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Transport Properties of Driven Inelastic Maxwell Mixtures

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Abstract. A granular binary mixture driven by a stochastic bath with friction is studied from the inelastic Boltzmann kinetic equation for inelastic Maxwell models. First, we focus on homogeneous steady state solutions, reached by the system due to the presence of the thermostat that compensates for the energy lost in collisions. At a macroscopic level, the homogeneous steady state is fully characterized by the partial granular temperatures of both species, which are determined and compared against molecular dynamics simulations of inelastic hard spheres. The comparison between theory and simulations shows an excellent agreement. Second, we solve the kinetic equation close to steady states by means of the Chapman–Enskog method adapted to dissipative dynamics. We consider the first-order approximation (Navier–Stokes hydrodynamic order) and compute explicitly the diffusion transport coefficients. The results obtained here for diffusion for inelastic Maxwell models agree with those derived for inelastic hard spheres when non-Gaussian corrections to the zeroth-order solution are neglected.

INTRODUCTION

In the last years, the Kinetic Theory of molecular gases has been properly modified to describe granular matter under rapid flow conditions. The main new ingredient of the theory is that collisions among grains are inelastic [1, 2]. Usually, the modifications involve (a) the introduction of collisional dissipation through a new form of the Boltzmann collision operator (which also takes into account the inelasticity in the scattering rules) and (b) the inclusion of a term in the streaming part of the kinetic equation that accounts for the energy injected into the system by the external sources. The two aspects have been modeled differently. A simple but realistic model for (a) considers smooth hard spheres (inelastic hard spheres, IHS) where particles lose a fraction of their translational kinetic energy after instantaneous collisions. In this model, the inelasticity is characterized by a (positive) constant coefficient of normal restitution. However, the resulting Boltzmann equation turns out to be quite complex so that one needs to consider sometimes uncontrolled approximations. Hence, it is useful to consider other analytically tractable models where exact results can be offered. This is the case of inelastic Maxwell models (IMM) where the collision rate of colliding particles is independent of their relative velocity. For (b), the simplest model assumes no influence of the grains on the external energy sources, which act as thermostats. In particular, as in Ref. [3], a general approach is to model the thermostat as a stochastic bath with friction. This is a simple and flexible way of keeping the system fluidized.

IMM are a simplification of IHS: they share the same collision rules, but (in a complete analogy with the elastic case) the former is the result of replacing the collision frequency of the latter (which is proportional to the relative velocity of the interacting particles) by an effective, velocity independent collision frequency. Thanks to this simplification the moments of the Boltzmann collision operator can be determined without the knowledge of the velocity distribution function. This nice mathematical property opens up the possibility of obtaining exact results and this is one of the main reasons for which IMM has been extensively employed in the past for studying both homogeneous and non-homogeneous situations. For instance, IMM have been considered for obtaining exact properties of the velocity distribution function of both the homogeneous cooling state [4–7] and the uniform shear flow [8–10], as well as for deriving the Burnett transport coefficients [11], among many other interesting results.

As it is well known, an isolated granular gas has a monotonically decreasing-in-time kinetic energy. In real conditions, however, since the dynamics stops after few collisions per particle, an external energy input is needed to keep the grains in rapid flow conditions. Some years ago, a general way of injecting energy by means of a stochastic bath with friction was proposed [12]. Its generality relies on the fact that this thermostat results as the limiting behavior

of different ways of kicking the grains [13]. This kind of thermostat has been used more recently [3] to fluidize a granular binary mixture modeled as IHS. The Navier–Stokes hydrodynamic equations along with expressions of the relevant transport coefficients have been derived (in the so-called leading Sonine approximation) for states close to steady homogeneous states [3, 14]. The derivation of the hydrodynamic equations needs not only the approximations required for the free evolving case, but also other related to the time dependence of the distribution functions close to steady states. In fact, the transport coefficients not only depend on the steady values of the hydrodynamic fields but also on quantities characterizing the vicinity of the perturbed state with respect to the reference steady base state. This latter point is far from being a trivial point since it is an important aspect to take into account for obtaining the transport properties of the system.

In this work we aim at reconsidering the studies of Refs. [3, 13], namely, we consider a granular binary mixture driven by means of a stochastic bath with friction but modeled as IMM. As said before, the use of IMM allows us to get the exact forms of the transport coefficients and compare them with those previously obtained for IHS [3]. This comparison can be seen as a way of assessing the reliability of IMM as a model for describing granular flows in mixtures.

The paper is organized as follows. First, we consider steady homogeneous states and determine the partial temperatures in terms of the parameter space of the problem. Then, the model is solved by means of the Chapman–Enskog method up to first order in spatial gradients and the transport coefficients associated with the mass flux are explicitly computed. Finally, we end the paper with a brief description of the results and conclusions.

