Advanced applications of Finite-Size Scaling

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Summary. We report some results obtained in the framework of spin systems, using Finite-Size Scaling techniques and Monte Carlo Simulations. We focus on the high precision measurements of Critical Exponents in three dimensional systems of interest in Condense Matter, and the issue of triviality in four dimensional systems relevant for Quantum Field Theory.

1 Dedicatoria

Dedicamos este escrito al profesor Alberto Galindo con motivo de su septuagésimo cumpleaños. Hemos recogido algunos resultados en el área de simulaciones numéricas en sistemas estadísticos obtenidos en trabajos realizados en el Departamento de Física Teórica I a lo largo de los ultimos años, durante los cuales él fue director del departamento.

Aunque este tema de investigación es, de los realizados en el departamento, quizá de los más lejanos a las líneas seguidas por Alberto Galindo, queremos destacar la gran influencia que nos ha ejercido en todos los aspectos. Empezando por que ha sido profesor en la licenciatura de todos nosotros (y eso imprime carácter), pasando por las estrechas relaciones que hemos tenido con él, desde las tareas docentes y organizativas a las relaciones personales.

La trayectoria científica de Alberto ha tenido dos sedes importantes, Madrid y Zaragoza. Es notable que, aunque ninguno de los autores ha estudiado (ni nacido o vivido) en Zaragoza, mantenemos unos estrechsimos vínculos con el Departamento de Física Teórica de la Universidad de Zaragoza (especialmente con los profesores José Luis Alonso, Andrés Cruz y Alfonso Tarancón).

En tiempos muy recientes, los autores de este trabajo nos hemos involucrado en la creación y puesta en marcha del *Instituto de Biocomputación y Física de Sistemas Complejos* (BIFI) de la Universidad de Zaragoza. También en este empeño, el apoyo de Alberto ha resultado crucial.

2 Introduction

The scope of this paper is to review part of the activities of the Statistical Mechanics group of our Department during the last decade. Specifically we will show our main results on spin model systems at criticality. In the following section, we describe the finite-size techniques we have developed and used. Next we report the high precision numerical values of critical couplings and critical exponents, obtained for several three dimensional spin models which have been reference numbers for years [1]. In section 5 we describe our work on diluted Ising model. We first address the three dimensional case, whose properties were rather controversial. We were able of predicting a single universality class for this problem, which was confirmed later in experiments [2]. We end with the description of our results in four dimensional spin systems, at the upper critical dimension. We have modified the standard Finite-Size Scaling approach, to cover this case. The final goal is to understand the issue of triviality in quantum field theories.

3 Finite-Size Scaling

In a nutshell Finite-Size Scaling [3, 4, 5, 6, 7, 8] (FSS) aims to solve the paradox that real systems (which are finite) do show phase transitions, while statistical-mechanics predicts that all the thermodynamic properties of a finite system are smooth (analytical!) functions of temperature, pressure or whatsoever control parameter is of relevance for the problem at hand.

Consider any intensive quantity, O, (e.g. energy, magnetization density, magnetic susceptibility, etc.), behaving in the thermodynamic limit as[5]

$$\langle O\rangle_\infty(t) \propto |t|^{-x_O}\,,\ \text{when } t\to 0\,. \tag{1}$$

In practice we can only compute the finite size mean value $\langle O\rangle_L$ where L a characteristic length of the finite system. For a box geometry, $L = V^{1/d}$, while for a strip geometry L is the strip width. For a layer geometry, L is the thickness of the film.

The basic assumption of the FSS *Ansatz* [4, 5] is that the finite size behaviour is governed by the ratio L/ξ_∞, where ξ_∞ is the correlation length of the infinite system. If this ratio is large, the system has basically reached its thermodynamic limit. If it is small, we will be in the FSS regime.

On a first thought, finite size effects may look like a nuisance for the data analysis.[6] Yet, those effects carry the same information that is contained in the infinite volume divergences. They turn out [9, 10] to be precious for the investigation of critical phenomena: one concentrates on the temperature at which

[5]We define the reduced temperature as $t \equiv \frac{\beta_c-\beta}{\beta_c} \approx \frac{T-T_c}{T_c}$ (not exactly the standard definition).

