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IMPROVEMENT OF A KRYLOV-BOGOLIUBOV METHOD THAT USES JACOBI ELLIPTIC FUNCTIONS

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An improved version of a Krylov-Bogoliubov method that gives the approximate solution of the non-linear cubic oscillator $\ddot{x}+c_1x+c_3x^3+\varepsilon f(x,\dot{x})=0$ in terms of Jacobi elliptic functions is described. Compact general expressions are given for the time derivatives of the amplitude and phase similar to those obtained by the usual Krylov-Bogoliubov method (which gives the approximate solution in terms of circular functions). These expressions are especially simple for quasilinear ($c_3=0$) and quasi-pure-cubic ($c_1=0$) oscillators. Two types of cubic oscillators have been used as examples: the linear damped oscillator $f(x,\dot{x})=\dot{x}$, and the van der Pol oscillator $f(x,\dot{x})=(\alpha-\beta x^2)\dot{x}$. The approximate solutions of these quasilinear and quasi-pure-cubic oscillators are simple and accurate. The influence of the non-linearity on the rate of variation of the amplitude of these two types of cubic oscillators was also studied.

1. INTRODUCTION

The usual Krylov-Bogoliubov (K-B) method (dating from 1937) is widely used for determining approximate solutions to quasi-linear differential equations of the form

$$\ddot{x}+c_1x+\varepsilon f(x,\dot{x})=0, \tag{1.1}$$

where ε is a small constant coefficient. It is able to give the steady state periodic solution and the transient solution of equation (1.1). As the (generating) solution of equation (1.1) with $\varepsilon=0$ (generating equation) is $x(t)=A\cos(\omega t-\phi)$ with A and ϕ constant, the K-B approximate solution is the same but with A and ϕ time dependent: $x(t)=A(t)\cos(\omega t-\phi(t))\equiv A\cos\psi$. The approximate expressions for the amplitude $A(t)$ and phase $\phi(t)$ are obtained by solving

$$\dot{A}=\frac{\varepsilon}{\omega}\frac{1}{2\pi}\int_0^{2\pi} f(A\cos\psi,-A\omega\sin\psi)\sin\psi\,\mathrm{d}\psi,$$

$$\dot{\phi}=-\frac{\varepsilon}{A\omega}\frac{1}{2\pi}\int_0^{2\pi} f(A\cos\psi,-A\omega\sin\psi)\cos\psi\,\mathrm{d}\psi. \tag{1.2}$$

Unfortunately, the basic method is applicable only to weakly non-linear oscillators. So, for non-linear oscillators of the form

$$\ddot{x}+b\dot{x}+c_1x+\varepsilon f(x,\dot{x})=0,\qquad 0<\varepsilon\ll 1, \tag{1.3}$$

several extensions of the K-B method have been constructed (see references [1-4]). Many other oscillators have the form

$$\ddot{x}+F(x)+\varepsilon f(x,\dot{x})=0,\qquad 0<\varepsilon\ll 1, \tag{1.4}$$

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where $F(x)$ is an odd non-quasi-linear force. If $F(x) = c_1x + c_3x^3 + O(\varepsilon)$ then equation (1.4) becomes

$$\ddot{x} + c_1x + c_3x^3 + \varepsilon f(x, \dot{x}) = 0, \qquad 0 < \varepsilon \ll 1. \tag{1.5}$$

The generating equation, equation (1.5) with $\varepsilon = 0$, has solutions (generating solutions) in terms of Jacobi elliptic functions. To our knowedge the first papers devoted to solving the oscillator class (1.5) with $c_1 > 0$ and $c_3 > 0$ by methods of Krylov-Bogoliubov type to provide approximate solutions in terms of Jacobi elliptic functions were those of Barham and Soudack [5-8]. Christopher [9] developed a more accurate version, but only for oscillators with $\varepsilon f(x, \dot{x}) = \varepsilon\dot{x}$ and $c_1 - (\varepsilon/2)^2 > 0$ and $c_3 > 0$. Christopher and Brocklehurst [10] then extended this version to equation (1.4) with $c_1 > 0$ and $c_3 > 0$. Yuste and Bejarano in reference [11] have shown that the Christopher method of reference [9] can be extended to oscillators with $c_1 > 0$, $c_3 < 0$ and $c_1 < 0$, $c_3 > 0$, and in reference [12] improved the Christopher–Brocklehurst method of reference [10] and showed that it is also valid for $c_1 > 0$, $c_3 < 0$ and $c_1 < 0$, $c_3 > 0$. This last version of the method is precise and not too complicated. However, simple expressions for the time derivatives of the variable parameters similar to those obtained by the usual K–B method as in equation (1.2) have not yet been obtained. That will be done in the present communication. We will show in section 3 that the expressions of the usual K–B method are simply particular cases of the method of K–B type presented here (which will be called the EKB method). The EKB method is especially simple when $c_3 = 0$ (quasilinear oscillators) and when $c_1 = 0$ (quasi-pure-cubic oscillator). For $c_3 = 0$ the present method coincides with the normal K–B method that uses circular functions. Simple and accurate solutions are obtained for the case $c_1 = 0$ in two examples: a cubic oscillator with linear damping $f(x, \dot{x}) = \dot{x}$ and a van der Pol cubic oscillator $f(x, \dot{x}) = (\alpha - \beta x^2)\dot{x}$.

2. CUBIC OSCILLATOR SOLUTIONS

In this section we study some properties of the solution of equation (1.4) with $\varepsilon = 0$ (generating equation): that is, of the equation

$$\ddot{x} + c_1x + c_3x^3 = 0. \tag{2.1}$$

Its solution is

$$x(t) = A \operatorname{cn}(\omega t - \phi, m), \tag{2.2}$$

with

$$\omega^2 = c_1 + c_3A^2 = c_1(1+\nu), \qquad m = c_3A^2/[2(c_1 + c_3A^2)] = \nu/[2(1+\nu)], \tag{2.3, 2.4}$$

where A and ϕ are constants determined by the initial conditions, and ν is the nonlinear factor $\nu = c_3A^2/c_1$. We define the oscillator energy by $En = \dot{x}^2 + V(x)$, where the potential is $V(x) = c_1x^2 + c_3x^4/2$. The maximum (or minimum) potential is given by $V_m = -c_1^2/2c_3$. It is useful to distinguish four cases: (i) cubic hard oscillator, $c_1 \geq 0$, $c_3 \geq 0$ or $0 \leq m \leq 1/2$ or $0 \leq \nu \leq \infty$; (ii) cubic soft oscillator, $c_1 \geq 0$, $c_3 \leq 0$, $En \leq V_m$ or $m \leq 0$ or $-1 \leq \nu \leq 0$; (iii) cubic soft-hard oscillator with $En \leq 0$, $c_1 \leq 0$, $c_3 \geq 0$, $En \leq 0$ or $1 \leq m$ or $-2 \leq \nu \leq -1$; (iv) cubic soft-hard oscillator with $En \geq 0$, $c_1 \geq 0$, $En \geq 0$ or $1/2 \leq m \leq 1$ or $\nu \leq -2$. These cases are illustrated in Figure 1.

