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Comment on "Kinetic theory models for granular mixtures with unequal granular temperature: Hydrodynamic velocity" [Phys. Fluids 33, 043321 (2021)]

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In a recent study, Solsvik and Manger[1] (referred to as the SM-theory) have proposed a kinetic theory for granular mixtures where the velocity distribution functions for each species $f_i(\mathbf{r}, \mathbf{v}; t)$ are assumed to be Maxwellian distributions. Since energy equipartition is broken for inelastic collisions, those distributions are defined in terms of the partial temperatures T_i, which are in general different from the granular temperature T. In addition, in contrast to the previous studies based on the same assumption, the distributions f_i take into account the differences in the mean flow velocities $\mathbf{U}_i$ of the species. Following this approach, the authors evaluate some of the collision integrals appearing in the balance equations for the momentum and kinetic energy. In particular, they obtain corrections to the collision contributions to the momentum and heat fluxes, which are of order $|\mathbf{U}_i - \mathbf{U}_j|^2$ and $|\mathbf{U}_i - \mathbf{U}_j|^4$.

On the other hand, a different way of analyzing transport properties in granular mixtures has been carried out in recent years by Garzó, Dufty, and Hrenya (the GDH-theory).[2] In contrast to the SM-theory, the GHD-theory solves the Enskog kinetic equation by means of the Chapman–Enskog method up to the first order in spatial gradients (Navier–Stokes hydrodynamic order). The transport coefficients are given in terms of the solutions of a set of coupled linear integral equations, which are approximately solved by considering the leading terms in a Sonine polynomial expansion. The GDH-theory extends to granular mixtures the results derived years ago for monocomponent granular gases by established kinetic theory models.[3] These theories extend to arbitrary inelasticity the results obtained for nearly elastic systems in the seminal works of Jenkins and Savage[4] and Lun *et al.*[5] The accuracy of the GDH-theory for granular mixtures has been tested with computer simulations and even with real experiments. In the case of computer simulations, the results obtained in the GDH-theory for the tracer diffusion coefficient[6] and the shear viscosity coefficient of a heated granular mixture[7] show a very good agreement with simulations for conditions of practical interest. In addition, the hydrodynamic profiles derived from the GHD-theory for a single granular fluid compare well with experiments of a three-dimensional system of mustard seeds fluidized by vertical vibrations of the container.[8] All these studies clearly show the applicability of the GDH-theory for densities outside the low-density regime and values of inelasticity beyond the quasi-elastic limit.

One of the main deficiencies of the SM-theory is that it does not attempt to solve any kinetic equation since it supposes that the distribution functions of each species are *local* Maxwellian distributions even for inhomogeneous states. In this sense, the SM approach could be considered as a reasonable approach to estimate the collisional transfer contributions to the momentum and heat fluxes but not their kinetic contributions. In particular, the SM-theory yields vanishing Navier–Stokes transport coefficients for *dilute* granular mixtures, which is of course a wrong result.[9]

Therefore, it would be convenient to assess the degree of reliability of the SM-theory by comparing its predictions for the collisional contributions to the fluxes with those obtained from the GDH-theory to first-order in spatial gradients. Here, for the sake of concreteness, we will address our attention to the bulk η_b and shear viscosity η coefficients where computer simulations have clearly shown the accuracy of the GDH-theory, even for strong inelasticity.

According to Eqs. (41) and (77) of Ref. 1, to first order in gradients, the bulk viscosity η_b for a binary mixture of inelastic hard spheres in the SM-theory can be identified as

$$\eta_b^{\rm SM} = \frac{\sqrt{2\pi}}{9}\sum_{i,j=1}^{2} n_i n_j \sigma_{ij}^4 \chi_{ij} m_{ij}^2 (1+\alpha_{ij}) \left(\frac{T_i^{(0)}}{m_i} + \frac{T_j^{(0)}}{m_j}\right)^{3/2} \left(\frac{1}{T_i^{(0)}} + \frac{1}{T_j^{(0)}}\right), \tag{1}$$

