Additional Homeworks. Gravitation and Cosmology. Year 2017/2018

1. Consider the following space-time

$$ds^2 = \frac{a^2}{x^2}(-dt^2 + dx^2)\,,\ t \in \mathbb{R},\, x \in \mathbb{R}^+\,.$$

   Compute all the geodesics (space, null and time-like ones).

2. Compute all the geodesics of the following metric

$$ds^2 = (dx^1)^2 - \frac{1}{(x^2)^2}(dx^2)^2\,.$$

3. Show

$$V_{\alpha;\nu\kappa} - V_{\alpha;\kappa\nu} = V_\sigma R^\sigma{}_{\alpha\nu\kappa}\,.$$

4. A Killing vector ξ satisfies $\mathcal{L}_\xi \boldsymbol{g} = 0$, where $\mathcal{L}_\xi$ is the Lie-derivative along the vector ξ and $\boldsymbol{g}$ is the metric. Show that

$$\xi_{(\alpha;\beta)} = 0\,.$$

   Build some metrics with explicit symmetries and check the previous equation.

5. Let ξ_1 and ξ_2 be two Killing vectors. Show that
   (a) $[\xi_1, \xi_2]$ is a Killing vector.
   (b) $\lambda_1\xi_1 + \lambda_2\xi_2$ is also a Killing vector ($\lambda_1, \lambda_2 \in \mathbb{R}$).
   (c) Check the results (a) and (b) by building some metrics with symmetries.

6. (a) Show that the interval ds^2 of a plane space-time (t, x, y, z) which is rotating with angular speed Ω about the z axis of a inertial frame can be written as

$$ds^2 = -[1 - \Omega^2(x^2 + y^2)]dt^2 + 2\Omega(ydx - xdy)dt + dx^2 + dy^2 + dz^2\,.$$

   (b) Compute the differential equations which follow the time-like geodesics.
   (c) Show that the Newtonian limit of the time-like geodesics are the Newton equations written in a non-inertial frame. Identify and name all the terms.

7. Consider a planet orbiting in the equatorial plane of the Sun.
   (a) Assuming the Sun is oblate, compute its gravitational potential to order $O(1/r^3)$ in the equatorial plane.
   (b) Write the perturbed Kepler problem and solve it perturbatively to shown that the perihelion precesses.
   (c) Consider Mercury as an example and compute for it the precession angular speed of the perihelion.
   (d) Find in the literature the Sun's oblateness and compare your result with the measured Mercury precession.

   Hint: Check, for example, Goldstein's book.

8. Consider a planet orbiting the Sun in presence of interplanetary dust. One can show that the total potential (Sun+dust) is (assuming a spherical and uniform distribution of the dust):

$$V(r) = -\frac{GM}{r} + \frac{1}{2}Cr^2\,,$$

   where M is the solar mass and C is a positive constant.

(a) Could you relate the constant C to the density of interplanetary dust?

(b) Write the perturbed Kepler problem and solve it perturbatively ($C \ll GM/l^3$, l is a measure of the size of the Mercury orbit) to shown that the perihelion precesses.

(c) Compute the precession angular speed of the perihelion.

Hint: Check, for example, Goldstein's book.

9. Consider the following metric:

$$ds^2 = dudv + \log(x^2 + y^2)du^2 - dx^2 - dy^2$$

with $0 < x^2 + y^2 < 1$. Write the equations of the geodesics and show that $K = x\dot{y} - y\dot{x}$ is a conserved quantity.

10. (a) Let χ^μ be a vector satisfying

$$\nabla_\mu \chi_\nu + \nabla_\nu \chi_\mu = c g_{\mu\nu}$$

being c a positive constant and let γ be a null geodesic with tangent vector u^μ. Show that $u^\mu \chi_\mu$ is constant along the geodesic γ.

(b) Let $T_{\mu\nu}$ and ξ be a symmetric conserved tensor ($\nabla_\mu T^{\mu\nu} = 0$) and a Killing vector respectively. Show that the current defined as $J^\mu \equiv T^{\mu\nu}\xi_\nu$ is conserved ($\nabla_\mu J^\mu = 0$).

Hint: A Killing vector satisfies $\xi_{(\alpha;\beta)} = 0$ (see Problem 4).