The period of the solution of equation (2.2) is $T = 4K/\omega$ with

$$\begin{aligned} K &= \mathrm{K}(m) \text{ for cases (i) and (iv)}, \\ K &= (1-m)^{-1/2}\mathrm{K}(-m/(1-m)) \text{ for case (ii)}, \\ K &= \tfrac{1}{2}m^{-1/2}\mathrm{K}(1/m) \text{ for case (iii)}, \end{aligned} \tag{2.7}$$

where $\mathrm{K}(z)$ is the complete elliptic integral of the first kind.

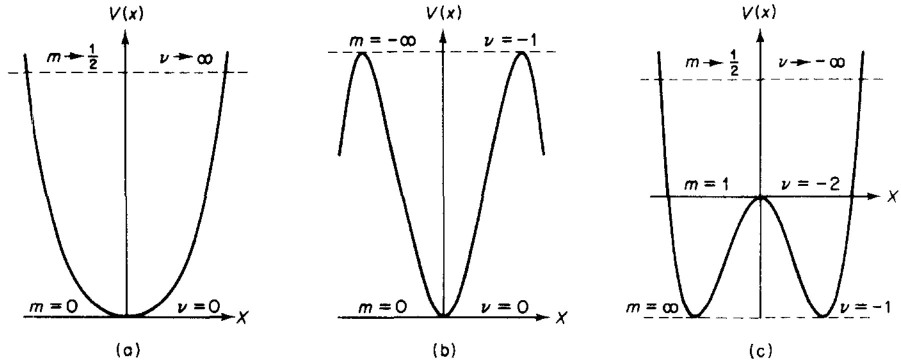

Figure 1. (a) Hard cubic potential (b) soft cubic potential and (c) soft-hard cubic potential.

3. THE K-B METHOD USING JACOBI ELLIPTIC FUNCTIONS

We follow here the presentation of reference [12]. As usual in the methods of K-B type, the form of the trial solution of the equation (1.5) is the same as the form of its generating solution. Then the trial solution is given by equation (2.2) but with A, ϕ, ω and m now time dependent:

$$x(t) = A(t)\,\mathrm{cn}\left(\int_0^t \omega(s)\,\mathrm{d}s - \phi(t), m(t)\right) = A(t)\,\mathrm{cn}\,(\psi(t), m(t)). \tag{3.1}$$

Then the task of finding the solution $x(t)$ is transformed into finding four functions $A(t)$, $\omega(t)$, $\phi(t)$ and $m(t)$ so that expression (3.1) satisfies equation (1.5). That is, although one is free to choose these four functions, one must impose a first obvious constraint: constraint 1; equation (3.1) must be a solution of equation (1.5). Three additional constraints can be imposed to further restrict the arbitrariness. The following one is usual in the K-B method: constraint 2; the time derivative of the trial solution must have the same form as the time derivative of the generating solution,

$$x = A\omega\,\mathrm{cn}_\psi = -A\omega\,\mathrm{sn}\,\mathrm{dn}. \tag{3.2}$$

The notation is $f_\beta(\alpha, \beta) = \partial f/\partial\beta$. The other two constraints are similar to the second: the relationships between frequency, parameter and amplitude must be the same for the trial solution as for the generating solution—see equations (2.3) and (2.4). Therefore

$$\text{constraint 3; } \omega^2 = c_1 + c_3A^2: \tag{3.3}$$

$$\text{constraint 4; } m = c_3A^2/[2(c_1 + c_3A^2)]. \tag{3.4}$$

Differentiating equation (3.1) with respect to t and using constraint 2 one finds

$$\dot{A}\,\mathrm{cn} - A\dot{\phi}\,\mathrm{cn}_\psi + A\dot{m}\,\mathrm{cn}_m = 0. \tag{3.5}$$

Differentiating expression (3.2), substituting the result into equation (1.5) and using constraint 3 and constraint 4 gives

$$\dot{A}\omega\,\mathrm{cn}_\psi + A\dot{\omega}\,\mathrm{cn}_\psi - A\omega\dot{\phi}\,\mathrm{cn}_{\psi\psi} + A\omega\dot{m}\,\mathrm{cn}_{\psi m} + \varepsilon f(A\,\mathrm{cn}, A\omega\,\mathrm{cn}_\psi) = 0. \tag{3.6}$$

Taking $\dot{\phi}$ from equation (3.5) and putting it into equation (3.6), one finds

$$\dot{A}\omega[(\mathrm{cn}_\psi)^2 - \mathrm{cn}\,\mathrm{cn}_{\psi\psi}] + \dot{\omega}A(\mathrm{cn}_\psi)^2 + \dot{m}A\omega[\mathrm{cn}_\psi\,\mathrm{cn}_{\psi m} - \mathrm{cn}_m\,\mathrm{cn}_{\psi\psi}] + \varepsilon f\,\mathrm{cn}_\psi = 0.$$

As $\mathrm{cn}_\psi = -\mathrm{sn}\,\mathrm{dn}$, $\mathrm{cn}_{\psi\psi} = \mathrm{cn}\,(1 - 2\mathrm{dn}^2)$, $\mathrm{cn}_\psi\,\mathrm{cn}_{\psi m} - \mathrm{cn}_m\,\mathrm{cn}_{\psi\psi} = -\mathrm{sn}^4/2$ and, from equations (3.3) and (3.4), $\dot{\omega}/\omega = \dot{m}/(1 - 2m)$, then equation (3.7) becomes

$$(1 - 2m\,\mathrm{sn}^2 + m\,\mathrm{sn}^4)(\dot{A}/A) + (\mathrm{sn}^2 - \mathrm{sn}^4/2)(\dot{\omega}/\omega) = (\varepsilon/A\omega)f\,\mathrm{sn}\,\mathrm{dn}. \tag{3.8}$$