KINETIC DESCRIPTION AND HOMOGENEOUS STEADY STATES

Boltzmann kinetic equation for driven granular mixtures

We consider a granular binary mixture driven by a stochastic bath with friction. As for the case of IHS [3, 13, 14], the nonlinear Boltzmann equation for the one-particle distribution function $f_i(\mathbf{r}, \mathbf{v}, t)$ of species i ($i = 1, 2$) having a position \mathbf{r} and a velocity \mathbf{v} at time t reads

$$\partial_t f_i + \mathbf{v} \cdot \nabla f_i - \frac{\gamma_b}{m_i^\beta} \Delta \mathbf{U} \cdot \frac{\partial f_i}{\partial \mathbf{v}} - \frac{\gamma_b}{m_i^\beta} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{V} f_i - \frac{1}{2} \frac{\xi_b^2}{m_i^\lambda} \frac{\partial^2 f_i}{\partial v^2} = \sum_{j=1}^2 J_{ij}[\mathbf{v}|f_i, f_j], \quad (1)$$

where m_i is the mass of the species i , γ_b is the drag or friction constant, ξ_b^2 is related to the strength of the stochastic part of the bath, and β and λ are constants of the model. In addition, $\Delta \mathbf{U} = \mathbf{U} - \mathbf{U}_g$ is the mean flow velocity of the solid particles \mathbf{U} with respect to the mean flow velocity of the interstitial gas \mathbf{U}_g , and $\mathbf{V} = \mathbf{v} - \mathbf{U}$ is the peculiar velocity. The difference between IMM and IHS lies on the form of the Boltzmann collision operator $J_{ij}[f_i, f_j]$. In the case of IMM, this operator is given by [9]

$$J_{ij}[\mathbf{v}_1|f_i, f_j] = \frac{\omega_{ij}}{n_j \Omega_d} \int d\mathbf{v}_2 \int d\widehat{\sigma} \left[\alpha_{ij}^{-1} f_i(\mathbf{v}'_1) f_j(\mathbf{v}'_2) - f_i(\mathbf{v}_1) f_j(\mathbf{v}_2) \right]. \quad (2)$$

Here, n_i is the number density of species i , $\omega_{ij} \neq \omega_{ji}$ is an effective collision frequency (to be chosen later) for collisions of type i - j , $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the total solid angle in d dimensions, and $\alpha_{ij} = \alpha_{ji} \leq 1$ refers to the (positive) constant coefficient of restitution for collisions between particles of species i with j . Moreover, the primes on the velocities denote the initial values $\{\mathbf{v}'_1, \mathbf{v}'_2\}$ that lead to $\{\mathbf{v}_1, \mathbf{v}_2\}$ following a binary collision:

$$\mathbf{v}'_1 = \mathbf{v}_1 - \mu_{ji} (1 + \alpha_{ij}^{-1}) (\widehat{\sigma} \cdot \mathbf{g}_{12}) \widehat{\sigma}, \quad \mathbf{v}'_2 = \mathbf{v}_2 + \mu_{ij} (1 + \alpha_{ij}^{-1}) (\widehat{\sigma} \cdot \mathbf{g}_{12}) \widehat{\sigma}, \quad (3)$$

where $\mathbf{g}_{12} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity of the colliding pair, $\widehat{\sigma}$ is a unit vector directed along the centers of the two colliding spheres, and $\mu_{ij} = m_i/(m_i + m_j)$. The collision frequencies ω_{ij} can be seen as free parameters in the model. Its dependence on the coefficients of restitution α_{ij} can be chosen to optimize the agreement with the results obtained from the Boltzmann equation for IHS. Of course, the choice is not unique and may depend on the property of interest. In particular, if ω_{ij} is taken to get the same cooling rates ζ_i as that for IHS (evaluated by using the Gaussian approximation) for diameters σ_i , ω_{ij} is defined as

$$\omega_{ij} = \frac{\Omega_d}{\sqrt{\pi}} x_j \left(\frac{\sigma_{ij}}{\sigma_{12}} \right)^{d-1} \left(\frac{\theta_i + \theta_j}{\theta_i \theta_j} \right)^{1/2} \nu_0. \quad (4)$$

Here, $\sigma_{ij} = (\sigma_i + \sigma_j)/2$, $\theta_i = M_i/\chi_i$, and $\chi_i = T_i/T$ where T_i is the partial temperature of species i and $T = x_1 T_1 + x_2 T_2$, T is the global granular temperature. In addition, $M_i = m_i/\bar{m}$, $\bar{m} = m_1 m_2 / (m_1 + m_2)$, and ν_0 is an effective collision frequency given by

$$\nu_0 = n \sigma_{12}^{d-1} \sqrt{\frac{2T}{\bar{m}}}, \quad (5)$$

$n = n_1 + n_2$ being the total number density of particles in the system.

Homogeneous steady states

Before considering inhomogeneous states, it is worthwhile to study first the homogeneous steady state. In this situation, the partial densities n_i are constant, the granular temperature T is spatially uniform, and, with an appropriate selection of the frame of reference, the mean flow velocities vanish ($\mathbf{U} = \mathbf{U}_g = \mathbf{0}$). Thus, after a transient regime, the system is expected to reach a *steady* state and hence, Eq. (1) becomes

$$-\frac{\gamma_b}{m_i^\beta} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{v} f_i - \frac{1}{2} \frac{\xi_b^2}{m_i^\lambda} \frac{\partial^2 f_i}{\partial v^2} = \sum_{j=1}^2 J_{ij}[\mathbf{v}|f_i, f_j]. \quad (6)$$