[6]Such data could be obtained from experiment or by solving a model on a finite sample.

the finite size effects are largest, there the FSS is studied. Most (if not all) of the relevant information can be extracted from the lattice size evolution of the intensive quantities at the critical point.

We will describe here the approach of Finite-Size Scaling (FSS) that has been developed at Madrid. The starting point is the standard scaling of the free energy with L:

$$f(t,h,\{u_j\},L^{-1}) = g(t,h,\{u_j\}) + b^{-d} f(b^{y_t}t, b^{y_h}h, \{u_j b^{y_j}\}, b/L). \tag{2}$$

At this point one takes the block size $b = L$, thus arriving to a single-site lattice. By performing the appropriate derivatives, all the critical quantities can be computed. The result can be cast in general form for a quantity O diverging like t^{-x_O} in the thermodynamical limit:

$$O(L,t) = L^{x_O/\nu}\left[F_O\left(\frac{L}{\xi(\infty,t)}\right) + \mathcal{O}(L^{-\omega},\xi^{-\omega})\right], \tag{3}$$

where F_O is a smooth scaling function. In usual applications one is interested in the $\xi\,\mathrm{G}_2\,L$ regime, thus $\xi^{-\omega}$ is safely neglected. Of course in Eq. (3), we have only kept the leading irrelevant eigenvalue, ω, but, in fact, other scaling corrections like

$$\{L^{y_j}\}, \{L^{y_j+y_i}\}, \ldots \;\; (i,j \geq 3) \tag{4}$$

are to be expected. In addition, other kind of terms are induced by the analytical part of the free energy, g. For the susceptibility (or related quantities like the Binder cumulant or the correlation-length, see below) one should take the second derivative with respect to the magnetic field, h, in Eq. (2).

Equation (3) is still not convenient for a numerical study, because it contains not directly measurable quantities like $\xi(\infty,t)$. Fortunately, it can be turned into an useful expression if a reasonable definition of the correlation length in a finite lattice, $\xi(L,t)$ (see below), is available:

$$O(L,t) = L^{x_O/\nu}\left[\widetilde{F}_O\left(\frac{\xi(L,t)}{L}\right) + \mathcal{O}(L^{-\omega})\right], \tag{5}$$

where $\widetilde{F}_O$ is a smooth function related with F_O and F_ξ.

To reduce the effect of the corrections-to-scaling terms, one could take measures only in large enough lattices. Even in the simplest models, as those in this paper, if one wants to obtain very precise results, the lattice sizes required can be unreachable. However, this is not the most efficient option. In the specific method we use, the scaling function is eliminated by taking measures of a given observable at the same temperature in two different lattice sizes (L_1, L_2). At the temperature where the correlation lengths are in the ratio $L_1 : L_2$, from Eq. (5) we can write the quotient of the measures of an observable, O, in both lattices as

$$Q_O\big|_{Q_\xi=\frac{L_1}{L_2}} = \left(\frac{L_1}{L_2}\right)^{x_O/\nu} + A_{Q_O} L_2^{-\omega} + \ldots\;, \tag{6}$$

where A_{Q_O} is a constant.

The great advantage of Eq. (6) is that to obtain the temperature where $Q_\xi = L_1/L_2$, only two lattices are required, and a very accurate and statistically clean measure of that temperature can be taken. In addition, the statistical correlation between Q_O and Q_ξ reduces the fluctuations.

To perform an extrapolation following Eq. (6), an estimate of ω is required. This can be obtained from the behaviour of dimensionless quantities that we define below, like the Binder cumulant or the correlation length in units of the lattice size, $\xi(L,t)/L$, which remain bounded at the critical point although their t-derivatives diverge. For a generic dimensionless quantity, g, we shall have a crossing

$$g(L, t^{\mathrm{cross}}(L,s)) = g(sL, t^{\mathrm{cross}}(L,s)).$$

The distance from the critical point, $t^{\mathrm{cross}}(L,s)$, goes to zero as [10]:

$$t^{\mathrm{cross}}(L,s) \propto \frac{1-s^{-\omega}}{s^{1/\nu}-1} L^{-\omega-1/\nu} \ . \tag{7}$$

From Eq. (7), a clean estimate of ω can be obtained provided that $|y_4| - \omega$ and $\gamma/\nu - \omega$ are large enough (say of order one).