At this point the procedure of the present paper diverges from that of previous papers [9-12]. We do not apply the averaging principle yet. Instead, from equations (3.3) and (3.4) we obtain

$$\dot{\omega}/\omega = 2m\dot{A}/A, \tag{3.9}$$

substitute it into equation (3.8), and find

$$\dot{A} = (\varepsilon/\omega)f(A\ \mathrm{cn},\ A\omega\ \mathrm{cn}_\psi)\ \mathrm{sn}\ \mathrm{dn}. \tag{3.10a}$$

But from equation (3.5) $\dot{\phi} = (A\ \mathrm{cn} + Am\ \mathrm{cn}_m)/A\ \mathrm{cn}_\psi$, and from equations (3.3) and (3.4) $\dot{m} = 2m(1-2m)\dot{A}/A$. Using these relations and the equation (3.10a), one finds

$$\dot{\phi} = -(\varepsilon/A\omega)f(A\ \mathrm{cn},\ A\omega\ \mathrm{cn}_\psi)[\mathrm{cn} + 2m(1-2m)\ \mathrm{cn}_m]. \tag{3.10b}$$

So the task of obtaining the solution $x(t)$ of equation (1.5) has been transformed into the equivalent one of obtaining the two solutions $A(t)$ and $\phi(t)$ of the system of equations (3.10) (the expressions for ω and m are obtained by substituting this solution $A(t)$ into relations (3.3) and (3.4) of constraints 3 and 4). These equations are usually quite complicated. But a comparison of the expressions (3.10) with their counterparts in the normal K-B method [13, 14] shows them to have the same form. It is at this point that we return to the usual procedure in the methods of slowly varying parameters and apply the averaging principle. This is, we transform (3.10) to the averaged system (key system):

$$\dot{A} = (\varepsilon/\omega)\langle f(A\ \mathrm{cn},\ A\omega\ \mathrm{cn}_\psi)\ \mathrm{sn}\ \mathrm{dn}\rangle, \tag{3.11a}$$

$$\dot{\phi} = -(\varepsilon/A\omega)\langle f(A\ \mathrm{cn},\ A\omega\ \mathrm{cn}_\psi)[\mathrm{cn} + 2m(1-2m)\ \mathrm{cn}_m]\rangle, \tag{3.11b}$$

where

$$\langle\cdots\rangle \equiv \frac{1}{4K}\int_0^{4K}\cdots\ \mathrm{d}\psi$$

is the operation of averaging over a period. As is well known, the solutions for this averaged system are closer to those of the exact system when A and ϕ change little in a period: i.e., when $\dot{A}$ and $\dot{\phi}$ are small (notice that $\dot{A}$ and $\dot{\phi}$ are of order ε) and the effective frequency $\dot{\psi} = \omega - \dot{\phi}$ is large.

When the oscillator is quasilinear, i.e., when $c_3 = 0$ and therefore $m = 0$ (and $\omega^2 = c_1$), the system (3.11) becomes especially simple,

$$\begin{aligned} \dot{A} &= \frac{\varepsilon}{\omega}\frac{1}{2\pi}\int_0^{2\pi} f(A\cos,\ -A\omega\sin)\sin\ \mathrm{d}\psi, \\ \dot{\phi} &= -\frac{\varepsilon}{A\omega}\frac{1}{2\pi}\int_0^{2\pi} f(A\cos,\ -A\omega\sin)\cos\ \mathrm{d}\psi, \end{aligned} \tag{3.12}$$

since $\mathrm{cn}\ (\psi, 0) = \cos\psi$, $\mathrm{sn}\ (\psi, 0) = \sin\psi$, $\mathrm{dn}\ (\psi, 0) = 1$ and $\mathrm{K}(0) = \pi/2$. These are the well known relations (1.2) of the normal K-B method [13, 14] but have been obtained here as a particular case of the general expressions (3.11) of the present elliptic method.

When the equation is quasi-pure-cubic, i.e., when $c_1 = 0$ and therefore $m = 1/2$ (and $\omega^2 = c_3A^2$), the system is also simple,

$$\begin{aligned} \dot{A} &= \frac{\varepsilon}{\omega}\frac{1}{4K}\int_0^{4K} f(A\ \mathrm{cn},\ -A\omega\ \mathrm{sn}\ \mathrm{dn})\ \mathrm{sn}\ \mathrm{dn}\ \mathrm{d}\psi, \\ \dot{\phi} &= -\frac{\varepsilon}{A\omega}\frac{1}{4K}\int_0^{4K} f(A\ \mathrm{cn},\ -A\omega\ \mathrm{sn}\ \mathrm{dn})\ \mathrm{cn}\ \mathrm{d}\psi, \end{aligned} \tag{3.13}$$

with cn = cn (ψ, 1/2), sn = sn (ψ, 1/2), dn = dn (ψ, 1/2) and $K = \mathrm{K}(1/2) = 1{\cdot}85407\ldots$. In reference [15] we gave a method of slowly varying amplitude and phase for this class of quasi-pure-cubic oscillators. But it has the defect that the frequency ω was considered constant and it was not clear how to determine it. The present method must be considered the correct version of the method expounded in [15].

In the next two sections we give two illustrative examples.

