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ON DUFFING OSCILLATORS WITH SLOWLY VARYING PARAMETERS

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Abstract—A method of Krylov–Bogoliubov type, which gives the approximate solution in terms of Jacobi elliptic functions, is used for the study of perturbed Duffing oscillators with slowly varying parameters: $\frac{\mathrm{d}}{\mathrm{d}t}[\mu(\tau)\dot{x}] + c_1(\tau)x + c_3(\tau)x^3 + \varepsilon f(x, \dot{x}, \tau) = 0$. This method is a natural generalization of the usual Krylov–Bogoliubov method that is only valid when $c_3(\tau)x^3$ is of ε order. Two examples are given. One is a pure cubic oscillator ($c_1 = 0$) with variable mass and linear damping, $f(x, \dot{x}) = \dot{x}$, for which a simple accurate approximate solution is found. The other is a pendulum with variable length and damping proportional to the velocity for which an approximate analytical expression for the rate of variation of the oscillation amplitude is obtained, and successfully compared with the numerically calculated result and with that obtained using the normal Krylov–Bogoliubov method.

1. INTRODUCTION

It is well known that most methods of obtaining approximate solutions of non-linear oscillators are only applicable to weak cases: $\ddot{x} + c_1 x + \varepsilon f(x, \dot{x}) = 0$, where $c_1 > 0$ and ε is a small parameter. Much effort has therefore been put into extending these methods to other interesting and more general classes of oscillators. References can be found in [1]. The author and colleagues have constructed some methods to obtain approximate solutions in terms of Jacobi elliptic functions for the class of perturbed Duffing oscillators (strongly non-linear oscillators)

$$\ddot{x} + c_1 x + c_3 x^3 + \varepsilon f(x, \dot{x}) = 0, \tag{1.1}$$

where c_1 and c_3 are *arbitrary* and ε is a small parameter. These methods are a method of harmonic balance [2, 3], a Galerkin method [4], a weighted mean–square method of "cubication" [5] and a Krylov–Bogoliubov (KB) method [2, 6, 7, 8]. The use of elliptic functions to solve many problems of non-linear oscillations approximately or exactly is well documented in, for example, references [1, 9, 10]. Recently Coppola and Rand [11] have implemented another method of KB type. This method, which we will call the elliptic KB (EKB) method, is completely equivalent to that expounded by Bravo Yuste and Díaz Bejarano [8] when $c_1 = 0$ or $c_3 = 0$. When $c_1 \neq 0$ and $c_3 \neq 0$ both methods lead to the same equation for the oscillation amplitude. However the method of Coppola and Rand is preferable because the phase equation is suitable for the averaging procedure even for $c_1 \neq 0$ and $c_3 \neq 0$.

In this paper we shall use a generalization of the EKB method [12] that makes it applicable to perturbed Duffing oscillators with *slowly varying parameters*

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mu(\tau)\dot{x}] + c_1(\tau)x + c_3(\tau)x^3 + \varepsilon f(x, \dot{x}, \tau) = 0, \tag{1.2}$$

where $\mu(\tau)$, $c_1(\tau)$, $c_3(\tau)$ are the slowly varying parameters, ε is a small constant parameter and $\tau = \varepsilon t$ is the "slow time". Two oscillators will be studied: one is a pure cubic oscillator, $c_1 = 0$, with variable mass and linear damping, $f(x, \dot{x}) = \dot{x}$, and the other is a pendulum with variable length and damping proportional to the velocity.

Contributed by A. H. Nayfeh.