3.1 Observables

In the prototypical case we consider a nearest-neighbor interaction. The spins live in the nodes of a (hyper)cubic lattice in d dimensions, of size L (the volume being $V = L^d$), with periodic boundary conditions. The Hamiltonian is

$$-H = \beta \sum_{\langle i,j \rangle} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \,, \tag{8}$$

and the partition function

$$Z = \sum_{\{\boldsymbol{\sigma}_i\}} \mathrm{e}^{-H} \,. \tag{9}$$

We have been deliberately vague about the nature of the spins, because the same framework covers a wide range of models and physical situations. For instance, the spins $\boldsymbol{\sigma}$, could be just ± 1 (Ising model), or unit vectors of N components (O(N) non-linear σ-model, the $N = 1$ case will be the Ising model). The system has a global O(N) invariance. If β (which plays the role of an inverse temperature) is positive, the model is ferromagnetic. On the other hand, if $\beta < 0$ it is antiferromagnetic.

An interesting generalization of model (8) consists in considering the *square* of the interaction $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$:

$$-H' = \beta \sum_{\langle i,j \rangle} (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)^2 \,. \tag{10}$$

Due to Elitzur theorem, the local Z_2-invariance of model (10) cannot be spontaneously broken and the model is naturally defined in terms of the real-projective space RP^{N-1}.

The quantities that we measure are basically the energy $(-H)$ and those related with the spin field. We have the momentum-dependent magnetization

$$\mathcal{M}(\boldsymbol{k}) = \frac{1}{V}\sum_i \boldsymbol{\sigma}_i \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_i}\,, \tag{11}$$

From it one may define the magnetization

$$M = \langle \|\mathcal{M}(0)\| \rangle\,, \tag{12}$$

the susceptibility

$$\chi = V\langle \|\mathcal{M}(0)\|^2 \rangle\,, \tag{13}$$

and the finite-lattice correlation-length [11]

$$\xi = \left(\frac{\chi/F - 1}{4\sin^2(\pi/L)}\right)^{1/2}\,, \tag{14}$$

where

$$F = \frac{V}{d}\left\langle \|\mathcal{M}(2\pi/L,0,\ldots,0)\|^2 \,+\, \mathrm{permutations}\right\rangle\,. \tag{15}$$

In the RP^{N-1} case one generalizes in the obvious way the previous formulae, considering the tensor field

$$\tau_i^{\alpha\beta} = \sigma_i^\alpha \sigma_j^\beta - \frac{\delta^{\alpha\beta}}{N}\,, \tag{16}$$

(the squared norm for a hermitean matrix, $\boldsymbol{\tau}$, is simply $\mathrm{tr}(\boldsymbol{\tau}\,\boldsymbol{\tau}^\dagger)$).

One may also consider the probability distribution function of the order parameter, $\mathcal{M}(0)$, through its cumulants. For instance, one considers the Binder parameter, which is simply the fourth derivative of the free-energy with respect of the magnetic field, divided by the square of the second-derivative. Expressing this in terms of rotationally invariant quantities, one finds (for $\mathrm{O}(N)$ models)

$$g_4 = \frac{N+2}{2} - \frac{N}{2}\frac{\langle \|\mathcal{M}(0)\|^4 \rangle}{\langle \|\mathcal{M}(0)\|^2 \rangle^2}\,, \tag{17}$$

Notice that g_4 is trivially related with the kurtosis of the probability distribution function of $\mathcal{M}(0)$. Away from the critical-point (when $T > T_\mathrm{c}$, otherwise one should take the *connected part*), the Central Limit Theorem tells us that this distribution is Gaussian, and g_4 tends to zero in the thermodynamical limit. To control this, one introduces the renormalized-coupling constant:

$$g_L^\mathrm{R} = \frac{L^d}{\xi^d} g_4\,. \tag{18}$$

For an interacting field theory, g_∞^R must remain finite and non-vanishing even if ξ tends to ∞.