For elastic collisions ($\alpha = 1$), as expected, a solution to Eq. (6) is given by Maxwellian distributions characterized by a common temperature T_b given by

$$T_b = \frac{\xi_b^2}{2\gamma_b(2\bar{m})^{\lambda-\beta-1}}. \quad (7)$$

Equation (7) defines a ‘‘bath temperature’’ T_b . Its name may be justified since it is determined by the two thermostat parameters and, additionally, it can be considered as a remnant of the temperature of the interstitial molecular fluid. For inelastic collisions ($\alpha \neq 1$), the total kinetic energy is not conserved and to date an exact solution to (6) is not known. However, an indirect information on the velocity distributions f_i are provided by their kurtosis or fourth cumulant $a_2^{(i)}$. These quantities measure the deviation of f_i from its Maxwellian form

$$f_{i,M}(\mathbf{v}) = n_i \left(\frac{m_i}{2T_i} \right)^{d/2} \exp\left(-\frac{m_i v^2}{2T_i}\right). \quad (8)$$

A careful study on the fourth cumulants $a_2^{(i)}$ have been carried out in Ref. [13] for IHS. The results show that in general the magnitude of those cumulants is very small and hence, a good estimate for the velocity moments (such as the partial temperatures T_i) of the distributions f_i can be obtained by considering the Maxwellian distributions (8) at different temperatures.

In the steady state, one of the most relevant quantities are the partial temperatures

$$T_i = \frac{m_i}{dn} \int d\mathbf{v} V^2 f_i(\mathbf{v}). \quad (9)$$

The equations defining the (reduced) partial temperatures $T_i^* = T_i/T_b$ can be obtained by multiplying both sides of Eqs. (6) by $m_i v^2$ and integrating over velocity. After some algebra, one gets [3]

$$T^* \left[1 - (M_i/2)^{\lambda-1-\beta} T_i^* \right] \xi^* = M_i^{\lambda-1} \zeta_i^* T_i^*, \quad i = 1, 2 \quad (10)$$

where $T^* = T/T_b = x_1 T_1^* + x_2 T_2^*$, $\xi^* = \xi_b^2 / (\nu_0 T \bar{m}^{\lambda-1})$, and $\zeta_i^* = \zeta_i / \nu_0$. Here,

$$\zeta_i = -\frac{m_i}{dn_i T_i} \sum_{j=1}^2 \int d\mathbf{v} V^2 J_{ij}[\mathbf{v}|f_i, f_j] \quad (11)$$

is the cooling rate associated with the partial temperature T_i . Equation (10) still applies for IHS. However, in contrast to IHS, the exact form of the partial cooling rates ζ_i^* can be determined for IMM without the knowledge of the distributions f_i . They are given by [9]

$$\zeta_i^* = \frac{4\pi^{(d-1)/2}}{d\Gamma\left(\frac{d}{2}\right)} \sum_{j=1}^2 x_j \mu_{ji} \left(\frac{\sigma_{ij}}{\sigma_{12}} \right)^{d-1} \left(\frac{\theta_i + \theta_j}{\theta_i \theta_j} \right)^{1/2} (1 + \alpha_{ij}) \left[1 - \frac{\mu_{ji}}{2} (1 + \alpha_{ij}) \frac{\theta_i + \theta_j}{\theta_j} \right]. \quad (12)$$

The partial temperatures T_i^* can be determined by substituting Eq. (12) into the coupled equations (10). It must be noted that the exact results obtained for IMM in the homogeneous state for the partial temperatures are completely equivalent to those previously derived for IHS when f_i is replaced by its Maxwellian form (8). Figure 1 shows a comparison between the numerical solution of Eq. (12) and molecular dynamics (MD) simulations of a mixture of inelastic hard spheres [13] for a dilute system. It is quite apparent that the theoretical results derived for IMM shows an excellent agreement with MD simulations of IHS.

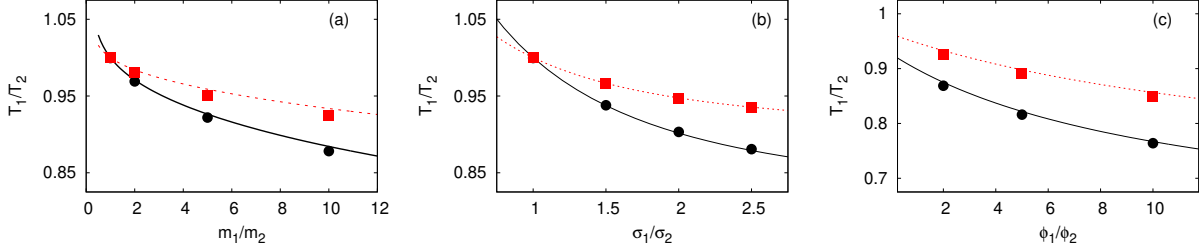


FIGURE 1. Plot of the temperature ratio T_1/T_2 for a three dimensional ($d = 3$) system with volume fraction $\phi = 0.00785$ and two different values of the (common) coefficient of restitution [$\alpha_{11} = \alpha_{22} = \alpha_{12} \equiv \alpha = 0.8$ (solid lines and circles) and 0.9 (dashed lines and squares)] as a function of: (a) the mass ratio m_1/m_2 for $\sigma_1/\sigma_2 = \phi_1/\phi_2 = 1$, (b) the size ratio σ_1/σ_2 for $m_1/m_2 = \phi_1/\phi_2 = 1$, and (c) the composition ratio ϕ_1/ϕ_2 for $m_1/m_2 = 8$ and $\sigma_1/\sigma_2 = 2$. The volume fraction $\phi = \phi_1 + \phi_2$ where $\phi_i = n_i \pi \sigma_i^3 / 6$ for hard spheres. The lines are the theoretical predictions and the symbols refer to the MD simulation results. The parameters for theory and simulations are $\gamma_b = 0.1$, $\xi_b^2 = 0.2$, $\beta = 1$, and $\lambda = 2$.