2. THE EKB METHOD FOR DUFFING OSCILLATORS WITH SLOWLY VARYING PARAMETERS

For ease of reference we shall give here the principal expressions of the EKB method for oscillators with slowly varying parameters [12]. The proposed solution of the EKB method for the oscillator (1.2) is given by

$$x(t) = A\,\mathrm{cn}\left(\int_0^t \omega(s)\,\mathrm{d}s - \phi, m\right) = A\,\mathrm{cn}(\psi, m) = A\,\mathrm{cn}(4K\varphi, m), \tag{2.1}$$

with $\dot{x}(t) = -\omega A\,\mathrm{sn}(\psi, m)\,\mathrm{dn}(\psi, m)$ and where the frequency, ω, and modulus, m, are

$$\mu\omega^2 = c_1 + c_3 A^2 = c_1(1+\nu), \tag{2.2}$$

$$m = c_3 A^2/[2(c_1 + c_3 A^2)] = \nu/[2(1+\nu)], \tag{2.3}$$

with ν, the non-linearity factor, given by

$$\nu = c_3 A^2/c_1. \tag{2.4}$$

The functions $A(t)$ and $\varphi(t)$ are the solutions of

$$\dot{A} = -\frac{\dot{\mu}A}{2\mu}\langle \mathrm{sn}^2\,\mathrm{dn}^2\rangle - \frac{\dot{c}_1 A}{2\omega^2\mu}\langle \mathrm{sn}^2\rangle - \frac{\dot{c}_3 A^3}{2\omega^2\mu}\langle \mathrm{sn}^2 - \mathrm{sn}^4/2\rangle + \frac{\varepsilon}{\omega\mu}\langle f\,\mathrm{sn}\,\mathrm{dn}\rangle, \tag{2.5a}$$

$$\dot{\varphi} = \left\langle \frac{\omega}{4K} - \frac{\dot{A}\,\mathrm{cn}}{4KA\,\mathrm{cn}_\psi} - \frac{\varphi K_m \dot{m}}{K} - \frac{\mathrm{cn}_m \dot{m}}{4K\,\mathrm{cn}_\psi}\right\rangle, \tag{2.5b}$$

where $f = f(A\,\mathrm{cn}\,\psi, -A\,\omega\mathrm{sn}\,\psi\,\mathrm{dn}\,\psi, \tau)$ and

$$\langle \ldots \rangle \equiv \frac{1}{4K}\int_0^{4K} \ldots \,\mathrm{d}\psi, \tag{2.6}$$

is the operation of averaging over the period $4K$ of the elliptic functions that appear in equations (2.5). We use the notation $F_\alpha(\alpha, \beta) = \partial F(\alpha, \beta)/\partial\alpha$. From the transformation properties of the Jacobi elliptic functions with respect to their modulus [13], one can deduce (see Appendix) that the value of K must be given by

$$\begin{aligned} &K = K(m) \quad \text{for} \quad 0 \leqslant m \leqslant 1, \\ &K = (1-m)^{-1/2} K(-m/(1-m)) \quad \text{for} \quad m < 0, \\ &K = \tfrac{1}{2} m^{-1/2} K(1/m) \quad \text{for} \quad m > 1, \end{aligned} \tag{2.7}$$

where $K(z)$ is the complete elliptic integral of the first kind of modulus z. Equation (2.5b) for $\dot{\varphi}$ could be written in a more explicit form as in reference [12], but we shall not give it here because it is very complex and not useful for the present work.

When the oscillator is quasilinear, i.e. when $c_3 = 0$ and therefore $m = 0$ (and $\mu\omega^2 = c_1$) the system (2.5) becomes especially simple

$$\begin{aligned} \dot{A} &= -\frac{1}{2}\frac{A}{\mu\omega}\frac{\mathrm{d}}{\mathrm{d}t}(\mu\omega) + \frac{\varepsilon}{\mu\omega}\frac{1}{2\pi}\int_0^{2\pi} f(A\cos\psi, -A\omega\sin\psi, \tau)\sin\mathrm{d}\psi, \\ \dot{\phi} &= -\frac{\varepsilon}{\mu A\omega}\frac{1}{2\pi}\int_0^{2\pi} f(A\cos\psi, -A\omega\sin\psi, \tau)\cos\psi\,\mathrm{d}\psi, \end{aligned} \tag{2.8}$$

