Additional Homeworks. Gravitation and Cosmology. Year 2021/2022

1. Show
$$V_{\alpha;\nu\kappa} - V_{\alpha;\kappa\nu} = V_\sigma R^\sigma{}_{\alpha\nu\kappa} \,.$$

2. Show
$$\nabla_\mu V^\mu = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} V^\mu) \,,$$
where $g \equiv -\det(g_{\mu\nu})$.

3. Consider the four velocity, $\boldsymbol{u}$, of a free falling observer in a Schwarzschild geometry. Show, by computing explicitly the covariant derivative, that
$$\nabla_{\boldsymbol{u}} \boldsymbol{u} = 0 \,.$$

4. A Killing vector ξ satisfies $\mathcal{L}_\xi \boldsymbol{g} = 0$, where $\mathcal{L}_\xi$ is the Lie-derivative along the vector ξ and $\boldsymbol{g}$ is the metric. Show that
$$\xi_{(\alpha;\beta)} = 0 \,.$$

5. Consider the metric
$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 = -(x^1)^2 \mathrm{d}(x^2)^2 + \mathrm{d}(x^1)^2 \,.$$
Compute:
   - The differential equations of the geodesics: $x^1(\tau)$ and $x^2(\tau)$.
   - The Christoffel's symbols (Lagrangian method).
   - The differential equation of the geodesic $x^1(x^2)$.
   - Is plane this space?
   - A Killing vector and its associated conserved quantity.
   - Show that this Killing vector satisfies the following equations
   $$\nabla_i x_j + \nabla_j x_i = 0 \,.$$

6. Consider the following stationary and spherically symmetric metric
$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 = -f_1(r)\mathrm{d}t^2 + \frac{1}{f_1(r)}\mathrm{d}r^2 + f_2^2(r)\mathrm{d}\Omega_2^2 \,,$$
with $\mathrm{d}\Omega_2^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$.
   - Compute two Killing vectors, related with t and ϕ, and their associated conserved quantities (denoted by e and l, respectively)
   - Show that the particles move in an equatorial orbit following the equation
   $$\dot{r}^2 + V(r) = e^2 \,,$$
   con
   $$V(r) = f_1(r) \left(\epsilon + \frac{l^2}{f_2^2(r)} \right) \,,$$
   with $\epsilon = 0$ for massless particles ($\dot{r} = \mathrm{d}r/\mathrm{d}\lambda$, where λ is an affine parameter) or $\epsilon = 1$ for massive particles ($\dot{r} = \mathrm{d}r/\mathrm{d}\tau$).
   - Determine the conditions the functions f_1 and f_2 should satisfy in order to have null geodesics with the coordinate r constant.

7. Find the function $f(v)$ so that the metric

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}z + f(z)\mathrm{d}y^2 + 2\mathrm{d}y\mathrm{d}x$$

is solution of the Einstein equations in the vacuum. Determine the conditions $f(z)$ should satisfy in order to have a plane space-time.

8. Consider two observers at rest in a gravitational field given by the following metric (anti-de Sitter space):

$$\mathrm{d}s^2 = -\left(1 + \frac{r^2}{l^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 + \frac{r^2}{l^2}} + r^2\mathrm{d}\Omega_2^2\,,$$

l being a parameter with dimensions of length. The first observer has spatial coordinates $r_1 = a$, $\theta_1 = \pi/2$, $\phi_1 = 0$ and the second $r_2 = b$, $\theta_2 = \pi/2$, $\phi_2 = 0$, with $b \ll a \ll l$. The first observer emits light which propagates in the direction $\theta = \pi/2$ and $\phi = 0$. This light is detected by the second observer. What is the redshift as seen by the second observer (blue or red)?

9. The metric of Reissner-Nordstrøm

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2\,,$$

is a solution of the Einstein equations with $\Lambda = 0$ and the energy-momentum tensor produced by a electromagnetic field. This metric describes the external gravitational field of a charged star or a static charged black hole. The parameters M and Q are related with the mass and charge of the star or black hole. We will assume hereafter that $M^2 > Q^2$, why?

   - Show that a chargeless massive particle falling radially can never reach $r = 0$. Compare this situation with a particle falling in the Schwarzschild metric.
   - Consider a particle initially is at rest at infinity. Compute the minimum radial coordinate of the particle during its falling, $r_{\min}$. Check that $r_{\min} < r_-$, where r_- is the smallest root of the equation

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0\,.$$

10. Consider a Schwarzschild black hole with mass M.

   - Show that the light can follow a circular orbit with radial coordinate $r = 3M$.
   - Discuss the stability of this orbit.
   - Compute in coordinate time the period of this orbit.
   - Compute the period measured by an observer at rest in the infinity.
   - Consider an observer at rest in a fixed point of the orbit $r = 3M$. Compute the period measured by this observer?

11. Build the three possible maximally symmetric spaces in 1+3 dimensions. Help: You need to build the Minkowski, de Sitter and anti-de Sitter spaces.

12. Write an integral, as function of H_0 and the Ωs, to compute the proper distance to the visible horizon. What is this distance today (light-years)? Take: $h = 0.69$, $\Omega_M = 0.31$, $\Omega_\Lambda = 0.69$ and $\Omega_R = 0$. Help: Evaluate numerically the integral.

13. Show ($K = 0$):

$$\ddot{z} = \frac{\dot{z}^2}{1+z}\left(\frac{5}{2} + \frac{3p}{2\rho}\right)\,.$$

14. Consider a universe with $h = 0.7$, $\Omega_\Lambda = 0.55$, $\Omega_M = 0.45$ and $\Omega_R = 0$.

- Compute the age of the universe.
- Compute its curvature.
- Compute $m - M$ and the luminosity distance (d_L) for a star with $z = 4$?. How long was the light of this star emitted?

15. Show:

$$\Omega_k(z) = \frac{\Omega_k}{\Omega_M(1+z) + \Omega_R(1+z)^2 + \Omega_\Lambda(1+z)^{-2} + \Omega_k}\,.$$