CHAPMAN-ENSKOG SOLUTION: FIRST ORDER DISTRIBUTION FUNCTION

Let us assume now that we slightly perturb the homogeneous steady state by weak spatial gradients. The perturbation will give nonzero contributions to the mass, momentum, and heat fluxes. The objective here is to obtain the transport coefficients associated with the mass flux. In this situation, we solve the Boltzmann equation (1) by means of the Chapman-Enskog method [15] adapted to dissipative dynamics. As usual, the Chapman-Enskog method assumes the existence of a normal or hydrodynamic solution where the space and time dependence of f_i only occurs through a functional dependence on the hydrodynamic fields. This functional dependence can be made local in space through an expansion of f_i in powers of the gradients of the hydrodynamic fields, i.e., $f_i = f_i^{(0)} + f_i^{(1)} + \dots$. Here, only terms up to the first order in gradients will be considered.

In ordering the different level of approximations in the Boltzmann kinetic equation (1), one has to characterize the magnitude of the thermostat parameters γ_b and ξ_b^2 relative to the gradients as well as the term $\Delta \mathbf{U}$. Regarding the thermostat parameters, since both quantities do not create any flux in the system, they must be considered to be of zeroth-order in gradients. With respect to the term $\Delta \mathbf{U}$, since for homogeneous systems \mathbf{U} relaxes towards \mathbf{U}_g after a transient period, then the difference $\Delta \mathbf{U}$ must be considered to be at least of first order in gradients.

Zeroth-order solution

To zeroth-order, the Boltzmann equation (1) reads

$$\partial_t^{(0)} f_i^{(0)} - \frac{\gamma_b}{m_i^\beta} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{v} f_i^{(0)} - \frac{1}{2} \frac{\xi_b^2}{m_i^\lambda} \frac{\partial^2 f_i^{(0)}}{\partial v^2} = \sum_{j=1}^2 J_{ij}[\mathbf{v} | f_i^{(0)}, f_j^{(0)}]. \quad (13)$$

The balance equations to this order are $\partial_t^{(0)} x_1 = \partial_t^{(0)} U_i = 0$ and $T^{-1} \partial_t^{(0)} T = p^{-1} \partial_t^{(0)} p = -\Lambda^{(0)}$ where

$$\Lambda^{(0)} \equiv 2\gamma_b \sum_{i=1}^2 \frac{x_i \chi_i}{m_i^\beta} - \frac{\xi_b^2}{p} \sum_{i=1}^2 \frac{\rho_i}{m_i^\lambda} + \zeta^{(0)}. \quad (14)$$

Here, $p = (n_1 + n_2)T$ is the hydrostatic pressure and $\zeta^{(0)}$ is the total cooling rate to zeroth-order; in reduced form it is given by $\zeta_0^* = \zeta^{(0)}/\nu_0 = x_1\chi_1\zeta_1^* + x_2\chi_2\zeta_2^*$, where ζ_i^* is defined by Eq. (12). Equation (13) can be rewritten in a more convenient way when one takes into account the balance equations:

$$-\Lambda^{(0)}(T\partial_T + p\partial_p)f_i^{(0)} - \frac{\gamma_b}{m_i^\beta} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{V}f_i^{(0)} - \frac{1}{2} \frac{\xi_b^2}{m_i^\lambda} \frac{\partial^2 f_i^{(0)}}{\partial v^2} = \sum_{j=1}^2 J_{ij}[\mathbf{v}|f_i^{(0)}, f_j^{(0)}]. \quad (15)$$

As widely discussed before in driven granular systems [3, 16, 17], although we want to compute the transport coefficients of the driven granular mixture in steady state conditions, for small perturbations around the homogeneous steady state, the density and temperature are specified separately in the local reference state $f_i^{(0)}$. As a consequence, the temperature T and the pressure p are not expected in general to be *stationary* at any point of the system. This means that the zeroth-order time derivatives $\partial_t^{(0)}T$ and $\partial_t^{(0)}p$ are different from zero in the reference base state. This contrasts with the usual application of the Chapman–Enskog method to ordinary mixtures ($\alpha_{ij} = 1$) where the terms $\partial_t^{(0)}T$ and $\partial_t^{(0)}p$ are of first order in the spatial gradients. In the steady state ($\Lambda^{(0)} = 0$), Eq. (15) reduces to Eq. (6) whose solution is not exactly known. However, as Fig. 1 clearly shows, a good estimate for the partial temperatures can be obtained by considering the Maxwellian distribution $f_{i,M}$ for the zeroth-order solution $f_i^{(0)}$.