since $\mathrm{cn}(\psi, 0) = \cos\psi$, $\mathrm{sn}(\psi, 0) = \sin\psi$, $\mathrm{dn}(\psi, 0) = 1$, $K(0) = \pi/2$, $\langle \sin^2\rangle = 1/2$ and $\langle \sin\cos\rangle = 0$. These are well-known relations in the normal KB method for oscillators with slowly varying parameters [14] but here obtained as a particular case of the general expressions (2.5) of the EKB method. When the oscillator is quasi-pure-cubic, i.e. when $c_1 = 0$ and therefore $m = 1/2$ (and $\mu\omega^2 = c_3 A^2$), the system is also simple

$$\begin{aligned} \dot{A} &= -\frac{A}{6\mu c_3}\frac{\mathrm{d}}{\mathrm{d}t}(\mu c_3) + \frac{\varepsilon}{\mu\omega}\frac{1}{4K}\int_0^{4K} f(A\mathrm{cn}, -A\omega\mathrm{sn}\,\mathrm{dn}, \tau)\,\mathrm{sn}\,\mathrm{dn}\,\mathrm{d}\psi, \\ \dot{\phi} &= -\frac{\varepsilon}{\mu A\omega}\frac{1}{4K}\int_0^{4K} f(A\mathrm{cn}, -A\omega\mathrm{sn}\,\mathrm{dn}, \tau)\,\mathrm{cn}\,\mathrm{d}\psi, \end{aligned} \tag{2.9}$$

with $\mathrm{cn} = \mathrm{cn}(\psi, 1/2)$, $\mathrm{sn} = \mathrm{sn}(\psi, 1/2)$, $\mathrm{dn} = \mathrm{dn}(\psi, 1/2)$ and $K = K(1/2) = 1.85407..$ In these last two cases we have given the averaged expressions in terms of the more usual phase ϕ.

In the next two sections, we will study two perturbed Duffing oscillators with slowly varying parameters using the elliptic KB method.

3. A LINEAR DAMPED QUASI-PURE-CUBIC OSCILLATOR WITH VARIABLE MASS

The equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mu(\tau)\dot{x}] + c_3x^3 + \varepsilon\dot{x} = 0, \tag{3.1}$$

i.e. equation (1.2) with $c_1 = 0$, c_3 constant and $f(x, \dot{x}) = \dot{x}$. For this oscillator the system (2.5) is

$$\frac{\dot{A}}{A} = -\frac{1}{6\mu}\frac{\mathrm{d}\mu}{\mathrm{d}t} - \frac{\varepsilon}{3\mu}, \quad \dot{\phi} = 0, \tag{3.2a, b}$$

since $\langle \mathrm{sn}^2\,\mathrm{dn}^2 \rangle = 1/3$ (for $m = 1/2$) and $\langle \mathrm{sn}\,\mathrm{dn}\,\mathrm{cn} \rangle = 0$. Integrating these equations we easily find that $\phi(t) = \phi(0) \equiv \phi_0$ and

$$A(t) = A_0\left(\frac{\mu}{\mu_0}\right)^{1/6} \exp\left(-\frac{\varepsilon}{3}\int_0^t \frac{1}{\mu}\,\mathrm{d}t\right). \tag{3.3}$$

We use the convention that an expression with a subscript zero represents its value at the initial time $t = 0$. The approximate solution is

$$x(t) = A(t)\,\mathrm{cn}(\Omega(t) - \phi_0, 1/2), \tag{3.4}$$

with $\Omega(t) = \int_0^t \omega(s)\,\mathrm{d}s$ and $\mu\omega^2 = c_3A^2$. To give an explicit expression for $A(t)$ and $\Omega(t)$ it is necessary to know the mass variation law. We shall assume that the law is linear $\mu = \mu_0 + \mu_1\tau = \mu_0 + \varepsilon\mu_1 t$, where μ_1 is a constant. Then, from (3.3), we find

$$A(t) = A_0\left(\frac{\mu_0}{\mu_0 + \varepsilon\mu_1 t}\right)^{\beta}, \tag{3.5}$$

where $\beta = [(2 + \mu_1)/6\mu_1]$. Also

$$\Omega(t) \equiv \int_0^t \omega(s)\,\mathrm{d}s = \sqrt{c_3}\,A_0\mu_0^{\beta}\int_0^t (\mu_0 + \varepsilon\mu_1 t)^{-\beta-1/2}\,\mathrm{d}t. \tag{3.6}$$