4 Measurements of critical exponents

A successful determination of critical exponents rests on two feet. One must first produce high-quality Monte Carlo data, using state-of-the-art algorithms (such as cluster methods in O(N) models) and computers (sometimes using dedicated computers, such as the SUE machine[12]). We shall not give details here about this technicalities. The second foot of the calculation is to squeeze as much information of the Monte Carlo data as it is possible. This is were Finite-Size Scaling enters the stage.

Typically, a study starts by considering the crossing point of several dimensionless quantities as g_4 and ξ/L. Using Eq. (7), and a technically demanding joint fit of statistically correlated data, one manages to get very accurate determination of the critical point and, more importantly a fair estimate (10$\%$ error) of the leading corrections-to scaling exponent (see table 1). It is crucial to use more than one dimensionless quantity, because corrections to scaling amplitudes may have different signs, which constraints largely the joint fit.

With an estimate of ω in our hands, we can proceed to extrapolate the finite-lattice estimate of the critical exponents to infinite volume, using Eq. (6). In this way, the final estimate should be free of systematic errors.

This approach has been used in a large variety of models producing the results shown in table 1. Sometimes, detailed (and favorable) comparison with experimental measurements is possible.

Table 1. Summary of critical couplings and critical exponents for different spin systems in three dimensions obtained with Monte Carlo calculations [13, 14, 15, 16, 17].

Model	β_c	ν	η	ω
Ising	0.22165456(20)	0.6294(10)	0.0374(12)	0.87(9)
O(2)	0.454165(5)	0.670(10)	0.0424(25)	0.81(13)
O(3)	0.693002(12)	0.711(10)	0.0414(19)	0.71(16)
O(4)	0.935861(8)	0.758(4)	0.0359(9)	1.85(23)
AF RP^2	−2.4087(4)	0.783(11)	0.038(3)	0.85(4)
Percolation	0.3116081(11)	0.8765(18)	−0.0460(3)	1.62(13)
Diluted Ising	—	0.684(5)	0.037(4)	0.37(6)
Diluted Potts	—	0.690(5)	0.078(4)	~ 0.4

5 Diluted Models

The generalization of *regular* spin models to the more realistic case of systems where some kind of disorder is added, is of evident interest: in real samples one expects some extent of lattice defects that could change the properties of the system.

There is an important result by Harris [18] which states that the disorder only changes the Universality Class if the α exponent for the pure model is positive. Limiting ourselves to O(N) models, only for the Ising model in three dimensions, a positive critical exponent is found.

The model we consider is a diluted Ising model defined as a standard Ising model but with vacants (we make below some comments for the diluted Potts model). In every node of a (hyper)cubic lattice, we place a spin with probability p. We study quenched disorder, namely the position (and number) of vacants does not change with time. The value of p ranges from $p = 1$ (Ising model) to the percolation threshold, where the critical temperature is exactly zero (in three dimensions $p_{\mathrm{c}} = 0.3116081(11)$ [14]).

As we consider quenched disorder, in addition of the thermal average over spin configurations, one must afterward mediate over disorder realizations. This averaging over disorder realizations will be indicated by an overline. Technically, we have to carry out an independent simulation for each vacants configuration and then grand-average the results. Fortunately, cluster methods are also applicable for diluted systems and the thermalization time is negligible.

In addition to the previously defined quantities, the disorder average allows to define further dimensionless quantities such as

$$g_2 = \frac{\overline{\langle \mathcal{M}^2 \rangle^2 - \overline{\langle \mathcal{M}^2 \rangle}^2}}{\overline{\langle \mathcal{M}^2 \rangle}^2} \ . \tag{19}$$

Away from the critical point (when $T > T_{\mathrm{c}}$), the Central Limit Theorem implies that g_2 vanishes for large volume as L^{-d}. However, at the critical point g_2 remains bounded (see below) when $L \to \infty$. This means that the critical diluted model is not *self-averaging*. Consequently, a large number of disorder realizations must be considered. We typically generate 20000 disorder realizations at every p value and lattice size.