where n_i is the number density of species i, $\sigma_{ij} = (\sigma_i + \sigma_j)/2$, σ_i and m_i being the diameter and mass of particles of species i, $m_{ij} = m_i m_j/(m_i + m_j)$, χ_{ij} is the pair correlation function, α_{ij} is the coefficient of restitution for collisions i-j, and $T_i^{(0)}$ is the zeroth-order contribution to the partial temperature of species i. In addition, upon obtaining Eq. (1), use has been made of the fact that the velocity differences $|\mathbf{U}_i - \mathbf{U}_j|$ are at least of first order in spatial gradients so that $\nabla \mathbf{U}_i = \nabla \mathbf{U}_j = \nabla \mathbf{U}$ and nonlinear terms in $|\mathbf{U}_i - \mathbf{U}_j|$ are neglected in the Navier–Stokes approximation. Here, $\mathbf{U}$ is the mean flow velocity of the mixture. In the SM-theory, the collisional contribution η_c to the shear viscosity is simply given by

$$\eta_c^{\rm SM} = \frac{3}{5}\eta_b^{\rm SM}. \tag{2}$$

The expressions of $\eta_b^{\rm GDH}$ and $\eta_c^{\rm GDH}$ in the GDH-theory are more intricate. For d-dimensional granular mixtures, the bulk viscosity $\eta_b^{\rm GDH}$ can be written as

$$\eta_b^{\rm GDH} = \eta_b' + \eta_b'', \tag{3}$$

where[2,10]

$$\eta_b' = \frac{\sqrt{2}\pi^{(d-1)/2}}{d^2\Gamma\left(\frac{d}{2}\right)} \sum_{i,j=1}^{2} n_i n_j \sigma_{ij}^{d+1} \chi_{ij} m_{ij} (1+\alpha_{ij}) \left(\frac{T_i^{(0)}}{m_i} + \frac{T_j^{(0)}}{m_j}\right)^{1/2}, \tag{4}$$

$$\eta_b'' = -\frac{\pi^{d/2}}{d\Gamma\left(\frac{d}{2}\right)} \sum_{i,j=1}^{2} n_i n_j \sigma_{ij}^{d} \chi_{ij} \mu_{ji} (1+\alpha_{ij}) \varpi_i. \tag{5}$$

Here, $\mu_{ji} = m_j/(m_i + m_j)$ and ϖ_i refers to the first-order contributions to the partial temperatures T_i. The quantities ϖ_i have been determined in Ref. 10 in terms of the parameter space of the mixture. The collision contribution $\eta_c^{\rm GDH}$ to the shear viscosity is[2,11]

$$\eta_c^{\rm GDH} = \frac{2\pi^{d/2}}{d(d+2)\Gamma\left(\frac{d}{2}\right)} \sum_{i,j=1}^{2} n_i \sigma_{ij}^{d} \chi_{ij} \mu_{ij} (1+\alpha_{ij}) \eta_j^k + \frac{d}{d+2}\eta_b^{\rm GDH}, \tag{6}$$

where the kinetic contributions η_i^k obey the set of algebraic equations

$$\begin{aligned} \sum_{j=1}^{2} \left(\tau_{ij} - \frac{1}{2}\zeta^{(0)}\delta_{ij}\right)\eta_j^k &= n_i T_i^{(0)} + \sum_{j=1}^{2} \frac{\pi^{d/2}}{d(d+2)\Gamma\left(\frac{d}{2}\right)} \\ &\times n_i n_j \sigma_{ij}^d m_{ij} \chi_{ij} (1+\alpha_{ij}) \\ &\times \left[\mu_{ji}(3\alpha_{ij} - 1)\left(\frac{T_i^{(0)}}{m_i} + \frac{T_j^{(0)}}{m_j}\right) - 4\frac{T_i^{(0)} - T_j^{(0)}}{m_i + m_j}\right]. \end{aligned} \tag{7}$$