If $\mu_1 = 1$, the approximate solution is given by (3.4) where

$$A(t) = A_0\left(\frac{\mu_0}{\mu_0 + \varepsilon t}\right)^{1/2}, \tag{3.7}$$

and

$$\Omega(t) = \frac{1}{\varepsilon}\sqrt{c_3\mu_0}\,A_0 \ln[(\mu_0 + \varepsilon t)/\mu_0]. \tag{3.8}$$

If $\mu_1 \neq 1$, the approximate solution is also given by (3.4) but now with the amplitude A given by (3.5) and

$$\Omega(t) = \frac{3}{\varepsilon(\mu_1 - 1)}\sqrt{c_3}\,(A\sqrt{\mu} - A_0\sqrt{\mu_0}). \tag{3.9}$$

Figures 1–3 show plots of the approximate solution given by the above expressions and the numerical solution obtained using a fourth order Runge–Kutta method for different oscillators. In all cases the results are very good. Figure 1 is for an oscillator with $\mu_1 = 1$, Fig. 2 with $\mu_1 \neq 1$ ($\mu_1 = 0.2$), and Fig. 3 with $\mu_1 = -2$. This last case shows that the oscillation amplitudes do not change when the ratio between the rate of variation of the mass and the damping coefficient is $\dot{\mu}(t)/\varepsilon = -2$, in agreement with equation (3.2a).

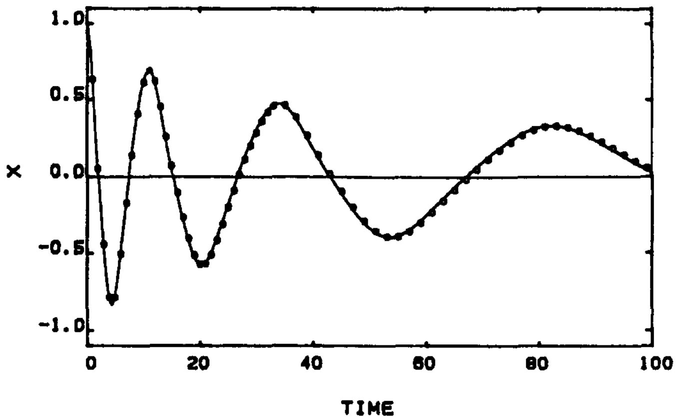

Fig. 1. Approximate (solid line) and numerical (o) solution of the linear damped quasi-pure-cubic oscillator with variable mass $\frac{d}{dt}[(1 + 0.1t)\dot{x}] + x^3 + 0.1\dot{x} = 0$, with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. The approximate solution is obtained using formulae (3.4), (3.7) and (3.8) since $\mu_1 = 1$. The numerical solution is obtained using a Runge–Kutta method of fourth order.

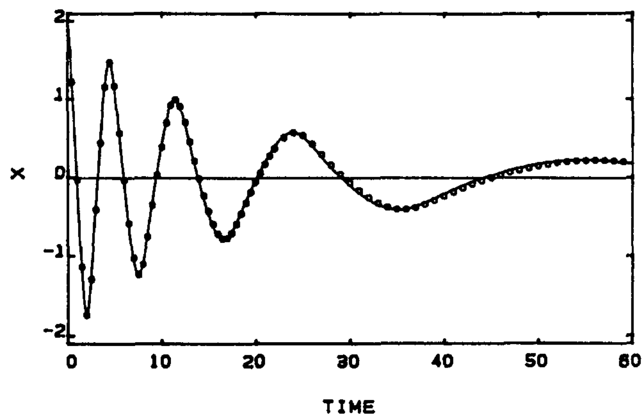

Fig. 2. Approximate (solid line) and numerical (o) solution of the linear damped quasi-pure-cubic oscillator with variable mass $\frac{d}{dt}[(1 + 0.04t)\dot{x}] + x^3 + 0.2\dot{x} = 0$, with initial conditions $x(0) = 2$ and $\dot{x}(0) = 0$. The approximate solution is obtained using formulae (3.4), (3.5), and (3.9) since $\mu_1 = 0.2 \neq 1$. The numerical solution is obtained using a Runge–Kutta method of fourth order.