First-order solution

To first-order in spatial gradients, the distribution $f_1^{(1)}$ obeys the kinetic equation [3]

$$\begin{aligned} \partial_t^{(0)}f_1^{(1)} - \frac{\gamma_b}{m_1^\beta} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{V}f_1^{(1)} - \frac{1}{2} \frac{\xi_b^2}{m_1^\lambda} \frac{\partial^2}{\partial v^2} f_1^{(1)} + \mathcal{L}_1 f_1^{(1)} + \mathcal{M}_1 f_2^{(1)} = \mathbf{A}_1 \cdot \nabla x_1 + \mathbf{B}_1 \cdot \nabla p + \mathbf{C}_1 \cdot \nabla T \\ + D_{1,k\ell} \frac{1}{2} \left(\nabla_k U_\ell + \nabla_\ell U_k - \frac{2}{d} \delta_{k\ell} \nabla \cdot \mathbf{U} \right) + E_1 \nabla \cdot \mathbf{U} + \mathbf{G}_1 \cdot \Delta \mathbf{U}. \end{aligned} \quad (16)$$

The linear Boltzmann collision operators \mathcal{L}_1 and \mathcal{M}_1 are defined as

$$\mathcal{L}_1 X = -\left(J_{11}[f_1^{(0)}, X] + J_{11}[X, f_1^{(0)}] + J_{12}[X, f_2^{(0)}] \right), \quad \mathcal{M}_1 X = -J_{12}[f_2^{(0)}, X]. \quad (17)$$

The coefficients of the field gradients on the right side of Eq. (16) are functions of \mathbf{V} and the hydrodynamic fields. They are given by

$$\mathbf{A}_1(\mathbf{V}) = -\mathbf{V} \frac{\partial f_1^{(0)}}{\partial x_1} + \frac{\gamma_b(m_2^\beta - m_1^\beta)}{\rho^2(m_1 m_2)^{\beta-1}} \frac{p}{T} D \frac{\partial f_1^{(0)}}{\partial \mathbf{V}}, \quad (18)$$

$$\mathbf{B}_1(\mathbf{V}) = -\mathbf{V} \frac{\partial f_1^{(0)}}{\partial p} - \rho^{-1} \frac{\partial f_1^{(0)}}{\partial \mathbf{V}} + \frac{\gamma_b(m_2^\beta - m_1^\beta)}{p(m_1 m_2)^\beta} D_p \frac{\partial f_1^{(0)}}{\partial \mathbf{V}}, \quad (19)$$

$$\mathbf{C}_1(\mathbf{V}) = -\mathbf{V} \frac{\partial f_1^{(0)}}{\partial T} + \frac{\gamma_b(m_2^\beta - m_1^\beta)}{T(m_1 m_2)^\beta} D_T \frac{\partial f_1^{(0)}}{\partial \mathbf{V}}, \quad (20)$$

$$D_{1,k\ell}(\mathbf{V}) = V_k \frac{\partial f_1^{(0)}}{\partial V_\ell}, \quad (21)$$

$$E_1(\mathbf{V}) = \frac{d+2}{d} p \frac{\partial f_1^{(0)}}{\partial p} + \frac{2}{d} T \frac{\partial f_1^{(0)}}{\partial T} + \frac{1}{d} \mathbf{V} \cdot \frac{\partial f_1^{(0)}}{\partial \mathbf{V}}, \quad (22)$$

$$\mathbf{G}_1(\mathbf{V}) = \frac{\gamma_b}{\rho} \frac{m_2^\beta - m_1^\beta}{(m_1 m_2)^\beta} (\rho_2 + D_U) \frac{\partial f_1^{(0)}}{\partial \mathbf{V}}. \quad (23)$$

The corresponding equation for $f_2^{(1)}$ is obtained from Eq. (16) by the change $1 \leftrightarrow 2$ (not for the mole fraction x_1 since $x_2 = 1 - x_1$). Equation (16) has the same structure as for IHS except for the form of the linearized Boltzmann collision operators \mathcal{L}_1 and \mathcal{M}_1 and the fact that the first-order contribution to the cooling rate vanishes for IMM.

Mass flux

We compute here the first-order contribution $\mathbf{j}_1^{(1)}$ to the mass flux. It is defined as

$$\mathbf{j}_1^{(1)} = \int d\mathbf{v} m_1 \mathbf{V} f_1^{(1)}(\mathbf{V}). \quad (24)$$

To get it, we multiply both sides of Eq. (16) by $m_1 \mathbf{V}$ and integrate over velocity. After some algebra, one achieves the result

$$\begin{aligned} \partial_t^{(0)} \mathbf{j}_1^{(1)} + \frac{\gamma_b}{m_1^\beta} \mathbf{j}_1^{(1)} + \nu_D \mathbf{j}_1^{(1)} &= - \left[p \frac{\partial}{\partial x_1} (x_1 \chi_1) + \frac{\gamma_b \rho_1 (m_2^\beta - m_1^\beta)}{\rho^2 (m_1 m_2)^{\beta-1}} \frac{p}{T} D \right] \nabla x_1 \\ &\quad - \left[x_1 \left(\chi_1 + p \frac{\partial \chi_1}{\partial p} \right) - \frac{\rho_1}{\rho} + \frac{\gamma_b \rho_1 (m_2^\beta - m_1^\beta)}{p (m_1 m_2)^\beta} D_p \right] \nabla p \\ &\quad - \left[p x_1 \frac{\partial \chi_1}{\partial T} + \frac{\gamma_b \rho_1 (m_2^\beta - m_1^\beta)}{T (m_1 m_2)^\beta} D_T \right] \nabla T \\ &\quad - \frac{\gamma_b \rho_1}{\rho} \frac{m_2^\beta - m_1^\beta}{(m_1 m_2)^\beta} (\rho_2 + D_U) \Delta \mathbf{U}. \end{aligned} \quad (25)$$