The expressions of the zeroth-order cooling rate $\zeta^{(0)}$ and the collision frequencies τ_{ij} are given by Eqs. (5.51) and (5.65)–(5.66), respectively of Ref. 11. In addition, the temperature ratio $T_1^{(0)}/T_2^{(0)}$ is determined by equating the partial cooling rates $\zeta_1^{(0)} = \zeta_2^{(0)} = \zeta^{(0)}$. It is important to remark that for elastic collisions ($\alpha_{ij} = 1$) and hard spheres ($d = 3$), Eqs. (3)–(7) of the GDH-theory agree with the results derived many years ago from the Enskog kinetic theory for multicomponent molecular mixtures.[12]

A comparison between Eqs. (1) and (2) and (3)–(7) shows that in general the results obtained for η_b and η_c from the SM-theory differ from those derived from the GDH-theory, even for elastic collisions. On the other hand, when both the first-order contributions to the partial temperatures ϖ_i and the kinetic contributions η_i^k are neglected in Eqs. (3) and (6), then the SM and GDH theories agree for elastic collisions for $d = 3$. To illustrate the differences between both theories for inelastic collisions, for the sake of simplicity, we consider the case of a common coefficient of restitution ($\alpha_{11} = \alpha_{22} = \alpha_{12} \equiv \alpha$) of an equimolar mixture [$x_1 = n_1/(n_1 + n_2) = \frac{1}{2}$] with $\sigma_1/\sigma_2 = 2$. Moreover, for spheres,

$$\chi_{ij} = \frac{1}{1-\phi} + \frac{3}{2}\frac{\phi}{(1-\phi)^2}\frac{\sigma_i\sigma_j M_2}{\sigma_{ij} M_3} + \frac{1}{2}\frac{\phi^2}{(1-\phi)^3}\left(\frac{\sigma_i\sigma_j M_2}{\sigma_{ij} M_3}\right)^2, \tag{8}$$

where ϕ is the total solid volume fraction and $M_s = \sum_i x_i \sigma_i^s$. Figures 1 and 2 show the α-dependence of the reduced coefficients $\eta_b(\alpha)/\eta_b(1)$ and $\eta_c(\alpha)/\eta_c(1)$ for $\phi = 0.1$ and two values of the mass ratio. Here, $\eta_b(1)$ and $\eta_c(1)$ refer to the bulk and shear viscosities, respectively, for elastic collisions. It is quite apparent that while the SM-theory reproduces quite well the dependence of η_b on α (in particular for $m_1/m_2 = 10$), important quantitative discrepancies appear in the case of η_c for strong dissipation. Much more significant differences between the SM and GDH theories are present in Fig. 3 where the density dependence of the ratio $\eta_c(\alpha, \phi)/\eta_c(1, \phi)$ is plotted for two values of α. While the SM-theory predicts a tiny density dependence

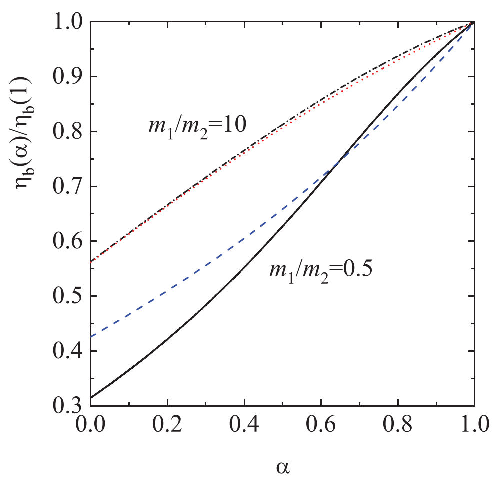

FIG. 1. Plot of the (reduced) bulk viscosity $\eta_b(\alpha)/\eta_b(1)$ vs the common coefficient of restitution α for $d = 3$, $x_1 = \frac{1}{2}$, $\sigma_1/\sigma_2 = 2$, $\phi = 0.1$, and two different values of the mass ratio: $m_1/m_2 = 0.5$ (solid line for the GDH-theory and dashed line for the SM-theory) and $m_1/m_2 = 10$ (dashed-dotted line for the GDH-theory and dotted line for the SM-theory).