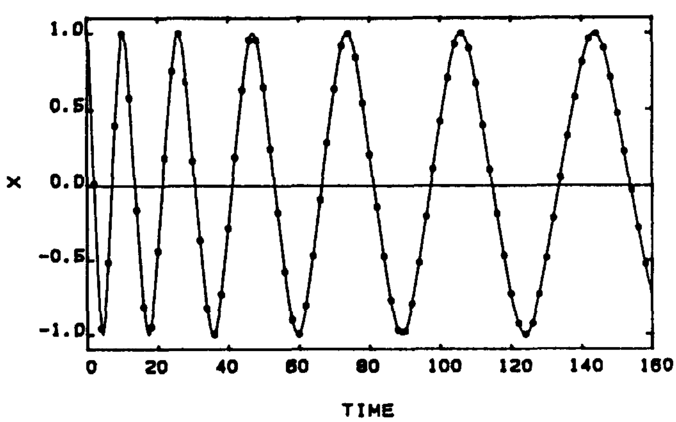

Fig. 3. Approximate (solid line) and numerical (o) solution of the linear damped quasi-pure-cubic oscillator with variable mass $\frac{d}{dt}[(1 + 0.2t)\dot{x}] + x^3 - 0.1\dot{x} = 0$, with initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$. The approximate solution is obtained using formulae (3.4), (3.5) and (3.9) since $\mu_1 = -2 \neq 1$. The oscillation amplitude is constant because $\mu_1 = -2$ implies $\beta = 0$. The numerical solution is obtained using a Runge–Kutta method of fourth order.

4. PENDULUM WITH LINEAR DAMPING AND VARIABLE LENGTH

The oscillator is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Ml^2(\tau)\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) + 2n\frac{\mathrm{d}}{\mathrm{d}t}[l(\tau)\theta] + mgl(\tau)\sin\theta = 0. \tag{4.1}$$

We follow closely the notation of reference [14]: θ is the angle of deviation of the pendulum from the vertical, g is the gravitational acceleration, M is the mass of the pendulum, $l(\tau)$ is the slowly varying length and $2n$ is the coefficient of friction. For not too large oscillations we can approximate $\sin\theta$ by the first two terms of the power series expansion, and then equation (4.1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Ml^2(\tau)\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) + mgl(\tau)\theta - \frac{1}{6}mgl(\tau)\theta^3 + \varepsilon f(\theta, \dot{\theta}, \tau) = 0, \tag{4.2}$$

where

$$\varepsilon f(\theta, \dot{\theta}, \tau) = 2nl(\tau)\dot{\theta} + 2n\theta\dot{l}(\tau). \tag{4.3}$$

Notice that, unlike the usual methods, it is not necessary to assume that $mgl(\tau)\theta^3/6 = 0(\varepsilon)$: the present method works for large oscillations. The approximate solution is given by equation (2.1) where $A(t)$ and $\varphi(t)$ are determined by integrating equations (2.5). For quasilinear or quasi-pure-cubic oscillators (such as the oscillator of Section 3) the integration is not too difficult because the averaged expressions *do not depend* on m, i.e. on the amplitude. When c_1 and c_3 are non-zero, however, the integration is very difficult because the averaged expressions depend on the amplitude in a non-trivial way. This problem also arises in Duffing oscillators with constant parameters [6, 7, 8]. However, as in those references, useful information can be obtained from equations (2.5) directly. For example, an analytical expression for the dependence of the relative amplitude variation rate, $\dot{A}/A$, on the amplitude of the oscillations has been found. For the oscillator (4.2) equation (2.5a) has the form

$$\frac{\dot{A}}{A}\left(-\frac{n}{Ml(\tau)} - \frac{3}{4}\frac{\dot{l}(\tau)}{l(\tau)}A\right)^{-1} = \alpha, \tag{4.4}$$