The field-theory for the diluted Ising model is a ϕ^4 theory with a random mass term:

$$S[\phi] = \int d^4x \ \left(\frac{1}{2} \left(\partial_\mu \phi\right)^2 + \frac{1}{2} m^2(x) \phi^2 + \frac{1}{4!} v \phi^4 \right) . \tag{20}$$

The mass term is a quenched, spatially-uncorrelated, stochastic variable with mean r, and variance Δ^2, so we will assume for simplicity that the distribution of $m^2(x)$ is Gaussian.

As we said above, one needs first to obtain the free-energy for a disorder realization then average over the random mass. The replica-trick [19] was invented to manage this kind of problems [19]: we introduce n replicas of the initial system, ϕ_i, with $i = 1, \ldots, n$. The average of the replicated partition function over the Gaussian disorder will be denoted by overlines.

$$F = \overline{\log \mathcal{Z}} = \lim_{n \to 0} \frac{1}{n} \left(\overline{\mathcal{Z}^n} - 1 \right) . \tag{21}$$

This is the starting point of the considerations in section 5.2.

Let us recall what happens on a disordered system whose pure counterpart suffers a first order transition. The answer is highly dependent on the spatial dimension. In $d = 2$ an infinitesimal dilution renders the phase transition second-order [20]. In $d = 3$ general arguments [21] suggest that the latent-heat decreases with increasing dilution, until a critical value is reached. The phase transition is second order from that dilution on. Our results for this Universality Class in the $d = 3$ three-state Potts model [17] case can be found in table 1.

5.1 Universality in diluted three dimensional Ising model

In this subsection we will consider only the three dimensional case. Results in two dimensions can be found in Ref. [22]. The four dimensional case is considered in the following subsection.

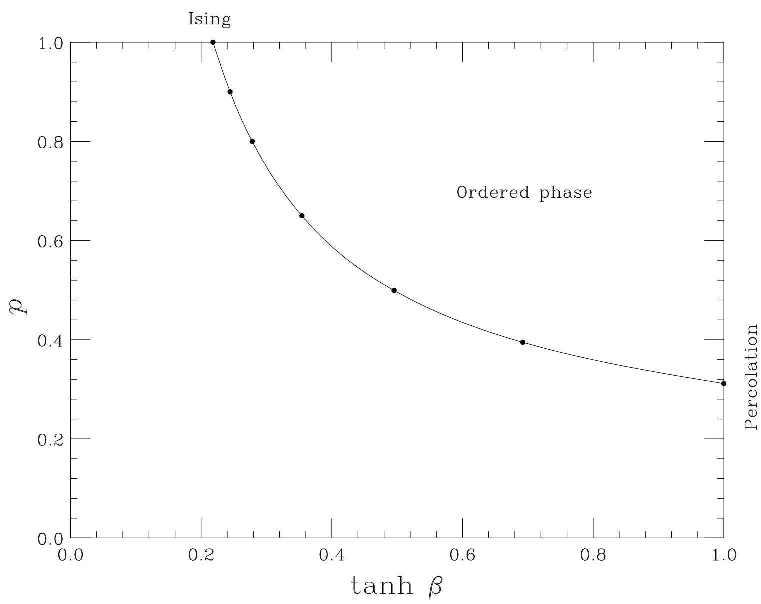

Fig. 1. Phase diagram (β, p) for the 3-d diluted Ising Model. The points are obtained from Monte Carlo simulations (the error bars are much smaller than the point size. The continuous line is just a smooth interpolation.

In figure 1 we sketch the phase diagram of the system, that includes the Ising universality class ($p = 1$) and the percolation one ($\beta = \infty$). What about the universality class of the rest of the line? When we study the problem, the situation was rather controversial. Previous estimates of the critical exponents showed a continuous variation along the line. However, we have found that this dependency is only due to corrections to scaling.

