{\cdot}00$.

4.3. OSCILLATOR TYPE III

In the example under consideration this is the most interesting case because of the very limited range of values of A/B for which there are limit cycles (see Figure 3). Here we have the double well ($E < 0$ part) of the quartic potential. The limit cycles are symmetrically situated in each well.

The limit cycles in the lower potential well are limited to the values $-2{\cdot}5000 < A/B < -1{\cdot}9994$.

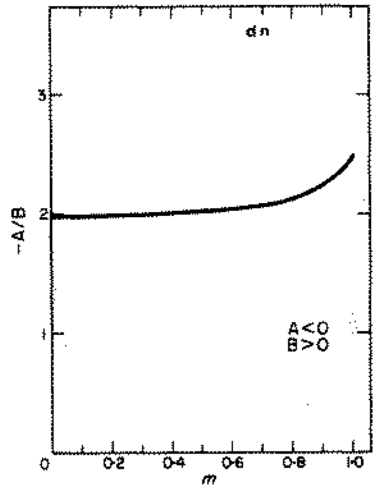

Figure 3. The ratio $-A/B$ versus m for the solution dn.

5. COMPARISON WITH NUMERICAL INTEGRATION

It is instructive to compare these simple analytical approximations with a numerical integration of the equations. We used the fourth order Runge–Kutta method and illustrate the results with the value $\varepsilon = 0{\cdot}1$.

The results for the limit cycles as shown in Figure 4 for the type I oscillator and for the case $A = 0{\cdot}20$, and $A/B = 1{\cdot}87$ from formula (3.5). The (a) curve is the analytical solution $X = 1{\cdot}9338\,\mathrm{cn}\,(0{\cdot}9998t;\, m = 0{\cdot}40)$. The (b) curve represents the results of the numerical integration with initial conditions $X(t=0) = 1{\cdot}9338$ and $\dot{X}(t=0) = 0{\cdot}00$.

Figure 5 confirms the non-existence of limit cycles in the case $A = -3{\cdot}0$ and $B = 1{\cdot}0$. The numerical integration result is for $X(t=0) = 1{\cdot}9541$ and $\dot{X}(t=0) = 0{\cdot}00$. One sees in Figure 1 that if $A/B = -3$ no limit cycle is possible if $B > 0$.

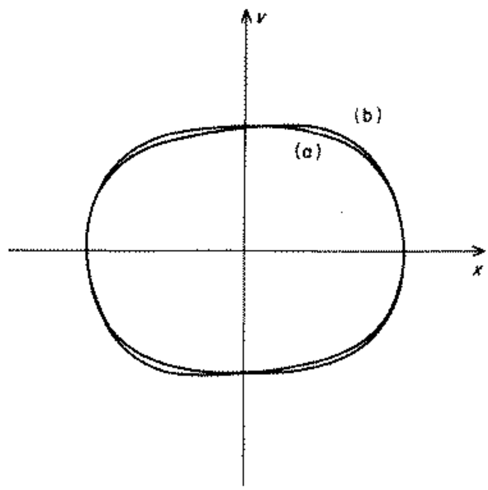

Figure 4. Limit cycles for oscillator type I: (a) analytical solution for $X = 1{\cdot}9338\,\mathrm{cn}\,(0{\cdot}9998t;\, m = 0{\cdot}40)$ and $A/B = 1{\cdot}87$; (b) numerical integration for initial conditions $X(t=0) = 1{\cdot}9338$ and $\dot{X}(t=0) = 0{\cdot}00$.