4. LINEAR DAMPED CUBIC OSCILLATOR

The equation is

$$\ddot{x} + c_1 x + c_3 x^3 + \varepsilon \dot{x} = 0. \tag{4.1}$$

Equation (3.11a) is then

$$\dot{A} = -\frac{\varepsilon}{\omega}\frac{1}{4K}\int_0^{4K} A\omega\, \mathrm{sn}^2\, \mathrm{dn}^2\, \mathrm{d}\psi = -\varepsilon A\langle \mathrm{sn}^2\, \mathrm{dn}^2\rangle \equiv -\tilde{\varepsilon}A, \tag{4.2a}$$

where $\tilde{\varepsilon} = \tilde{\varepsilon}A(m)$ and [16]

$$Q(m) \equiv \langle \mathrm{sn}^2\, \mathrm{dn}^2\rangle = \frac{1}{3m}\,\frac{(2m-1)E + (1-m)K}{K}$$

with K given by equation (2.7) and E given by $E = \mathrm{E}(m)$ for cases (i) and (iv), $E = (1-m)^{1/2}\mathrm{E}(-m/(1-m))$ for case (ii), and $E = \frac{1}{2}m^{1/2}[\mathrm{E}(1/m) - ((m-1)/m)\mathrm{K}(1/m)]$ for case (iii), where $\mathrm{E}(z)$ is the complete elliptic integral of the second kind. In Figure 2 the function $Q(\nu) = Q(m)$ is plotted versus the non-linearity factor. Equation (3.11b) becomes

$$\dot{\phi} = \varepsilon 2m(1-2m)\langle \mathrm{sn}\, \mathrm{dn}\, \mathrm{cm}_m\rangle. \tag{4.2b}$$

In obtaining expression (4.2b) the relation $\langle \mathrm{sn}\, \mathrm{cn}\, \mathrm{dn}\rangle = 0$ has been used.

We will now look at the simplest cases: equation (4.1) for $c_1 = 0$ and equation (4.1) for $c_3 = 0$. These two cases are the simplest because the elliptic parameter does not depend on the amplitude.

4.1. QUASILINEAR OSCILLATOR

In this oscillator $c_3 = 0$ and so $m = 0$ and $\omega^2 = c_1$ for all amplitudes. Equations (4.2) are now $\dot{A}/A = -\varepsilon Q(m=0) = -\tilde{\varepsilon}$. As $Q(m=0) = 1/2$, integrating these expressions gives the well known result [13] of the normal K–B method: $x(t) = A_0 \exp(-\varepsilon t/2)\cos(\omega_0 t - \phi_0)$, where $A_0 \equiv A(0)$, $\omega_0 \equiv \omega(0)$ and $\phi_0 \equiv \phi(0)$. This notation will be used in the following.

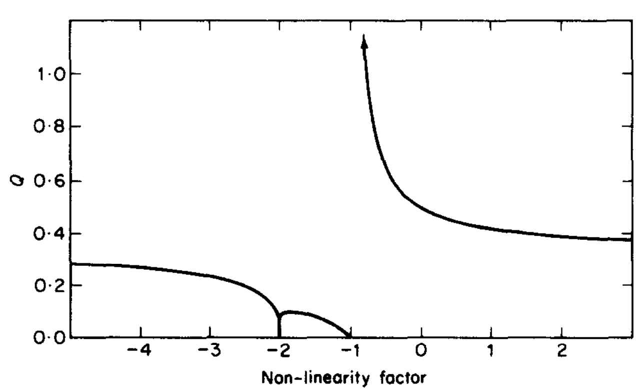

Figure 2. The $Q = \langle \mathrm{sn}^2\, \mathrm{dn}^2\rangle$ function versus the non-linearity factor.

4.2. QUASI-PURE-CUBIC OSCILLATOR

For this oscillator $c_1 = 0$ and then $m = 1/2$ and $\omega^2 = c_3 A^2$ for all amplitudes. Equations (4.2) are now $\dot{A}/A = -\varepsilon Q(m = 1/2) = -\tilde{\varepsilon}$ and $\dot{\phi} = 0$. Integrating these expressions gives $\phi(t) = \phi(0) = \phi_0$ and $A(t) = A_0 \exp(-\tilde{\varepsilon}t)$. The constants A_0 and ϕ_0 are obtained from the initial conditions. As $\psi(t) = \int_0^t \omega(s)\,\mathrm{d}s - \phi(t)$, on integrating one finds that the approximate solution (3.1) is given by

$$x(t) = A_0 \exp(-\tilde{\varepsilon}t)\,\mathrm{cn}\,[(\omega_0/\tilde{\varepsilon})(1 - \exp(-\tilde{\varepsilon}t)) - \phi_0, 1/2], \tag{4.3}$$

where $\tilde{\varepsilon} = \varepsilon/3$ because $Q(m = 1/2) = 1/3$.

In Figures 3–5 are plotted the approximate solution given by equation (4.3) and the numerical solution obtained by using a fourth order Rung-Kutta method. The results are very good and, as expected, better for smaller values of ε and larger values of the frequency.

It is not surprising that expression (4.3) is a good approximation, because it is an *exact* solution [17, 18] of the equation

$$\ddot{x} + (2\varepsilon^2/9)x + c_3 x^3 + \varepsilon\dot{x} = 0. \tag{4.4}$$

References [17] and [18] give only the case $c_3 = -2$ and in the non-standard form $-\mathrm{i}aK_1\,\mathrm{e}^{-at}\,\mathrm{sn}_{m=-1}(K_1\,\mathrm{e}^{-at} + K_2)$, where $a = \varepsilon/3$. Observe that if c_3 and ε are arbitrarily large the solution is exact if $c_1 = 2\varepsilon^2/9$, or alternatively c_1, c_3 arbitrary and $\varepsilon = (9c_1/2)^{1/2}$.

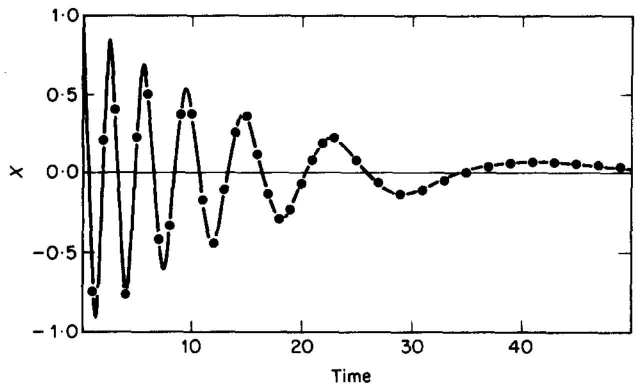

Figure 3. Approximate (solid line) and numerical (●) solution of the linear damped cubic oscillator $\ddot{x} + 10x^3 + 0{\cdot}2\dot{x} = 0$ with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. The approximate solution is obtained by using formula (4.3). The numerical solution is obtained using a Runge-Kutta method of fourth order.