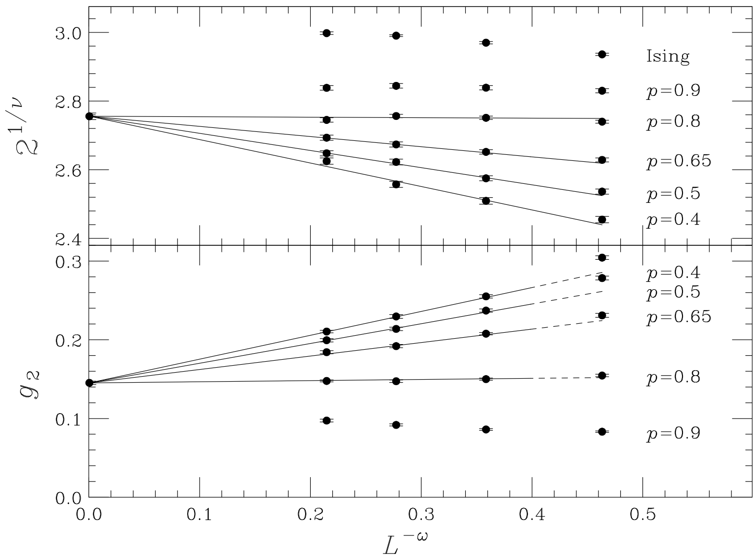

Fig. 2. Infinite volume extrapolation for the 3-d diluted Ising model for several values of the spin concentration p. Considering the leading corrections to scaling, we observe a single extrapolation for the ν exponent and g_2 cumulant for p in the range [0.4,0.8].

To make short a long story, we show in figure 2 our results for the ν exponent and g_2 cumulant, as a function of the lattice size for every p value, versus $L^{-\omega}$, as obtained using Eq. (6). We observe a clear size dependence but one that can be parametrized using the concept of corrections to scaling with a (very small) exponent $\omega \sim 0.4$. Performing a joint fit of both set of data (notice the statistical correlation) with a single limit for each quantity, we obtain a fair result just excluding the $p = 0.9$ data which is too close to the pure Ising limit. The scatter of previous results for the critical exponents along the critical line can be easily accounted for. Although the critical exponents are universal, the amplitudes for scaling corrections are not. In particular, note that the $p = 0.8$ data is quite close to be a *perfect action*: The scaling-corrections amplitude is compatible with zero, within our statistical accuracy.

The critical exponent ν estimate is in perfect agreement with recent experimental determinations ($\nu = 0.69(1)$[2]), that followed this theoretical computation.

5.2 Triviality in Scalar Quantum Field Theories: the four-dimensional diluted Ising model

Euclidean Quantum Field Theories (EQFT) are particular examples of general Statistical Mechanics Systems (SM). EQFT's live in the unstable manifolds of the critical points of these Statistical Mechanics Systems (this can be shown using the Wilson "triangle of renormalization" construction) [23]. Therefore, we can use the Statistical Mechanics framework to study problems which arise in EQFT, in particular the triviality issue: is non zero the renormalized coupling constant when the cut-off of the theory is sent to infinite? [24] The problem may be addressed in perturbation theory (PT), but SM provides us with a powerful non-perturbative technique. For instance, fixed points (where to define the EQFT) not accessible in PT [23] may be studied.

We focus in the renormalized coupling-constant (18) at the critical point. Obviously the Binder parameter should be computed in a finite geometry. The main idea is to expand the field $\phi(x)$ in Fourier modes. In a finite geometry the biggest contribution comes from the zero mode. It can be shown that it has to be treated non perturbatively while this is not necessary for the rest of the modes [25].

We will illustrate the above considerations in the example of the four dimensional diluted Ising model. The effective action follows from (20) and (21):

$$\overline{\mathcal{Z}^n} = \mathcal{Z}_{\mathrm{eff}} = \int d[\phi_i] \ \exp(-S_{\mathrm{eff}}[\phi_i]) \ , \tag{22}$$

with

$$S_{\mathrm{eff}}[\phi_i] = \int d^d x \left[\frac{1}{2}\sum_{i=1}^{n} \left(\partial_\mu \phi_i\right)^2 + \frac{r}{2}\sum_{i=1}^{n}\phi_i^2 + \frac{u}{4!}\left[\sum_{i=1}^{n}\phi_i^2\right]^2 + \frac{v}{4!}\sum_{i=1}^{n}\phi_i^4 \right], \tag{23}$$

where $u = -3\Delta^2$. This gives us a starting point for the analytical calculation. The $n \to 0$ limit should only be taken at the end. For $v = 0$ the action is $O(n)$-invariant. When $u = 0$ the action describes n decoupled Ising models. We remark that u is negative and proportional to the dilution. In our numerical simulation a site is occupied with probability p, so $\Delta^2 = p(1-p)$.