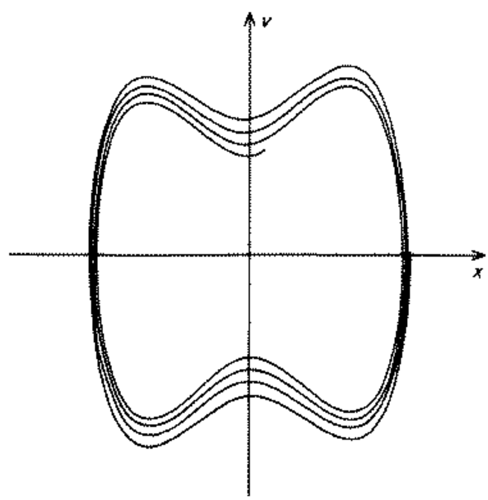

Figure 5. The non-existence of limit cycles for oscillator type I in the case $A/B = -3{\cdot}00$. Numerical integration for initial conditions $X(t=0) = 1{\cdot}9541$ and $\dot{X}(t=0) = 0{\cdot}00$.

Figure 6 shows the results for the type II oscillator for the case $A = 2{\cdot}00$ and $B = -0{\cdot}20$. Figure 2 shows that for the value $A/B = -10{\cdot}0$, one has $m = 0{\cdot}9990$. The (a) curve is the analytical result $X = 2{\cdot}2306\,\mathrm{sn}\,(1{\cdot}0000t;\ m = 0{\cdot}9990)$. The (b) curve gives the numerical integration results for $X(t=0) = 2{\cdot}2306$ and $\dot{X}(t=0) = 0{\cdot}00$.

Figure 7 checks the non-existence of a limit cycle for the case with $A = 6{\cdot}00$, $B = -1{\cdot}00$. We give the numerical integration results with initial conditions $X(t=0) = 2{\cdot}15$ and $\dot{X}(t=0) = 0{\cdot}00$.

The results for the case $A = -2{\cdot}18$, and $B = 1{\cdot}00$ are shown in Figure 8 for the type III oscillator. The values of formula (3.9) shown in Figure 3 give $m = 0{\cdot}86$. The (a) curve represents the analytical solution $X = 1{\cdot}3835\,\mathrm{dn}\,(1{\cdot}3835t;\ m = 0{\cdot}86)$. The (b) curve gives the numerical integration results for $X(t=0) = 1{\cdot}3835$ and $\dot{X}(t=0) = 0{\cdot}00$.

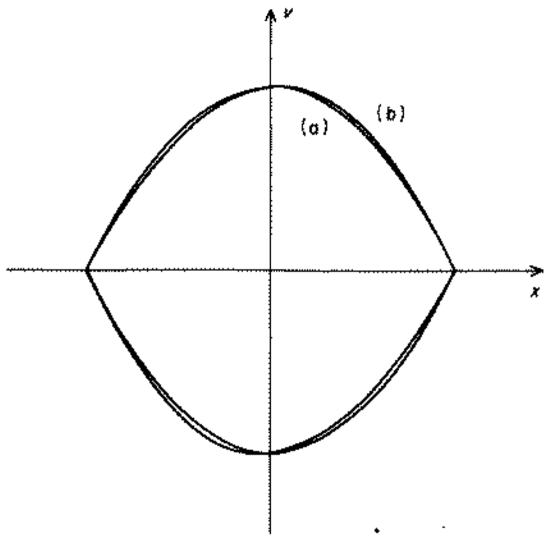

Figure 6. Limit cycles for oscillator type II: (a) analytical solution for $X = 2{\cdot}2306\,\mathrm{sn}\,(1{\cdot}00t;\ m = 0{\cdot}9990)$ and $A/B = -10$; (b) numerical integration for initial conditions $X(t=0) = 2{\cdot}2306$ and $\dot{X}(t=0) = 0{\cdot}00$.