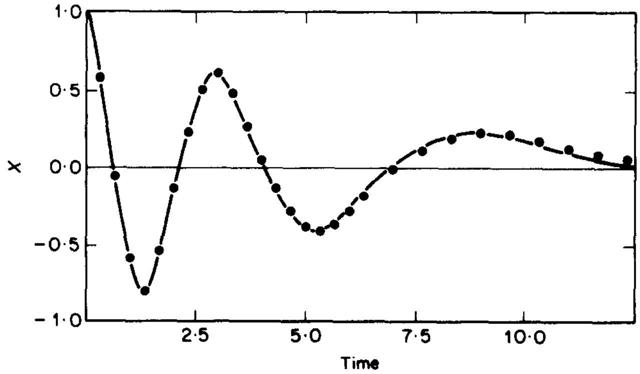

Figure 4. Approximate (solid line) and numerical (●) solution of the linear damped pure cubic oscillator $\ddot{x} + 10x^3 + 0{\cdot}5\dot{x} = 0$ with $x(0) = 1$ and $\dot{x}(0) = 0$. These solutions are obtained as indicated in the caption to Figure 3.

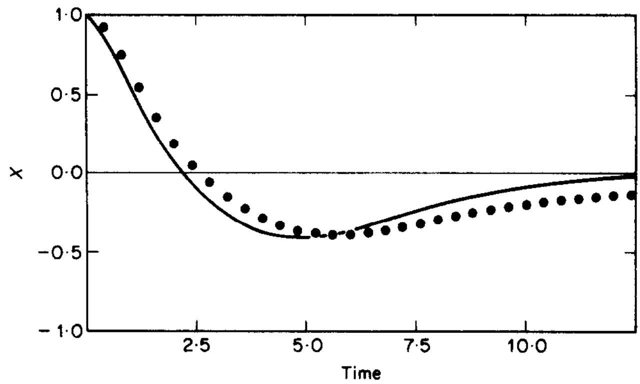

Figure 5. Approximate (solid line) and numerical (●) solution of the linear damped pure cubic oscillator $\ddot{x}+x^3+0{\cdot}5\dot{x}=0$ with $x(0)=1$ and $\dot{x}(0)=0$. These solutions are obtained as indicated in the caption to Figure 3.

Specifically, if $\varepsilon=0{\cdot}2$ as in Figure 3, the solution is exact for $c_1=0{\cdot}08/9$ and, as shown, it is a very good approximation for $c_1=0$. Notice that c_1 is of order ε^2 in equation (4.4).

Finally, it is of interest to note that the Emden equation $\mathrm{d}^2y/\mathrm{d}\xi^2+(2/\xi)\,\mathrm{d}y/\mathrm{d}\xi+c_3y^n=0$ for $n=3$, with the changes $x=\mathrm{e}^{-t}y(\xi=\mathrm{e}^{-t})$, gives $\mathrm{d}^2x/\mathrm{d}t^2+\mathrm{d}x/\mathrm{d}t+c_3x^3=0$: i.e., our methods can also be used to find approximate solutions of the Emden equation for $n=3$. Of course, these solutions will be better for larger c_3.

4.3. OSCILLATOR (4.1) WITH NON-ZERO c_1 AND c_3.

For the two cases of sections 4.1 and 4.2 we have obtained simple accurate expressions for $x(t)$ because $Q(m)$ did not depend on the amplitude and the integrations were easy. But when c_1 and c_3 are non-zero, the integrations are more difficult because $Q(m)$ depends on the amplitude in a non-trivial way. However, one can obtain useful information from equation (4.2) directly. From equation (4.2a) one sees that the relative amplitude variation $\dot{A}/A$ is proportional to $Q(\nu)$. Then from Figure 1 one can make some deductions about this amplitude variation rate: (i) it is smaller for cubic hard oscillators and cubic soft–hard oscillators than for linear ($\nu=0$, $Q=1/2$) oscillators; (ii) only for cubic soft oscillators is it larger than for linear oscillators; (iii) for oscillations with $\nu\leqslant-1$, i.e., oscillations near the bottom of the well of Figure 1(c), and for oscillations with $\nu\simeq-2$, i.e., for oscillations with $En\simeq0$ in the well of Figure 1(c), it is close to zero; (iv) for oscillations with $\nu\geqslant-1$, i.e., for oscillations with an energy near the top of the well of Figure 1(b), it is very large. These qualitative affirmations can be checked by numerically integrating equation (4.1) for the different cases. However a simple quantitative check is also possible if $Q(m)$ is quasi-constant in the integration interval $[0,\tau]$, say $Q(m)\simeq\langle Q\rangle$, where $\langle Q\rangle$ is a constant. Then an approximate integration of equation (4.2) gives

$$A(t)=A_0\exp\left(-2\langle Q\rangle t\varepsilon/2\right): \tag{4.5}$$

i.e., the amplitude decay has an exponential form with the exponent equal to that corresponding to a linear oscillator, $\varepsilon/2$, modified by a factor $2\langle Q\rangle$ that depends on the non-linearity of the oscillations. In Tables 1–3 the amplitudes of three example oscillators evaluated numerically by means of a fourth order Runge–Kutta method are given. The exponential fit to the data of Table 1 gives a curve $A(t)=a\exp(bt)$ with $a=1{\cdot}360$, $b=-0{\cdot}986\times10^{-3}$, and with standard errors in a and b given by $\sigma_a\simeq1{\cdot}7\times10^{-4}$, $\sigma_b\simeq6{\cdot}6\times10^{-6}$. The approximate curve given by expression (4.5) is in excellent agreement with the above results because $A_0=1{\cdot}360$ and $-\varepsilon\langle Q\rangle=-1{\cdot}0\times10^{-3}$, where the value

TABLE 1

Oscillation amplitude $A(t)$ of the soft-hard cubic oscillator $(En \leqslant 0)\ddot{x} - x + x^3 + 0{\cdot}01\dot{x} = 0$, with initial conditions $x(0) = 1{\cdot}360$ and $\dot{x}(0) = 0$

t	0·000	5·510	10·923	16·252	21·510	26·706	31·845	36·935	41·981
$A(t)$	1·360	1·352	1·345	1·338	1·331	1·324	1·317	1·311	1·305