We can compute the Binder cumulant isolating the contribution of the zero mode, ψ_i, to the action [26]. The effective action for the zero mode, in a L^4 volume and at the MF critical point (i.e. $r = 0$), is

$$S_{\mathrm{eff}}[\psi_i] = L^4 \left[\frac{u}{4!}\left(\sum_{i=1}^{n}\psi_i^2\right)^2 + \frac{v}{4!}\sum_{i=1}^{n}\psi_i^4 \right] \ , \tag{24}$$

and the partition function is

$$\mathcal{Z}_{\mathrm{eff}}(n) = \int \left(\prod_{i=1}^{n} d\psi_i\right) \exp(-S_{\mathrm{eff}}[\psi_i]) \ . \tag{25}$$

In the asymptotic regime (large L) the renormalization group implies that the relation $4u + 3v \simeq 0$ is satisfied with good precision [27]. Therefore:

$$\mathcal{Z}_{\rm eff}(n) = \frac{1}{\sqrt{3\pi}} \int \left(\prod_{i=1}^{n} d\psi_i \right) d\lambda \ \exp\left[-\frac{1}{3}\lambda^2 + \lambda \sum_i \psi_i^2 - \sum_i \psi_i^4 \right] . \tag{26}$$

A trivial computation tells that dimensionless ratios, like the Binder cumulant, do not depend on the specific value of v, thereby we have also fixed $v = 4!/L^d$ in the previous formula and in the rest of the section.

We can perform the integrals on the ψ variables

$$\mathcal{Z}_{\rm eff}(n) = \frac{1}{\sqrt{3\pi}} \int d\lambda \ e^{-\lambda^2/3} I_0(\lambda)^n \ , \tag{27}$$

where

$$I_m(\lambda) \equiv \int d\psi \exp\left[\lambda\psi^2 - \psi^4\right] \psi^m \ . \tag{28}$$

Now, we identify the moments of the magnetization in terms of the moments of the replicated variables (ψ_a)

$$\overline{\langle \mathcal{M}^{2m} \rangle} \to \langle \psi_a^{2m} \rangle \, . \tag{29}$$

with

$$\langle \psi_a^{2m} \rangle = \frac{1}{\sqrt{3\pi}} \int d\lambda \ \frac{I_{2m}(\lambda)}{I_0(\lambda)} \ e^{-\lambda^2/3} \, . \tag{30}$$

Evaluating numerically the previous integrals we obtain [26]

$$B^{\rm disordered} = 0.32455 \ldots . \tag{31}$$

The only source for triviality in (18) is the factor L/ξ. A perturbative RG calculation indicates [26]

$$\xi(L) \simeq \frac{L}{v(L)^{1/4}} \simeq L(\log L)^{1/8} \, , \tag{32}$$

where $v(L)$ is the coupling v renormalized at the scale L. The logarithm, which appears at the upper critical dimension, drives the renormalized coupling constant to zero in the thermodynamic limit. For the sake of completeness,

$$g_R^{\rm disordered} \simeq \frac{1}{\sqrt{\log L}} \, . \tag{33}$$

Hence, the theory is trivial.

We remark that this result relies in PT (we have computed [26] the behavior of the running coupling $v(L)$ starting from PT results) [27]. However, we have found a reasonable agreement between numerical simulations and PT [26].

6 Conclusions

We hope that this small note will be illustrative of the power of Finite-Size Scaling for the non perturbative study of Field Theory.

The questions that can be addressed are of a large variety, and the results are of remarkable accuracy. Universal quantities, freed of systematic errors, can be obtained by means of an infinite-volume extrapolation. One can even achieve tasks usually considered as impossible, as careful determinations of non-universal critical parameters. This is illustrated in the location of the perfect action (where the amplitude for the leading scaling-corrections vanishes) for the diluted Ising model).

The approach can be extended to the upper critical dimension as well. We have found agreement between analytical results and computer simulations in the $4d$ Ising model.

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