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THE RAYLEIGH METHOD WITH JACOBI ELLIPTIC FUNCTIONS

1. THE RAYLEIGH METHOD AND THE ELLIPTIC FUNCTIONS

The Rayleigh (one-term deflection function) method is an old method [1] that has been extensively used to find an upper approximation to the lowest or fundamental frequency of vibrating systems. It is sometimes presented as a generalization of the energy method, in which the frequency is obtained by equating either the maximum kinetic energy T_{max} with the maximum elastic energy U_{max} [2, 3], or the mean kinetic energy with the mean elastic energy [4, 5].

For a vibrating string [2, 3]

$$U_{max} = \frac{\tau}{2} \int_0^L (y')^2 \, \mathrm{d}x, \tag{1}$$

and $T_{max} = \omega^2 T^*$ with

$$T^* = \frac{\rho}{2} \int_0^L y^2 \, \mathrm{d}x. \tag{2}$$

Hence (the Rayleigh quotient) $\omega^2 = U_{max}/T^*$, where τ is the string's tension, ρ is the mass of the string per unit of length, L is the distance along the x axis between the ends of the string, and $y(x)$ is the string's deflection curve. There are similar expressions for torsional or longitudinal vibrations of bars [2, 5]. For beams one has [2, 5, 6]

$$U_{max} = \frac{\varepsilon I}{2} \int_0^L (y'')^2 \, \mathrm{d}x, \qquad T^* = \frac{\rho}{2} \int_0^L y^2 \, \mathrm{d}x, \tag{3, 4}$$

where ε is Young's modulus, I the beam moment of inertia, ρ the mass of the beam per unit of length, L is the distance along the x axis between the ends of the beam, and $y(x)$ is the beam deflection curve.

The frequency obtained with the Rayleigh method is the exact fundamental frequency if the trial deflection function $y(x)$ is the exact one. But if $y(x)$ is different from the exact deflection function, the frequency obtained from the Rayleigh quotient is always higher than the exact fundamental frequency. Often a trial deflection function that depends on an undetermined parameter, say γ, is used, and this parameter γ is chosen to minimize $\omega^2(\gamma)$. For example, Rayleigh himself [1, vol. I, p. 112] used the trial deflection function

$$y(x) = y_0[1 - (2|x|/L)^\gamma] \tag{5}$$

for the vibration of a stretched string (the origin of x is the string's middle point).

The Rayleigh method has been applied with a great variety of trial functions; polynomial and periodic (trigonometric) functions principally (especially in textbooks, see references [2–5]). For polynomial trial functions the exponent is the natural undetermined parameter—expression (5) is an example—or at least has been the most popular undetermined parameter since the work of Schmidt [7–9]. However, there is no similar natural undetermined parameter reported in the literature for the periodic (trigonometric) functions. The purpose of the present communication is to point out that the elliptic parameter k^2 can be taken as the natural undetermined parameter in the Rayleigh method (when periodic

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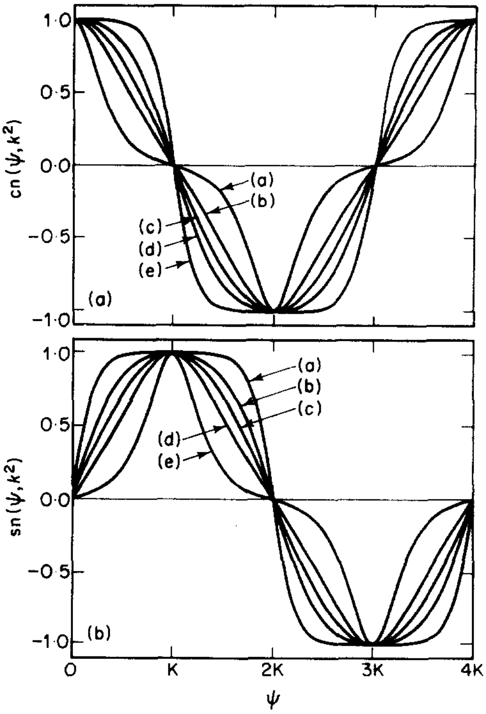

Figure 1. (A) The cn (ψ, k^2) functions and (B) the sn (ψ, k^2) functions for (a) $k^2 = 0{\cdot}998$, (b) $k^2 = 0{\cdot}8$, (c) $k^2 = 0$, (d) $k^2 = -4$ and (e) $k^2 = -499$.