TABLE 2

Oscillation amplitude $A(t)$ of the soft-hard cubic oscillator $(En \geqslant 0)\ddot{x} - x + x^3 - 0{\cdot}005\dot{x} = 0$ with initial conditions $x(0) = 1{\cdot}580$ and $\dot{x}(0) = 0$

t	0·000	3·809	7·576	11·302	14·988	18·636	22·248	25·823	29·362	32·868	36·340
$A(t)$	1·580	1·586	1·592	1·598	1·604	1·610	1·616	1·622	1·628	1·634	1·641

TABLE 3

Oscillation amplitude $A(t)$ of the soft cubic oscillator $\ddot{x} + x - x^3 + 0{\cdot}001\dot{x} = 0$ with initial conditions $x(0) = 0{\cdot}900$ and $\dot{x}(0) = 0$

t	0·000	5·277	10·493	15·652	20·759	25·819	30·835	35·809	40·744
$A(t)$	0·900	0·895	0·890	0·885	0·881	0·877	0·872	0·868	0·864

$\langle Q\rangle = 0{\cdot}10$ used was obtained as follows. We evaluate the intermediate non-linearity factor $\nu^* = [\nu(0) + \nu(\tau)]/2$, and simply set $Q(\nu^*) = \langle Q\rangle$; as $\nu(t=0) \simeq -1{\cdot}85$, $\nu(t = 41{\cdot}981) \simeq -1{\cdot}70$, then $\nu^* \simeq -1{\cdot}78$ and $\langle Q\rangle = Q(-1{\cdot}78) \simeq 0{\cdot}10$. Despite the simplicity of the calculation, the results are good. For the data of Table 2 the exponential fit gave $a = 1{\cdot}580$, $b = 1{\cdot}033 \times 10^{-3}$, $\sigma_a \simeq 1{\cdot}5 \times 10^{-4}$ and $\sigma_b \simeq 6{\cdot}8 \times 10^{-6}$. As $\nu(t=0) \simeq -2{\cdot}50$, $\nu(t = 36{\cdot}34) \simeq -2{\cdot}69$, then $\nu^* = -2{\cdot}60$, and as $\langle Q\rangle = Q(\nu^*) \simeq 0{\cdot}21$, one finds for the approximate amplitude $A_0 = 1{\cdot}580$ with $-\varepsilon\langle Q\rangle = 1{\cdot}05 \times 10^{-3}$, in good agreement with the exponential fit. For the data of Table 3, the exponential fit gave $a = 0{\cdot}900$, $b = -0{\cdot}999 \times 10^{-3}$, $\sigma_a \simeq 2{\cdot}5 \times 10^{-4}$, $\sigma_b \simeq 1{\cdot}0 \times 10^{-5}$ and the coefficients of expression (4.5) are $A_0 = 0{\cdot}900$ and $-\varepsilon\langle Q\rangle = -1{\cdot}0 \times 10^{-3}$ because $\nu(t=0) \simeq -0{\cdot}81$, $\nu(t = 40{\cdot}799) \simeq -0{\cdot}75$, giving $\nu^* \simeq -0{\cdot}78$ and then $\langle Q\rangle = Q(\nu^*) \simeq 1{\cdot}0$. The agreement with the numerical fit is again good.

5. THE VAN DER POL CUBIC OSCILLATOR

The oscillator is

$$\ddot{x} + c_1 x + c_3 x^3 = \varepsilon(\alpha - \beta x^2)\dot{x}. \tag{5.1}$$

By using equations (3.11) one obtains

$$\dot{A}/A = \varepsilon[\alpha\langle \mathrm{sn}^2\,\mathrm{dn}^2\rangle - \beta\langle \mathrm{sn}^2\,\mathrm{cn}^2\,\mathrm{dn}^2\rangle A^2] = \varepsilon(\tilde{\alpha} - \tilde{\beta}A^2) = \varepsilon\tilde{\alpha}(1 - A^2/A_c^2), \tag{5.2a}$$

$$\dot{\phi} = -(\varepsilon/A\omega)2m(1-2m)[\alpha\langle \mathrm{sn}\,\mathrm{dn}\,\mathrm{cn}_m\rangle - \beta\langle \mathrm{cn}^2\,\mathrm{sn}\,\mathrm{dn}\,\mathrm{cn}_m\rangle A^2], \tag{5.2b}$$

with [16]

$$\langle \mathrm{sn}^2\,\mathrm{cn}^2\,\mathrm{dn}^2\rangle \equiv R = (1/15m^2)[m_1(m-2)K + 2(m^2 + m_1)E]/K,$$

$\langle \mathrm{sn}^2\, \mathrm{dn}^2 \rangle \equiv Q$, $m_1 = 1 - m$, $\tilde{\alpha} = \alpha Q$, $\tilde{\beta} = \beta R$ and $A_c^2(m) = \tilde{\alpha}/\tilde{\beta}$. The function $R(\nu)$ is plotted in Figure 6.

Notice that as $\dot{A} = 0$ when $A^2 = A_c^2(A)$, then for these amplitudes the oscillator (5.1) has a limit cycle.

5.1. QUASILINEAR OSCILLATOR

In this simple case $c_3 = 0$, and then $m = 0$ and $\omega^2 = c_1$ for all amplitudes. As $Q(m=0) = 1/2$ and $R(m=0) = 1/8$, one has $\tilde{\alpha} = \alpha/2$, $\tilde{\beta} = \beta/8$ and $A_c^2 = 4\alpha/\beta$. Integrating equation (5.2b), one finds $\phi(t) = \phi(0) = \phi_0$. Integrating equation (5.2a) one obtains

$$A(t) = A_c A_0 \exp(\varepsilon \tilde{\alpha} t)/\{A_c^2 + A_0^2[\exp(2\tilde{\alpha}\varepsilon t) - 1]\}^{1/2}. \tag{5.3}$$

The solution (3.1) is then given by $x(t) = A(t)\cos(\omega_0 t - \phi_0)$, where $\omega_0^2 = c_1$ and A_0 and ϕ_0 are obtained from the initial conditions. This is the well known solution given by the normal Krylov-Bogoliubov method [13, 14].