Upon obtaining Eq. (25), use has been made of the result

$$\int d\mathbf{v} m_1 \mathbf{V} (\mathcal{L}_1 f_1^{(1)} + \mathcal{M}_1 f_2^{(1)}) = \nu_D \mathbf{j}_1^{(1)}, \quad (26)$$

where

$$\nu_D = \rho \frac{\omega_{12}}{dn_2} \frac{1 + \alpha_{12}}{m_1 + m_2} = \frac{2\pi^{(d-1)/2}}{d\Gamma(\frac{d}{2})} (1 + \alpha_{12}) \left(\frac{M_1 \chi_2 + M_2 \chi_1}{M_1 M_2} \right)^{1/2} (x_2 M_1^{-1} + x_1 M_2^{-1}) v_0. \quad (27)$$

The constitutive equation of the mass flux is

$$\mathbf{j}_1 = - \left(\frac{m_1 m_2 n}{\rho} \right) D \nabla x_1 - \frac{\rho}{p} D_p \nabla p - \frac{\rho}{T} D_T \nabla T - D_U \Delta \mathbf{U}, \quad \mathbf{j}_2 = -\mathbf{j}_1. \quad (28)$$

We want to determine the diffusion transport coefficients in the steady state, namely, when $\Lambda^{(0)} = 0$. Dimensional analysis shows that $D \propto T^{1/2}$, $D_p \propto D_T \propto T^{3/2}/p$, and $D_U \propto p/T$. Hence, in dimensionless form, the diffusion transport coefficients D , D_p and D_T can be written as

$$D = \frac{\rho T}{m_1 m_2 \nu_0} D^*, \quad D_p = \frac{n T}{\rho \nu_0} D_p^*, \quad D_T = \frac{n T}{\rho \nu_0} D_T^*, \quad (29)$$

In addition, to evaluate the time derivative of $\mathbf{j}_1^{(1)}$, one has to take into account the intermediate results

$$\partial_t^{(0)} \nabla p = \nabla (\partial_t^{(0)} p) = -\nabla (p \Lambda^{(0)}) = -\Lambda^{(0)} \nabla p + \mathcal{P}_{x_1} \nabla x_1 + \mathcal{P}_p \nabla p + \mathcal{P}_T \nabla T, \quad (30)$$

$$\partial_t^{(0)} \nabla T = \nabla (\partial_t^{(0)} T) = -\nabla (T \Lambda^{(0)}) = -\Lambda^{(0)} \nabla T + \mathcal{T}_{x_1} \nabla x_1 + \mathcal{T}_p \nabla p + \mathcal{T}_T \nabla T, \quad (31)$$

where

$$\mathcal{P}_{x_1} = \frac{p}{T} \xi_b^2 \frac{m_2^{\lambda-1} - m_1^{\lambda-1}}{(m_1 m_2)^{\lambda-1}} - p \frac{\partial \zeta^{(0)}}{\partial x_1} - 2\gamma_b p \frac{m_2^\beta - m_1^\beta}{(m_1 m_2)^\beta} \left(\chi_1 + x_1 \frac{\partial \chi_1}{\partial x_1} \right), \quad (32)$$

$$\mathcal{P}_p = - \left(2\gamma_b \sum_{i=1}^2 \frac{x_i \chi_i}{m_i^\beta} + 2\gamma_b p \frac{m_2^\beta - m_1^\beta}{(m_1 m_2)^\beta} x_1 \frac{\partial \chi_1}{\partial p} - \xi_b^2 \frac{1}{T} \sum_{i=1}^2 \frac{x_i}{m_i^{\lambda-1}} + \zeta^{(0)} + p \frac{\partial \zeta^{(0)}}{\partial p} \right), \quad (33)$$

$$\mathcal{P}_T = - \left(\xi_b^2 \frac{p}{T^2} \sum_{i=1}^2 \frac{x_i}{m_i^{\lambda-1}} + p \frac{\partial \zeta^{(0)}}{\partial T} + 2\gamma_b p \frac{m_2^\beta - m_1^\beta}{(m_1 m_2)^\beta} x_1 \frac{\partial \chi_1}{\partial T} \right), \quad (34)$$

$$\mathcal{T}_{x_1} = \xi_b^2 \frac{m_2^{\lambda-1} - m_1^{\lambda-1}}{(m_1 m_2)^{\lambda-1}} - T \frac{\partial \zeta^{(0)}}{\partial x_1} - 2\gamma_b T \frac{m_2^\beta - m_1^\beta}{(m_1 m_2)^\beta} \left(\chi_1 + x_1 \frac{\partial \chi_1}{\partial x_1} \right), \quad (35)$$

$$\mathcal{T}_p = - \left(2\gamma_b T \frac{m_2^\beta - m_1^\beta}{(m_1 m_2)^\beta} x_1 \frac{\partial \chi_1}{\partial p} + T \frac{\partial \zeta^{(0)}}{\partial p} \right), \quad (36)$$

$$\mathcal{T}_T = - \left(2\gamma_b \sum_{i=1}^2 \frac{x_i \chi_i}{m_i^\beta} + 2\gamma_b T \frac{m_2^\beta - m_1^\beta}{(m_1 m_2)^\beta} x_1 \frac{\partial \chi_1}{\partial T} + \zeta^{(0)} + T \frac{\partial \zeta^{(0)}}{\partial T} \right). \quad (37)$$