functions are used as trial functions), if the trigonometric functions are considered as particular cases of the Jacobi elliptic functions, since cos $(\psi) =$ cn $(\psi, k^2 = 0)$ and sin $(\psi) =$ sn $(\psi, k^2 = 0)$. In other words, the Jacobi elliptic functions cn (ψ, k^2) and sn (ψ, k^2) are a natural extension of the trigonometric functions in the Rayleigh method with an undetermined parameter. The graphical representation of these functions with $0 \leqslant k^2 < 1$ and $k^2 < 0$ is shown in Figure 1. Note that the plot of cn (ψ, k^2) displaced a quarter period to the right is the same as sn (ψ, k^2). This is because sn $(\psi, \mu^2) =$ cn $(\varphi - \mathrm{K}, k^2)$, where $\varphi = \psi/(1-k^2)$, $k^2 = -\mu^2/(1+\mu^2)$, and $\mathrm{K} = \mathrm{K}(k^2)$ is the complete elliptic integral of the first kind (for more details see reference [10]). The period of cn (ψ, k^2) and sn (ψ, k^2) is $4\mathrm{K}(k^2)$.

The next section presents some illustrations of the use of Jacobi elliptic functions as trial deflection functions. The results are satisfactory, especially for the beam examples.

2. ILLUSTRATIVE EXAMPLES

2.1. *String with fixed ends and a point mass at the middle*

This problem is mathematically equivalent to the longitudinal (or torsional) vibration of a bar clamped at both ends and having a disk with mass (or moment of inertia) in the middle of the bar. The circular trial function is [2, 4]

$$y(x) = y_0 \operatorname{sen}(\pi x/L) = y_0 \cos\left(\frac{\pi}{L}x - \frac{\pi}{2}\right) = y_0 \operatorname{cn}\left(\frac{2\mathrm{K}(0)}{L}x - \mathrm{K}(0), 0\right). \tag{6}$$

The elliptic trial function is therefore

$$y(x) = y_0 \operatorname{cn}\left(\frac{2\mathrm{K}(k^2)}{L}x - \mathrm{K}(k^2), k^2\right). \tag{7}$$

By substituting expression (7) into equation (2) and carrying out the integrations (see reference [11]) one obtains the maximum kinetic energy of the string:

$$\tfrac{1}{2}\omega^2\rho L y_0^2[(\mathrm{K}-\mathrm{E})/(k^2\mathrm{K})]=\tfrac{1}{2}\omega^2\rho L y_0^2 T_2^*,$$

where $\mathrm{K}=\mathrm{K}(k^2)$, and $\mathrm{E}=\mathrm{E}(k^2)$ is the complete elliptic integral of the second kind. The maximum kinetic energy of the point mass (of mass m) is $\frac{1}{2}m\omega^2 y^2(L/2)=\frac{1}{2}m\omega^2 y_0^2$. By using

$$(\mathrm{d}/\mathrm{d}\psi)\,\mathrm{cn}\,(\psi,k^2)=-\mathrm{sn}\,(\psi,k^2)\,\mathrm{dn}\,(\psi,k^2)$$

in equation (1) and carrying out the integrations [11], one finds that the maximum potential energy is

$$U_{max}=\tfrac{1}{2}(\tau/L)y_0^2 4\mathrm{K}[(1+k^2)\mathrm{E}-(1-k^2)\mathrm{K}]/(3k^2)\equiv\tfrac{1}{2}(\tau/L)y_0^2 H_2.$$

Equating U_{max} with the total maximum kinetic energy (the beam's maximum kinetic energy plus the maximum kinetic energy of the point mass) one finds

$$\omega^2=\{H_2/[T_2^*+(m/\rho L)]\}\tau/(\rho L^2)=\bar{\omega}^2\tau/(\rho L^2).$$

The values of $\bar{\omega}^2$ thus obtained with their optimum parameters k^2 are given in Table 1. Other values of $\bar{\omega}^2$ are given for comparison: the exact ones [2, 4] and those obtained by using the Rayleigh method with the polynomial trial function given by equation (5) and the circular trial function of equation (6).