5.2. QUASI-PURE-CUBIC OSCILLATOR

In this oscillator $c_1 = 0$, and therefore $m = 1/2$ and $\omega^2 = c_3 A^2$ for all amplitudes. Integrating equation (5.2b) one has $\phi(t) = \phi(0) \equiv \phi_0$. Integrating equation (5.2a), one finds again the expression (5.3) for the amplitude but not with $\tilde{\alpha} = \alpha Q(m = 1/2) = \alpha/3$ and $\tilde{\beta} = \beta R(m = 1/2) = \beta[2(E/K) - 1]/5 = 0{\cdot}091389\beta$. Then $A_c^2 = \alpha/\beta = 3{\cdot}6474\alpha/\beta$. The argument of the elliptic function of equation (3.1) is obtained by integrating $\omega(t)$:

$$\Omega(t) \equiv \int_0^t \omega(s)\,\mathrm{d}s = \frac{1}{\varepsilon}(c_3/\tilde{\alpha}\tilde{\beta})^{1/2} \ln\left\{\frac{A_0 \exp(\varepsilon\tilde{\alpha}t)[1 + A_c/A(t)]}{A_0 + A_c}\right\}. \tag{5.4}$$

The approximate solution is given by

$$x(t) = A(t)\,\mathrm{cn}\,(\Omega(t) - \phi_0, 1/2), \tag{5.5}$$

where $A(t)$ is given by equation (5.3) and $\Omega(t)$ by equation (5.4) with $\tilde{\alpha} = \alpha/3$ and $A_c = 1{\cdot}9098\sqrt{\alpha/\beta}$. Notice that, from equations (5.3) and (5.4), if $\varepsilon > 0$ and $t \to \infty$ one has

$$x(t) \to A_c\,\mathrm{cn}\,(\sqrt{c_3}A_c t - \phi_c, 1/2): \tag{5.6}$$

i.e., the motion tends to a limit cycle given by expression (5.6) with amplitude A_c, period $T = 4\mathrm{K}(1/2)/\sqrt{c_3}A_c) = 3{\cdot}8833/\sqrt{c_3}$ and ϕ_c a constant given by

$$\phi_c = (1/\varepsilon)(c_3/\tilde{\alpha}\tilde{\beta})^{1/2} \ln[2A_0/(A_0 + A_c)] - \phi_0.$$

In Figures 7-9 are plotted the approximate solution given by (5.5) and the fourth order Runge-Kutta numerical solution. The results are excellent even for large ε. The approximate solution is, of course, better for smaller ε and larger frequency.

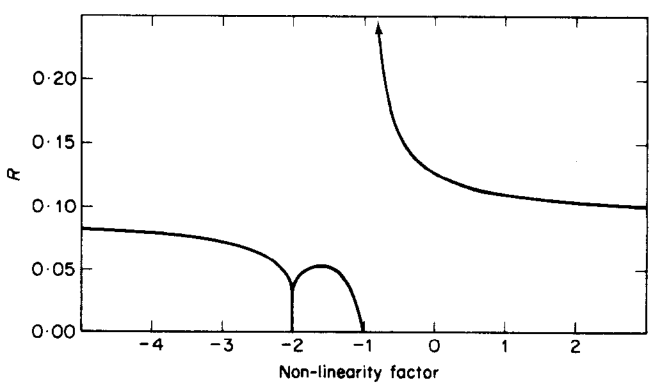

Figure 6. The $R = \langle \mathrm{sn}^2\, \mathrm{cn}^2\, \mathrm{dn}^2 \rangle$ function versus the non-linearity factor.

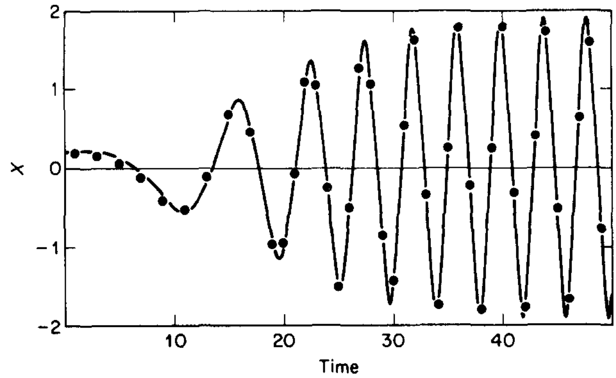

Figure 7. Approximate (solid line) and numerical (●) solution of the van der Pol pure cubic oscillator $\ddot{x}+x^3=0{\cdot}3(1-x^2)\dot{x}$ with initial conditions $x(0)=0{\cdot}2$ and $\dot{x}(0)=0$. The approximate solution is obtained by using the formula (5.5). The numerical solution is obtained by using a Runge-Kutta method of fourth order.

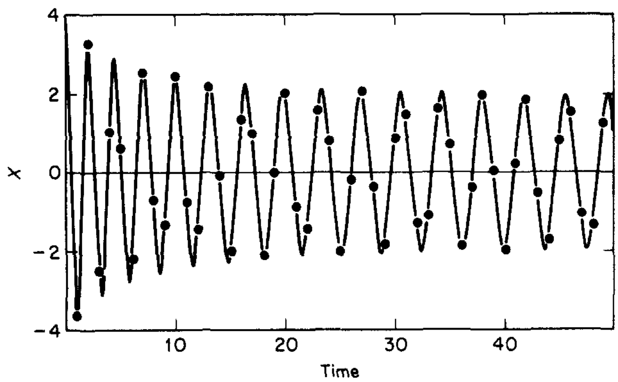

Figure 8. Approximate (solid line) and numerical (●) solution of the van der Pol pure cubic oscillator $\ddot{x}+x^3=0{\cdot}1(1-x^2)\dot{x}$ with $x(0)=4$ and $\dot{x}(0)=0$. These solutions are obtained as indicated in the caption to Figure 7.

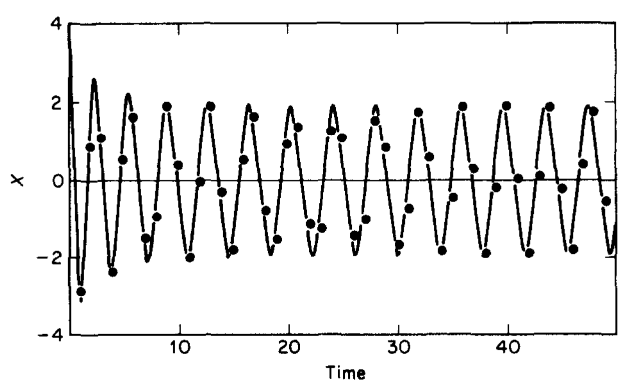

Figure 9. Approximate (solid line) and numerical (●) solution of the van der Pol pure cubic oscillator $\ddot{x}+x^3=0{\cdot}3(1-x^2)\dot{x}$ with $x(0)=4$ and $\dot{x}(0)=0$. These solutions are obtained as indicated in the caption to Figure 7.