Therefore, in the steady state ($\Lambda^{(0)} = 0$), the zeroth-order time derivative $\partial_t^{(0)} \mathbf{j}_1^{(1)}$ gives the contributions

$$\partial_t^{(0)} \mathbf{j}_1^{(1)} \rightarrow - \left(\frac{\rho}{p} \mathcal{P}_{x_1} D_p + \frac{\rho}{T} \mathcal{T}_{x_1} D_T \right) \nabla_{x_1} - \left(\frac{\rho}{p} \mathcal{P}_p D_p + \frac{\rho}{T} \mathcal{T}_p D_T \right) \nabla_p - \left(\frac{\rho}{p} \mathcal{P}_T D_p + \frac{\rho}{T} \mathcal{T}_T D_T \right) \nabla_T. \quad (38)$$

When $m_1 = m_2$ or $\gamma_b = 0$, then $D_U = 0$ as expected from the previous result obtained for IHS. The diffusion coefficients D , D_p , D_T , and D_U can be easily identified after inserting Eq. (38) into Eq. (25). As expected, while the coefficients D , D_p , and D_T are coupled, the coefficient D_U obeys an autonomous equation whose solution is

$$D_U = \left[\frac{\gamma_b}{m_1^\beta} \left(1 - \frac{\rho_1}{\rho} \frac{m_2^\beta - m_1^\beta}{m_2^\beta} \right) + \nu_D \right]^{-1} \frac{\gamma_b \rho_1 \rho_2}{\rho} \frac{m_2^\beta - m_1^\beta}{(m_1 m_2)^\beta}. \quad (39)$$

The coefficients D , D_p , and D_T verify the following set of linear algebraic equations:

$$\frac{\rho}{p} \mathcal{P}_{x_1} D_p + \frac{\rho}{T} \mathcal{T}_{x_1} D_T + \frac{m_1 m_2 n}{\rho} \left(\frac{\gamma_b}{m_1^\beta} + \nu_D \right) D = p \frac{\partial}{\partial x_1} (x_1 \chi_1) + \frac{\gamma_b \rho_1 (m_2^\beta - m_1^\beta)}{\rho^2 (m_1 m_2)^{\beta-1}} \frac{p}{T} D, \quad (40)$$

$$\frac{\rho}{p} \mathcal{P}_p D_p + \frac{\rho}{T} \mathcal{T}_p D_T + \frac{\rho}{p} \left(\frac{\gamma_b}{m_1^\beta} + \nu_D \right) D_p = x_1 \chi_1 \left(1 - \frac{\rho_1}{\rho x_1 \chi_1} \right) + x_1 p \frac{\partial \chi_1}{\partial p} + \frac{\gamma_b \rho_1 (m_2^\beta - m_1^\beta)}{p (m_1 m_2)^{\beta-1}} D_p, \quad (41)$$

$$\frac{\rho}{p} \mathcal{P}_T D_p + \frac{\rho}{T} \mathcal{T}_T D_T + \frac{\rho}{T} \left(\frac{\gamma_b}{m_1^\beta} + \nu_D \right) D_T = x_1 p \frac{\partial \chi_1}{\partial T} + \frac{\gamma_b \rho_1 (m_2^\beta - m_1^\beta)}{T (m_1 m_2)^{\beta-1}} D_T. \quad (42)$$

In dimensionless form, the explicit solutions to Eqs. (40)–(42) are

$$D_p^* = \frac{a_{23} a_{30} - a_{33} a_{20}}{a_{23} a_{32} - a_{22} a_{33}}, \quad D_T^* = \frac{a_{32} a_{20} - a_{22} a_{30}}{a_{23} a_{32} - a_{22} a_{33}}, \quad D^* = \frac{a_{10} - a_{12} (D_p^* + D_T^*)}{a_{11}}, \quad (43)$$

where we have introduced the quantities

$$a_{11} = \nu_D + \overline{m}^\beta \omega^* \xi^{*1/3} \frac{\rho_1 m_1^\beta + \rho_2 m_2^\beta}{\rho (m_1 m_2)^\beta}, \quad (44)$$

$$a_{12} = -2\omega^* \xi^{*1/3} M_1^{-\beta} \frac{m_2^\beta - m_1^\beta}{m_2^\beta} \frac{\partial}{\partial x_1} (x_1 \chi_1) + \xi^* M_1^{1-\lambda} \frac{m_2^{\lambda-1} - m_1^{\lambda-1}}{m_2^{\lambda-1}} - \frac{\partial \zeta_0^*}{\partial x_1}, \quad (45)$$

$$a_{10} = \frac{\partial}{\partial x_1} (x_1 \chi_1), \quad a_{22} = a_{11} + a_{23}, \quad a_{20} = x_1 \chi_1 - \frac{\rho_1}{\rho} + x_1 p \frac{\partial \chi_1}{\partial p}, \quad (46)$$

$$a_{23} = -2\omega^* \xi^{*1/3} M_1^{-\beta} \frac{m_2^\beta - m_1^\beta}{m_2^\beta} x_1 p \frac{\partial \chi_1}{\partial p} - \frac{p}{\nu_0} \frac{\partial \zeta^{(0)}}{\partial p}, \quad (47)$$

$$a_{20} = x_1 \chi_1 - \frac{\rho_1}{\rho} + x_1 p \frac{\partial \chi_1}{\partial p}, \quad (48)$$

$$a_{32} = -2\omega^* \xi^{*1/3} M_1^{-\beta} \frac{m_2^\beta - m_1^\beta}{m_2^\beta} x_1 T \frac{\partial \chi_1}{\partial T} - \xi^{*} \sum_i x_i M_i^{1-\lambda} - \frac{T}{\nu_0} \frac{\partial \xi^{(0)}}{\partial T}, \quad (49)$$

$$a_{33} = a_{11} + a_{32}, \quad a_{30} = x_1 T \frac{\partial \chi_1}{\partial T}. \quad (50)$$