TABLE 1

String with fixed ends and point-mass at the middle: comparison of the $\bar{\omega}^2$ values obtained in the various approximations (the optimum parameter values are given in parentheses)

$m/\rho L$	Exact	Circular	Elliptic	Polynomial
0·0	π	π	π (0·0)	3·146 (1·72)
0·1	2·858	2·868	2·863 (0·29)	2·860 (1·55)
0·2	2·628	2·655	2·641 (0·44)	2·630 (1·43)
0·5	2·154	2·221	2·185 (0·64)	2·155 (1·24)
1·0	1·720	1·814	1·761 (0·74)	1·722 (1·13)
5·0	0·866	0·947	0·898 (0·83)	0·866 (1·03)

2.2. *Clamped-free beam*

As the circular trial function is [5, 2]

$$y(x)=y_0\left[1-\cos\left(\frac{\pi}{2L}x\right)\right]=y_0\left[1-\mathrm{cn}\left(\frac{\mathrm{K}(0)}{L}x,0\right)\right],\tag{8}$$

we use

$$y(x)=y_0\left[1-\mathrm{cn}\left(\frac{\mathrm{K}(k^2)}{L}x,k^2\right)\right]\tag{9}$$

as the elliptic trial deflection function. These two trial functions (8) and (9) satisfy the kinematic conditions $y(0)=y'(0)=0$. By using the relation

$$(\mathrm{d}^2/\mathrm{d}\psi^2)\,\mathrm{cn}=-(1-2k^2)\,\mathrm{cn}-2k^2\,\mathrm{cn}^3,$$

where $\mathrm{cn}\equiv\mathrm{cn}\,(\psi,k^2)$, in equation (3) and carrying out the integrations one finds

$$U_{max}=U_1\tfrac{1}{2}y_0^2 EI/L^3\tag{10}$$

with

$$U_1 = (1-2k^2)^2 C_2 + 4k^2(1-2k^2)C_4 + 4k^4 C_6. \tag{11}$$

and where (see reference [11])

$$C_2 = \int_0^K \mathrm{cn}^2(\psi, k^2)\,\mathrm{d}\psi = (\mathrm{E} - k_1^2\mathrm{K})/k^2,$$

$$C_4 = \int_0^K \mathrm{cn}^4(\psi, k^2)\,\mathrm{d}\psi = [2(2k^2-1)C_2 + k_1^2\mathrm{K}]/(3k^2),$$

$$C_6 = \int_0^K \mathrm{cn}^6(\psi, k^2)\,\mathrm{d}\psi = [4(2k^2-1)C_4 + 3k_1^2 C_2]/(5k^2)$$

and $k_1^2 = 1-k^2$. By substituting equation (9) into equation (4) and carrying out the integrations, one has [11]

$$T^* = T_1^* \tfrac{1}{2} y_0^2 \rho L, \tag{12}$$

with

$$T_1^* = [\mathrm{K} + C_2 - (2/k)\arcsin(k)]/\mathrm{K}. \tag{13}$$

Therefore $\omega^2 = \bar{\omega}^2 \varepsilon I/(\rho L^4)$ with $\bar{\omega}^2 = U_1/T_1^*$. The minimum value $\bar{\omega}^2 = 3{\cdot}520$ is found for $k^2 = 0{\cdot}40$. When $k^2 = 0$, i.e., by using equation (8), one finds [2, 3, 5, 6] $\bar{\omega}^2 = 3{\cdot}664$. The exact value is $\bar{\omega}^2 = 3{\cdot}516$ [3, 5].

2.3. *Clamped-free beam carrying a point mass at the free end*

The elliptic trial deflection function used is the same as in the last example. The kinematic conditions are again satisfied. The expression for U_{max} is again given by equations (10) and (11). The value of T_{max} is obtained by adding the maximum kinetic energy of the point mass, given by $\frac{1}{2}m\omega^2 y^2(L) = \frac{1}{2}m\omega^2 y_0^2$, where m is the value of the point mass, to the maximum beam kinetic energy, given by equations (12) and (13). Then the Rayleigh quotient gives

$$\omega^2 = \{U_1/[T_1^* + (m/\rho L)]\}\varepsilon I/(\rho L^4) \equiv \bar{\omega}^2 \varepsilon I/(\rho L^4).$$

The values of $\bar{\omega}^2$ and the optimum parameters k^2 obtained with the present method are given in Table 2. Other $\bar{\omega}^2$ values are given for comparison: the exact ones [12], those obtained by using the Rayleigh method with the exact solution of the clamped-free beam as trial deflection function [12], and those obtained by using the Rayleigh method with the circular expression (8) as trial deflection function.

TABLE 2

Clamped-free beam with point-mass at the free end: comparison of the $\bar{\omega}^2$ values obtained in the various approximations (the optimum parameter values k^2 of the elliptic method are given in parentheses)

$m/\rho L$	Exact	Circular	Elliptic	Stephen [12]
0·2	2·613	2·671	2·616 (0·31)	2·621
0·4	2·168	2·204	2·170 (0·27)	2·181
0·6	1·892	1·919	1·894 (0·25)	1·907
0·8	1·701	1·722	1·702 (0·24)	1·716
1·0	1·557	1·575	1·559 (0·23)	1·573

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