with [15]

$$\alpha/2 = \alpha(\sigma^2)/2 = \langle \mathrm{sn}^2\,\mathrm{dn}^2\rangle = \left((\sigma^2+1)\frac{E(\sigma^2)}{K(\sigma^2)} + \sigma^2 - 1\right)\Big/[3\sigma^2(1-\sigma^2)], \tag{4.5}$$

and

$$\sigma^2 = \sigma^2(A) = A^2/(12 - A^2). \tag{4.6}$$

The function $E(\sigma^2)$ is the complete elliptic integral of the second kind. The relative amplitude variation rate $\dot{A}/A$ has the property of being independent of the amplitude in a linear oscillator with linear damping. This quantity is closely related to the useful magnitude $\ln(A_{n+1}/A_n)$, or logarithmic decrement [16, 17, 18], where A_n and A_{n+1} are the amplitudes of two successive oscillation maxima.

Equation (4.4) with $\alpha = 1$ agrees with the expression obtained in the usual KB method [14]. In other words: the relative amplitude variation rate given by the EKB method is equal to that obtained from the usual KB method (as well as other methods, such as the method of multiple scales [1]), multiplied by the factor $\alpha = 2\langle \mathrm{sn}^2\,\mathrm{dn}^2\rangle$. When $A \to 0$, then $\sigma^2 \to 0$ and $\alpha \to 1$, i.e for small amplitudes the EKB method and the usual methods agree. Figure 4 shows the excellent comparison between the analytical results and numerical calculations. The usual methods (for which $\alpha = 1$ for any amplitude) are only suitable for small oscillations.

Finally, some words about the procedure of finding $\dot{A}$ numerically. First, we calculate approximately the amplitudes A evaluating the maximum of the function $|x(t)|$ obtained by means a fourth order Runge–Kutta method. With these values of $A(t_i)$, we proceed to find the values of the derivative at each time t_i. It should be noted that the points of Fig. 4 are given for selected amplitudes in order to make the figure simple and illustrative; these selected amplitudes *are not* successive amplitudes $A(t_i)$, $A(t_{i+1})$ obtained in the numerical

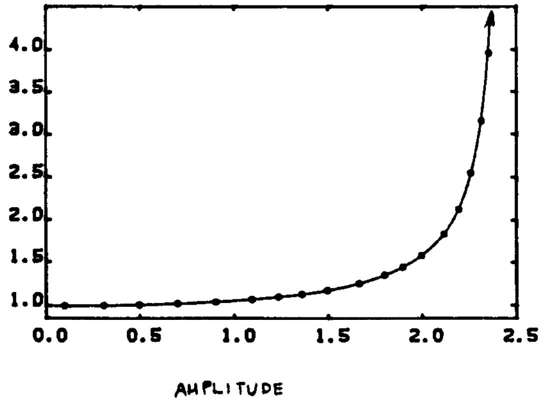

Fig. 4. The pendulum function $(\dot{A}/A)[-n/Ml(\tau) - 3\dot{l}(\tau)A/4l(\tau)]^{-1}$ vs the amplitude A (in radians), obtained numerically (o), and analytically, equation (4.4), by the EKB method (solid line). The numerical values are obtained by integrating the oscillators (4.1) with $M = 1$, $\varepsilon = 0.1$ and $l(t) = 0.1 + 0.001t$. Fourth order Runge–Kutta integration procedure for A and a Gregory–Newton interpolation technique for $\dot{A}$ were used in the numerical calculation. Different values of the damping coefficient, n, were used in order to calculate $\dot{A}$ with sufficient precision.

integration. In order to find the numerical derivative we have used the formulae of the Gregory–Newton interpolation procedure [18]. Clearly the estimates of the derivative $\dot{A}$ will be better when the differences between successive amplitudes $A(t_i)$, $A(t_{i+1})$ and times t_i, t_{i+1} are small. For large oscillations with amplitudes close to the limit oscillation amplitude $\theta_1 = \sqrt{6}$, this favorable situation only occurs when the friction coefficient n is small. So, for the four points closest to θ_1 in Fig. 4, the amplitudes were evaluated solving numerically the oscillator (4.1) with $M = 1$, $\varepsilon = 0.1$, $l(t) = 0.1 + 10^{-3}t$, and a value of the friction coefficient very small: $n = 10^{-10}$. For not so large amplitudes it is not necessary to use so small a value of n. So, to evaluate the first seven points on Fig. 4 we used $n = 10^{-2}$, for the following four points $n = 10^{-3}$ and for the following three points $n = 10^{-8}$.