11. Compute $a^\mu = Du^\mu/D\tau$ for a stationary particle in the Schwarzschild geometry. Compute its norm (a^2). Show that the force needed to hold the particle at rest is infinite as $r = 2M$.

12. Consider the following $4+1$ metric

$$ds_5^2 = e^{2A(v)}\eta_{\alpha\beta}dx^\alpha dx^\beta + dv^2$$

where v is the fourth spatial coordinate. Find the differential equation which is satisfied by $A(v)$ if the metric is solution of the Einstein equations in the vacuum with cosmological constant ($\Lambda \neq 0$). What is the sign of Λ? What happens if $\Lambda = 0$?

13. Consider a $4+1$ space-time.

- What is the Newtonian gravitational potential?
- Show that the metric

$$ds^2 = -\left(1 - \frac{r_s^2}{r^2}\right)dt^2 + \left(1 - \frac{r_s^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2$$

is a solution of the Einstein equations in the vacuum (is the $4+1$ analogous of the Schwarzschild solution). Moreover, $d\Omega_3^2$ is the metric of the 3-sphere S^3 and r_s is a constant.

14. Find the function $f(v)$ so that the metric

$$ds^2 = 2dudv + f(v)dw^2 + 2dwdz$$

is solution of the Einstein equations in the vacuum. Determine the conditions $f(v)$ should satisfy in order to have a plane space-time.

15. Compute the non zero components (modulo symmetries) of the Riemann tensor in the Schwarzschild geometry. Write these Riemann tensor components in the associated orthonormal basis.

Consider a radially falling observer in a Schwarzschild black hole. Write the geodesic deviation equations in the orthonormal basis associated to the falling observer.

Hint: Perform an (instantaneous) Lorentz transformation between the orthonormal basis of an observer at infinite and that of the falling observer.

16. Consider the following metric

$$ds^2 = -(1 - G(x,y))dt^2 - 2G(x,y)dtdz + (1 + G(x,y))dz^2 + dx^2 + dy^2 .$$

- Write the geodesic equations and the Christoffel's symbols.
- Identify two Killing vectors and write the two associated conserved quantities. Compute the norms of the these Killing vectors (time, null or spacelike?).
- Assuming vacuum,write the Einstein equations to obtain differential equations for the function G and find some solution for G.
- If $G(x,y) = \Phi(x^2 + y^2)$, find a third Killing vector, and write an explicit solution for the function G.

17. Consider two observers at rest in a gravitational field given by the following metric (anti-de Sitter space):

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{l^2}} + r^2 d\Omega_2^2 ,$$

l being a parameter with dimensions of length. The first observer has spatial coordinates $r_1 = a$, $\theta_1 = \pi/2$, $\phi_1 = 0$ and the second $r_2 = b$, $\theta_2 = \pi/2$, $\phi_2 = 0$, with $b \ll a \ll l$. The first observer emits light which propagates in the direction $\theta = \pi/2$ and $\phi = 0$. This light is detected by the second observer. What is the redshift as seen by the second observer (blue or red)?

18. The metric of Reissner-Nordstrøm

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 ,$$

is a solution of the Einstein equations with $\Lambda = 0$ and the energy-momentum tensor produced by a electromagnetic field. This metric describes the external gravitational field of a charged star or a static charged black hole. The parameters M and Q are related with the mass and charge of the star or black hole. We will assume hereafter that $M^2 > Q^2$, why?

- Show that a chargeless massive particle falling radially can never reach $r = 0$. Compare this situation with a particle falling in the Schwarzschild metric.
- Consider a particle initially is at rest at infinity. Compute the minimum radial coordinate of the particle during its falling, $r_{\min}$. Check that $r_{\min} < r_-$, where r_- is the smallest root of the equation

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0 .$$

19. Consider a Schwarzschild black hole with mass M.

- Show that the light can follow a circular orbit with radial coordinate $r = 3M$.
- Discuss the stability of this orbit.
- Compute in coordinate time the period of this orbit.
- Compute the period measured by an observer at rest in the infinity.
- Consider an observer at rest in a fixed point of the orbit $r = 3M$: compute the period measured by this observer?