Here, we have introduced the dimensionless quantity

$$\omega^* = \frac{\gamma_b}{\bar{m}^\beta} \left(\frac{\bar{m}^\lambda}{2\xi_b^2} \right)^{1/3} (n\sigma_{12}^{d-1})^{-2/3}. \quad (51)$$

As for homogeneous states, the expressions of the diffusion transport coefficients coincide with those approximately derived for IHS in the first Sonine approximation. The explicit expressions for the derivatives appearing in Eqs. (43) can be found in Ref. [3].

To illustrate the influence of inelasticity of collisions on mass transport, Fig. 2 shows the dimensionless coefficients D^* , D_p^* , and D_T^* as a function of the common coefficient of normal restitution ($\alpha_{ij} \equiv \alpha$) for a two dimensional system and several values of the mass ratio. It is quite apparent that the influence of collisional dissipation on mass transport is in general important.

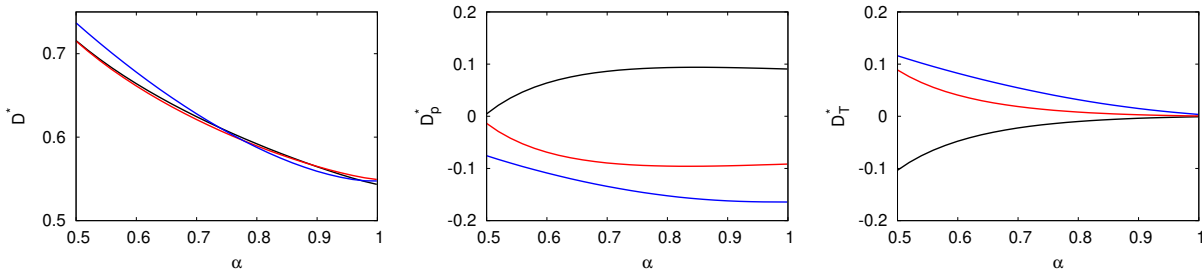


FIGURE 2. Reduced diffusion coefficients D^* , D_p^* , and D_T^* as a function of the (common) coefficient of restitution α for an equimolar binary mixture ($x_1 = \frac{1}{2}$) of hard disks ($d = 2$) with $\phi = 0.00758$, $\sigma_1/\sigma_2 = 1$, and three different values of the mass ratio: $m_1/m_2 = 0.5$ (black), $m_1/m_2 = 2$ (red), and $m_1/m_2 = 4$ (blue). The volume fraction for hard disks is $\phi = \sum_i n_i \pi \sigma_i^2 / 4$. The parameters of the driven model are $\gamma_b = 0.1$, $\xi_b^2 = 0.2$, $\beta = 1$, and $\lambda = 2$.

CONCLUSIONS

In this paper, we have studied some dynamic properties of granular mixtures driven by a stochastic bath with friction. In addition, to obtain the exact forms of these properties, IMM have been considered instead of the more realistic model of IHS. As for elastic collisions, in IMM the collision rate of two colliding particles is independent of their relative velocity so that the collisional moments of the Boltzmann operator can be exactly computed without the knowledge of the distribution functions. Our study has been carried out in two steps. First, we have analyzed the homogeneous steady state where the partial temperatures (measuring the mean kinetic energy of each species) have been determined in terms of the parameter space of the system. Our exact results show that the equations for the partial temperatures are the same as those obtained for IHS when non-Gaussian corrections to the distribution functions are neglected. Then, the diffusion transport coefficients have been also obtained from the application of the Chapman-Enskog method to the Boltzmann equation. A surprising result is that the expressions found for IMM agree with those obtained for IHS [3] in the first Sonine approximation.

As an application of the general calculation, we have evaluated the temperature ratio T_1/T_2 and the diffusion coefficients D , D_p , and D_T for several systems. Figure 1 shows that T_1/T_2 is a decreasing function of the mass ratio m_1/m_2 if the remaining quantities are kept equal for both species. It is worthwhile to note that this is not an obvious result since the thermostat differentiates between particles of different masses. Similar results are found if the temperature ratio is plotted versus both the diameter ratio σ_1/σ_2 and the composition ratio ϕ_1/ϕ_2 . The comparison of these theoretical results against MD simulations of IHS [13] shows an excellent agreement, even for quite disparate

values of the mass and/or diameter ratios. The effect of the mass ratio on the diffusion coefficients has been also explored in Fig. 2. We observe that the (scaled) diffusion coefficient D^* is a decreasing function of the common coefficient of normal restitution with a slight dependence on the mass ratio. A different behavior is observed with the (scaled) pressure diffusion coefficient D_p^* and the (scaled) thermal diffusion coefficient D_T^* , since in both cases there is a significant dependence on the mass ratio. Thus, while D_p^* decreases (increases) with increasing inelasticity when $m_1 > m_2$ ($m_1 < m_2$), the opposite happens for D_T^* .

In summary, we can conclude that the exact results obtained here for IMM give support again to the use of this interaction model as a reliable model for describing transport properties of driven granular mixtures of IHS.

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