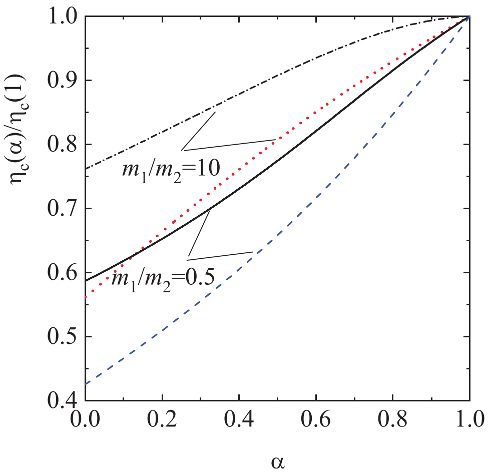

FIG. 2. The same as in Fig. 1 but for the (reduced) collisional shear viscosity $\eta_c(\alpha)/\eta_c(1)$.

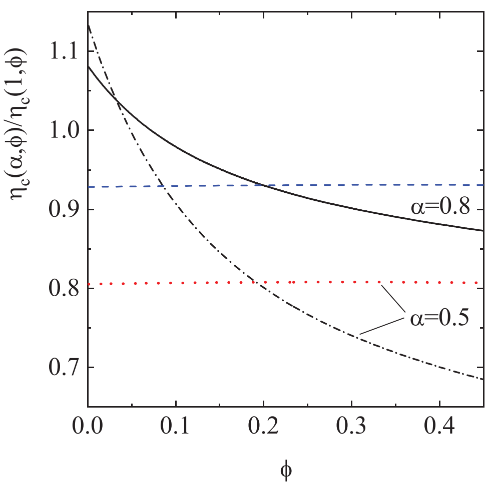

FIG. 3. Plot of $\eta_c(\alpha, \phi)/\eta_c(1, \phi)$ vs the volume fraction ϕ for $d = 3$, $x_1 = \frac{1}{2}$, $\sigma_1/\sigma_2 = 2$, $m_1/m_2 = 10$, and two values of α: $\alpha = 0.8$ (solid line for the GDH-theory and dashed line for the SM-theory) and $\alpha = 0.5$ (dashed-dotted line for the GDH-theory and dotted line for the SM-theory).

of the ratio $\eta_c(\alpha, \phi)/\eta_c(1, \phi)$ for any α, the GDH-theory shows that this ratio decreases with increasing density. The weak influence of the overall solid volume fraction ϕ on the (reduced) collisional shear viscosity in the SM-theory is essentially due to the fact that the density dependence of the ratio $\eta_c(\alpha, \phi)^{\text{SM}}/\eta_c(1, \phi)^{\text{SM}}$ is *only* via the partial temperatures $T_i^{(0)}$ (whose dependence on ϕ is very small). On the other hand, in the GDH-theory, the dependence of $\eta_c(\alpha, \phi)^{\text{GDH}}/\eta_c(1, \phi)^{\text{GDH}}$ on ϕ not only occurs through $T_i^{(0)}$ but also through the kinetic contributions η_i^k. In particular, in the limiting case of mechanically equivalent particles ($m_1 = m_2$, $\sigma_1 = \sigma_2$, and $\alpha_{ij} = \alpha$), while the SM-theory predicts that $\eta_c(\alpha, \phi)^{\text{SM}}/\eta_c(1, \phi)^{\text{SM}} = (1 + \alpha)/2$ (namely, it is independent of ϕ), the GDH-theory yields the result $\eta_c(\alpha, \phi)^{\text{GDH}}/\eta_c(1, \phi)^{\text{GDH}} = A(\alpha, \phi)[(1 + \alpha)/2]$, where the function A exhibits in general a complex dependence on both α and ϕ. In this context, it is worthwhile recalling that the results of the GDH-theory for the shear viscosity η agree quite well with computer simulations for moderate densities and/or strong inelasticity (see, for instance, Figs. 6–8 of Ref. 7).

In summary, as expected I have shown that the SM-theory is not able to completely capture the dependence of the bulk and shear viscosities on inelasticity in binary granular mixtures at moderate densities. However, despite these inadequacies, the SM-theory can be still considered as a valuable approach for estimating the collisional contributions to the fluxes since it captures at least qualitatively well (see Figs. 1 and 2) the α-dependence of η_b and η_c.

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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