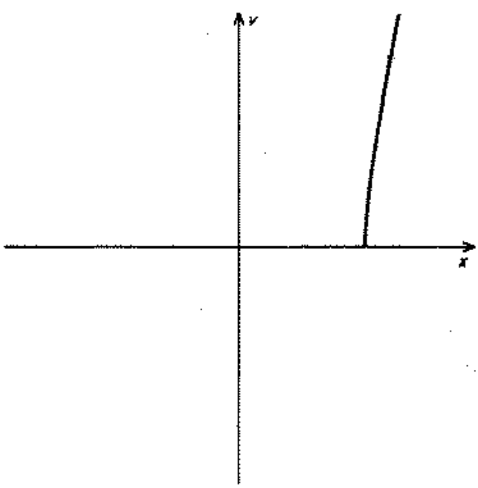

Figure 7. The non-existence of limit cycles for oscillator type II in the case $A/B = -6{\cdot}00$. Numerical integration for initial conditions $X(t=0) = 2{\cdot}2306$ and $\dot{X}(t=0) = 0{\cdot}00$.

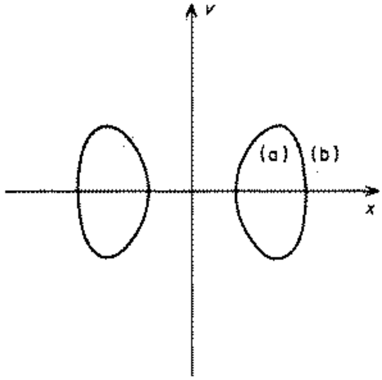

Figure 8. Limit cycles for oscillator type III: (a) analytical solution for $X = 1{\cdot}3835\,\mathrm{dn}\,(1{\cdot}3835t;\, m = 0{\cdot}860)$ and $X(t=0) = 1{\cdot}8260$ and $\dot{X}(t=0) = 0{\cdot}00$.

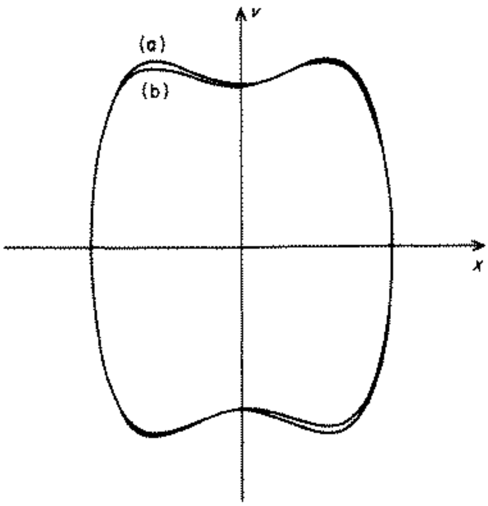

Figure 9. Limit cycles for oscillator type I: (a) analytical solution for $X = 1{\cdot}8260\,\mathrm{cn}\,(2{\cdot}1310t;\, m = 0{\cdot}74)$ and $A/B = -2{\cdot}18$; (b) numerical integration for initial conditions $X(t=0) = 1{\cdot}8260$ and $\dot{X}(t=0) = 0{\cdot}00$.

Figure 9 represents the results for the type I oscillator for $A = -2{\cdot}18$ and $B = 1{\cdot}00$. The (a) curve is the analytical result $X = 1{\cdot}8260\,\mathrm{cn}\,(2{\cdot}1310t;\, m = 0{\cdot}74)$. The (b) curve gives the numerical integration result for $X(t=0) = 1{\cdot}8260$ and $\dot{X}(t=0) = 0{\cdot}00$.

6. CONCLUSIONS

We have described how to generalize the method of harmonic balance to obtain first order approximations to the periodic solution of differential equations of type (1.3), using elliptic functions.

We have applied the method to the study of the limit cycles of equation (1.4). The presence of zero, one, or three limit cycles depends on the value of A/B. For known A and B, Figures 1–3 show the number of limit cycles and the approximate analytic solutions.

The maximum amplitude of the oscillations is found to depend only on m, $\mathrm{E}(m)$, and $\mathrm{K}(m)$.

We have compared the results with numerical integration results (Figures 4–9) and this comparison shows the analytic approximation to be very good.

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