5. CONCLUSIONS

In this paper we have used a version of a Krylov–Bogoliubov method designed to solve perturbed Duffing oscillators with slowly varying parameters, $\frac{d}{dt}[\mu(\tau)\dot{x}] + c_1(\tau)x + c_3(\tau)x^3 + \varepsilon f(x, \dot{x}, \tau) = 0$, where the approximate solution is given in terms of Jacobi elliptic functions. For quasilinear or quasi-pure-cubic oscillators the elliptic modulus of the proposed solution is not time-dependent and the expressions for the time derivative of the amplitude and phase are simpler than for the general oscillator. We thus obtained a simple and accurate approximate solution for a linear damped quasi-pure-cubic oscillator with variable mass. But if $c_1 \neq 0$ and $c_3 \neq 0$, the elliptic function modulus is in general time-dependent, and it is then not possible to integrate the EKB equations of the amplitude and phase. Even in these cases however, one can obtain useful information from these equations, especially from the amplitude equation. We have shown this in the present paper by the example of a pendulum of variable length and linear damping, deriving a very good expression for the influence of the oscillation amplitude on the amplitude variation rate.

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APPENDIX

In this Appendix we will show that if the period of the Jacobi elliptic functions of (2.5) is denoted by $4K$, then K must be given by (2.7). We shall use the $\mathrm{cn}(\psi, m)$ function (we would find the same results if we used other elliptic functions). It is well known [13] that, when $0 \leqslant m \leqslant 1$, the period of $\mathrm{cn}(\psi, m)$ is $4K(m)$, and then the justification of (2.7a) is obvious. The proof of (2.7b) starts from the following relation [13]

$$\mathrm{cn}(\psi, m) = \mathrm{cd}(\psi/\sigma_1, \sigma^2), \tag{A.1}$$

where $\sigma_1^2 = 1 - \sigma^2$, and $\sigma^2 = -m/(1-m)$. Notice that if m is negative then σ^2 lies between zero and one. If $4K$ is the period of $\mathrm{cn}(\psi, m)$, then $\mathrm{cn}(\psi + 4K, m) = \mathrm{cn}(\psi, m)$ or, equivalently, using (A.1), $\mathrm{cd}\left(\dfrac{\psi}{\sigma_1} + \dfrac{4K}{\sigma_1}, m\right) = \mathrm{cd}(\psi/\sigma_1, m)$. As the period of $\mathrm{cd}(\psi, \sigma^2)$ is $4K(\sigma^2)$ then $4K/\sigma_1 = 4K(\sigma^2)$, i.e. we find that K must be given by equation (2.7b) when $m < 0$. The proof of (2.7c) is similar. We start from the relation [13]

$$\mathrm{cn}(\psi, m) = \mathrm{dn}(\psi/\eta, \eta^2), \tag{A.2}$$

where $\eta^2 = 1/m$. Notice that if $m > 1$ then η^2 lies between zero and one. From (A.2) if $\mathrm{cn}(\psi + 4K, m) = \mathrm{cn}(\psi, m)$ then $\mathrm{dn}\left(\dfrac{\psi}{\eta} + \dfrac{4K}{\eta}, \eta^2\right) = \mathrm{dn}(\psi/\eta, \eta^2)$. But the period of $\mathrm{dn}(\psi, \eta^2)$ is $2K(\eta^2)$ so that $4K/\eta = 2K(\eta^2)$, i.e. we find that K must be given by equation (2.7c) when $m > 1$.