TABLE 4

Oscillation amplitude for the soft-hard cubic oscillator $(En > 0)\ddot{x} - x + x^3 = 0{\cdot}03(1 - x^2)\dot{x}$ *with* $x(0) = 1{\cdot}58$ *and* $\dot{x}(0) = 0$, $A_n(t)$ *evaluated by numerical integration, and* $A_a(t)$ *by using the approximate formula* (5.5)

t	0·000	3·804	7·558	11·262	14·923	18·541	22·118	25·658	29·162	32·631	36·068	39·475
$A_n(t)$	1·580	1·587	1·595	1·602	1·609	1·615	1·622	1·629	1·635	1·641	1·647	1·653
$A_a(t)$	1·580	1·588	1·596	1·603	1·611	1·617	1·624	1·630	1·636	1·642	1·648	1·653

5.3. OSCILLATOR (5.1) WITH NON-ZERO c_1 AND c_3.

Now integration of the system of equations (5.2) is not easy because $\tilde{\alpha}$, $\tilde{\beta}$, $\langle \mathrm{sn\,dn\,cn}_m \rangle$ and $\langle \mathrm{cn}^2\,\mathrm{sn\,dn\,cn}_m \rangle$ are complicated functions of the amplitude. However, valuable information about the behaviour of the oscillations can be obtained without carrying out these integrations. For example, equation (5.2a) serves (i) to determine whether there exist limit cycles or limit points, (ii) to evaluate the amplitude of these limit cycles and (iii) to determine the stability of these limit cycles and/or limit point. This task will not be carried out in this paper (however, it is of interest to note that the results for the limit cycles agree with those obtained in reference [19] by using a method of harmonic balance). Instead we will now discuss the goodness of expression (5.3) for evaluating the amplitude for cubic oscillators with c_1 and c_3 non-zero. If in the interval of integration $[0, \tau]$, $Q(\nu)$ and $R(\nu)$ are quasi-constant, say $Q(\nu) = \langle Q \rangle$ and $R(\nu) = \langle R \rangle$ with $\langle Q \rangle$ and $\langle R \rangle$ constants, then the integration of equation (5.2a) can be approximated by equation (5.3), where $\tilde{\alpha} = \alpha\langle Q \rangle$ and $\tilde{\beta} = \beta\langle R \rangle$. We have checked this expression by comparing the values of the amplitude that it gives with the amplitudes obtained by numerical integration (with a fourth order Runge–Kutta method). The constants $\langle Q \rangle$ and $\langle R \rangle$ are obtained as in section 4: $\langle Q \rangle = Q(\nu^*)$ and $\langle R \rangle = R(\nu^*)$. For obtaining $A_a(t)$ in Table 4 we took $\nu^* \simeq -2{\cdot}62$ and thus used $\langle Q \rangle = Q(-2{\cdot}62) \simeq 0{\cdot}21$ and $\langle R \rangle = R(-2{\cdot}62) \simeq 0{\cdot}065$ in the approximate formula (5.6). In Table, $\nu^* \simeq -1{\cdot}78$ and thus $\langle Q \rangle \simeq 0{\cdot}10$ and $\langle R \rangle \simeq 0{\cdot}051$. In Table 6, $\nu^* \simeq -0{\cdot}78$ and thus $\langle Q \rangle \simeq 1{\cdot}0$ and $\langle R \rangle \simeq 0{\cdot}22$. As was expected the results are better for small ε.

5. CONCLUSIONS

In this paper we have described an improved version of a Krylov–Bogoliubov elliptic method (given in reference [12]) designed to solve non-linear oscillator equations of the class given by equation (1.1). We have obtained compact general expressions, equations (3.11), for the time derivatives of the amplitude and phase similar to those equations (1.2), obtained in the usual Krylov–Bogoliubov method. These expressions are especially simple when the oscillator is quasilinear or quasi-pure-cubic since the elliptic parameter is not time dependent and the expressions are simpler than for the general cubic oscillator. For quasilinear oscillators the elliptic parameter is constant (equal to zero) and the expressions become the usual ones of the usual Krylov–Bogoliubov method. Finally, we have obtained simple accurate approximate solutions for two examples of quasi-pure-cubic oscillators: a linear damped oscillator and a van der Pol oscillator. Also we have shown by means of these examples that very useful information (the influence of the non-linearity of the oscillations on the amplitude variation rate, existence and stability of the limit cycles and/or limit points) can be obtained from the key relationships (3.11), specifically from (3.11a).

TABLE 5

Oscillation amplitude for the soft-hard cubic oscillator ($En < 0$) $\ddot{x} - x + x^3 = 0{\cdot}1(1 - x^2)\dot{x}$ *with* $x(0) = 1{\cdot}30$ *and* $\dot{x}(0) = 0$, $A_n(t)$ *evaluated by numerical integration, and* $A_a(t)$ *by using the approximate formula* (5.5)

t	0·000	5·011	10·050	15·120	20·226	25·373	30·566	35·814	41·127	46·518	52·006	57·622
$A_n(t)$	1·300	1·304	1·309	1·314	1·319	1·324	1·330	1·336	1·343	1·351	1·359	1·368
$A_a(t)$	1·300	1·310	1·319	1·327	1·335	1·342	1·349	1·355	1·360	1·367	1·370	1·375

TABLE 6

Oscillation amplitude for the soft cubic oscillator $\ddot{x} + x - x^3 = 0{\cdot}001(1 - x^2)\dot{x}$ *with* $x(0) = 0{\cdot}86$ *and* $\dot{x}(0) = 0$, $A_n(t)$ *evaluated by numerical integration, and* $A_a(t)$ *by using the approximate formula* (5.5)

t	0·000	4·891	9·812	14·764	24·768	29·824	34·920	40·056	45·238	50·466	55·746	61·082
$A_n(t)$	0·860	0·863	0·866	0·870	0·877	0·880	0·884	0·888	0·892	0·896	0·900	0·904
$A_a(t)$	0·860	0·864	0·867	0·871	0·878	0·882	0·885	0·889	0·893	0·897	0·901	0·905

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